On homogeneous structures of real hypersurfaces in non-flat complex space forms

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Abstract. This paper contains a survey about real hypersurfaces in non-flat complex space forms. We mainly discuss relations between the classes of generalized symmetric spaces and homogeneous real hypersurfaces in non-flat complex space forms. The homogeneous structure tensors play the central role in our discussion.

1. Introduction

In Riemannian Geometry, there are several important classes of manifolds which have good properties. The following 6 classes are considered as the most important examples:
(1) Riemannian symmetric spaces;
(2) Naturally reductive homogeneous spaces;
(3) Riemannian manifolds all of whose geodesics are orbits of one-parameter subgroups of isometries (called g.o. spaces);
(4) Commutative spaces;
(5) Weakly symmetric spaces;
(6) D’Atri spaces.
(For definitions, see later of this section).

The inclusion relations of these classes are:

\[ (2) \cup (3) \cup (4) \subset (6), \]
\[ (2) \subset (3), \]
\[ (5) \subseteq (4), \]
\[ (1) \subset (2) \cap (4) \cap (5). \]

On the other hand, when we focus our attention on real hypersurfaces in a non-flat complex space form $\mathcal{M}_n(c)$ with $n \geq 3$, we know the fact that there are no real hypersurfaces with parallel Ricci tensor (see [6]).
In particular, there are no Riemannian locally symmetric real hypersurfaces in $\mathbb{M}_n(c)$ ($n \geq 3, c \neq 0$). So, it is an important and an interesting problem to investigate the inclusion relations of the above 5 classes of spaces in real hypersurfaces in $\mathbb{M}_n(c)$. Concerning of this problem we have:

**Conclusion** In Hopf real hypersurfaces of a non-flat complex space form, Naturally reductive $\Leftrightarrow$ D’Atri $\Leftrightarrow$ g.o. $\Leftrightarrow$ weakly symmetric $\Leftrightarrow$ of type (A).

Finally of this section, we give the definitions of several classes of spaces.

**Definition 1.1.** Let $M = G/K$ be a Riemannian homogeneous space and $(\cdot, \cdot)$ its metric tensor, where $G$ is a transitive group of isometries of $M$ and $K$ its isotropy subgroup at some point $p \in M$. Then $M$ is said to be a naturally reductive Riemannian homogeneous space if there exists a subspace $m$ of the Lie algebra $g$ of $G$ which satisfies the following conditions:

(i) $g = \mathfrak{t} \oplus m$,

(ii) $\text{Ad}(K)m \subset m$,

(iii) $(\mathfrak{[X,Y]}_m, Z) + (Y, \mathfrak{[X,Z]}_m) = 0$, $X, Y, Z \in \mathfrak{m}$,

where $\mathfrak{t}$ is the Lie algebra of $K$ and $\mathfrak{[X,Y]}_m$ denotes the $m$-component of $\mathfrak{[X,Y]}$.

**Definition 1.2.** Let $\mathfrak{I}_0(M)$ denote the largest connected group of isometries of a Riemannian manifold $(M, g)$. Then a homogeneous Rimannian manifold $(M, g)$ is called a commutative space if all $\mathfrak{I}_0(M)$-invariant differential operators commute.

**Definition 1.3.** ([3]) A Riemannian manifold $M$ is called a weakly symmetric space if for any two points $p, q$ in $M$ there exists an isometry of $M$ mapping $p$ to $q$ and $q$ to $p$.

**Definition 1.4.** A Riemannian manifold $M$ is called a D’Atri space if each geodesic symmetry $s_m$ at a point $m$ is divergence-preserving. More precisely, for any vector field $X$ on $M$, the following equation holds:

$$\text{div} ((s_m)_*X) \circ s_m = \text{div} X.$$ 

2. Homogeneous real hypersurfaces in non-flat complex space forms

In this section, we examine the homogeneous real hypersurfaces in non-flat complex space forms $\mathbb{M}_n(c)$ ($n \geq 2, c \neq 0$). A real hypersurface in $\mathbb{M}_n(c)$ is called homogeneous if it is an orbit of a subgroup of the isometry group of $\mathbb{M}_n(c)$.

First, we summarize how these are constructed. Let $(U, K)$ be an almost effective Hermitian symmetric pair of compact type of rank 2. Further, let $\mathfrak{u} = \mathfrak{t} \oplus \mathfrak{p}$ be the canonical decomposition of the Lie algebra $\mathfrak{u}$ of $U$ where $\mathfrak{t}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is the vector subspace of $\mathfrak{u}$. For the Killing form $B$ of $\mathfrak{u}$, we define a
positive definite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ by $\langle X, Y \rangle = -B(X, Y)$ for $X, Y \in \mathfrak{p}$. Let $S^{2n+1}(1)$ be the unit sphere in $\mathfrak{p}$ with respect to $\langle \cdot, \cdot \rangle$ whose center is 0. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and $a \in \mathfrak{a}$ a unit regular element. Then a homogeneous real hypersurface $M[a]$ in $\mathbb{C}P_n$ is induced through the following commutative diagram:

$$
\begin{array}{ccc}
\text{Ad}(K)a = M'_a & \longrightarrow & S^{2n+1}(1) \\
\pi & \downarrow & \pi \\
M[a] & \longrightarrow & \mathbb{C}P_n(4)
\end{array}
$$

where $\pi : S^{2n+1}(1) \to \mathbb{C}P_n(4)$ is the Hopf fibration (for more details, see [21]).

Homogeneous real hypersurfaces in $\mathbb{C}P_n$ were completely classified by R. Takagi (see [22]). The following table gives the classification of them:

<table>
<thead>
<tr>
<th>type</th>
<th>$(U, K)$</th>
<th>$\dim M[a]$</th>
<th>$\Sigma_+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>$(SU(p+1) \times SU(q+1), \ S(U(1) \times U(p)) \times S(U(1) \times U(q)))$</td>
<td>$2(p+q) - 3$</td>
<td>$BC_1 \cup BC_1$</td>
</tr>
<tr>
<td>(B)</td>
<td>$(SO(p+2), SO(2) \times SO(p))$</td>
<td>$2p - 3$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>(C)</td>
<td>$(SU(p+2), S(U(2) \times U(p)))$</td>
<td>$4p - 3$</td>
<td>$BC_2$</td>
</tr>
<tr>
<td>(D)</td>
<td>$(SO(10), U(5))$</td>
<td>$17$</td>
<td>$BC_2$</td>
</tr>
<tr>
<td>(E)</td>
<td>$(E_6, SO(10) \cdot S^1)$</td>
<td>$29$</td>
<td>$BC_2$</td>
</tr>
</tbody>
</table>

Here $\Sigma_+$ denotes the set of restricted roots of $(U, K)$.

Another expressions of them are the following:

**Theorem 2.1.** ([22]) Let $M$ be a homogeneous real hypersurface of $\mathbb{C}P_n(4)$. Then $M$ is locally congruent to one of the following spaces:

(A) a tube of radius $r$ over a totally geodesic $\mathbb{C}P_k(4)$ ($0 \leq k \leq n - 1$), $0 < r < \frac{\pi}{2}$;

(B) a tube of radius $r$ over a complex quadric $Q_{n-1}$, $0 < r < \frac{\pi}{4}$;

(C) a tube of radius $r$ over $\mathbb{C}P_1 \times \mathbb{C}P_{n-2}$, and $n \geq 5$ is odd, $0 < r < \frac{\pi}{4}$;

(D) a tube of radius $r$ over a complex Grassmann $G_{2,5}(\mathbb{C})$, and $n = 9$,

$0 < r < \frac{\pi}{4}$;

(E) a tube of radius $r$ over a Hermitian symmetric space $SO(10)/U(5)$, and $n = 15$, $0 < r < \frac{\pi}{4}$.

On the other hand, Hopf hypersurfaces with constant principal curvatures in a complex hyperbolic space $\mathbb{C}H_n$ are classified by J. Berndt [3]. They are all homogeneous real hypersurfaces.

**Theorem 2.2.** ([3], Theorem 1) Let $M$ be a connected real hypersurface of $\mathbb{C}H_n$ ($n \geq 2$) with constant principal curvatures whose structure vector $\xi$ is principal. Then $M$ is orientable and holomorphic congruent to an open part of one of the following hypersurfaces:

(A) a horoshpere in $\mathbb{C}H_n$;
(A) a tube of radius \( r \in \mathbb{R}_+ \) over a totally geodesic \( \mathbb{C}H_k \) (0 \( \geq k > n - 1 \));
(B) a tube of radius \( r \in \mathbb{R}_+ \) over a totally geodesic totally real \( \mathbb{R}H_n \).

Secondly, we review some characterization theorems of homogeneous real hypersurfaces.

Let \( M \) be a connected orientable real hypersurfaces of a non-flat complex space form \( \overline{M}_n(c) \). Let \( J \) be the complex structure of \( \overline{M}_n(c) \). Then we denote by \( g \) the induced Riemannian metric on \( M \) and by \( \nu \) a unit normal vector field along \( M \) in \( \overline{M}_n(c) \). The almost contact metric structure \((\phi, \xi, \eta, g)\) on \( M \) is defined by

\[
\xi = -J\nu, \quad \eta(X) = g(X, \xi), \quad \phi X = (JX)^T, \quad X \in T M,
\]

where \( TM \) denotes the tangent bundle of \( M \) and \((\ )^T\) the tangential component of a vector.

Let \( A \) and \( S \) denote the shape operator and the Ricci tensor \( M \), respectively.

Homogeneous real hypersurfaces of type \((A)\) are characterized by

**Theorem 2.3.** ([19], [12], [13]) For a real hypersurface of a non-flat complex space form \( \overline{M}_n(c) \) (\( n \geq 2, c \neq 0 \)), the following conditions are equivalent:

(i) \( \phi A = A\phi \),
(ii) \( (\nabla X A)Y = -\frac{c}{4} \{ \eta(Y)\phi X + g(\phi X, Y)\xi \} \),
(iii) \( \|\nabla A\|^2 = c^2(n - 1) \),
(iv) \( M \) is of type \((A)\).

Recently, we proved the following theorem:

**Theorem 2.4.** ([18], [17]) Let \( M \) be a connected real hypersurface in a non-flat complex space form \( \overline{M}_n(c) \) (\( n \geq 2, c \neq 0 \)) on which the structure vector \( \xi \) is principal. Then the shape operator \( A \) satisfies

\[
g((\nabla^2 X Y)A)Z, W) = -\frac{c}{4} \{ g(\phi AX, W)g(\phi Y, Z) + g(\phi AX, Z)g(\phi Y, W) \}
\]

for any \( X, Y, Z, W \in \{\xi\}^\perp \) if and only if \( M \) is locally congruent to the homogeneous real hypersurface of type \((A)\).

The following two theorems give characterizations of homogeneous real hypersurfaces of type \((A)\) and \((B)\).

**Theorem 2.5.** ([8]) Let \( M \) be a connected real hypersurface in a complex space form \( \overline{M}_n(c) \) (\( n \geq 2, c \neq 0 \)) on which the structure vector \( \xi \) is principal. Then the shape operator \( A \) satisfies

\[
g((\nabla X A)Y, Z) = 0
\]

for any \( X, Y, Z \in \{\xi\}^\perp \) if and only if \( M \) is locally congruent to one of homogeneous real hypersurfaces of type \((A)\) and \((B)\).
Theorem 2.6. ([20]) Let $M$ be a connected real hypersurface in a non-flat complex space form $\mathbb{M}_n(c)$ $(n \geq 2, c \neq 0)$ on which the structure vector $\xi$ is principal. Then the Ricci tensor $S$ satisfies
\[ g((\nabla_X S)Y, Z) = 0 \]
for any $X, Y, Z \in \{\xi\}^\perp$ if and only if $M$ is locally congruent to one of homogeneous real hypersurfaces of type $(A)$ and $(B)$.

The following two theorems give characterizations of real hypersurfaces of type $(B)$.

Theorem 2.7. ([7]) Let $M$ be a real hypersurface of $\mathbb{C}P_n$, $n \geq 3$, on which the structure vector $\xi$ is principal with principal curvature $\alpha$. If $\alpha Tr A \leq \alpha^2 - (n-1)c$, then
\[ \|\nabla A\|^2 \geq \frac{c}{4}(n-1)(7c + 6\alpha^2), \]
where the equality holds if and only if $M$ is locally congruent to a real hypersurface of type $(B)$.

Theorem 2.8. ([17]) Let $M$ be a connected real hypersurface in $\mathbb{M}_n(c)$ $(n \geq 2, c \neq 0)$ whose structure vector $\xi$ is principal with principal curvature $\alpha \neq 0$. Then the shape operator $A$ satisfies
\[ g((\nabla_X^2 A)Y, Z) = \frac{\alpha}{4} \left\{ g(\phi A X, Y)g((4\phi A + \frac{\alpha}{\phi})Z, W) + g(\phi A X, Z)g((4\phi A + \frac{\alpha}{\phi})Y, W) + g(\phi A X, W)g((4\phi A + \frac{\alpha}{\phi})Y, Z) \right\}, \]
for $X, Y, Z, W \in \{\xi\}^\perp$ if and only if $M$ is locally congruent to one of the homogeneous real hypersurfaces of type $(A_0)$ and $(B)$.

Final of this section we give the following three Lemmas for later use.

Lemma 2.1. The shape operator of a real hypersurface of type $(A)$ satisfies the following relations:
\[ A^2 - \alpha A - \frac{\alpha}{4} I = -\frac{\alpha}{4} \eta \otimes \xi, \]
\[ (\nabla_X A)Y = -\frac{\alpha}{4} \left\{ \eta(Y)\phi X + g(\phi X, Y)\xi \right\}, \]
where $I$ denotes the identity mapping of $TM$.

Lemma 2.2. The shape operator of homogeneous real hypersurfaces of type $(A_0)$ and $(B)$ satisfies the following relations:
\[ \phi A + A\phi = -\frac{\alpha}{4} \phi, \]
\[ A^2 + \frac{\alpha}{4} A - \frac{\alpha}{4} I = (\alpha^2 + \frac{\alpha}{4} c)\eta \otimes \xi, \]
\[ (\nabla_X A)Y = -\frac{\alpha}{4} \left\{ (2\eta(X)(A\phi - \phi A)Y + \eta(Y)(A\phi - 3\phi A)X \right. \]
\[ + g((A\phi - 3\phi A)X, Y)\xi \right\}. \]
Lemma 2.3. ([23]) The tangent space of a homogeneous real hypersurface of type (B) in \( \mathbb{C}P_n \) can be decomposed as follows:

\[
TM = \mathbb{R}\xi \oplus T_x \oplus T_{-\frac{1}{2}}, \quad A\xi = \frac{-4x}{x^2 - 1}\xi, \quad 0 < x < 1,
\]

where \( T_\lambda \) denotes the eigenspace of the shape operator with the principal curvature \( \lambda \). Further, we have \( \phi T_x = T_{-\frac{1}{2}} \).

3. Homogeneous structures and applications

In this section we mention about homogeneous structures on homogeneous real hypersurfaces of non-flat complex space forms. We start with the following:

Theorem 3.1. ([1]) A connected, complete and simply connected Riemannian manifold \( M \) is homogeneous if and only if there exists a tensor field \( T \) of type (1, 2) on \( M \) such that

(i) \( g(T_X Y, Z) + g(Y, T_X Z) = 0 \),
(ii) \( (\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_X Y, Z) - R(Y, T_X Z) \),
(iii) \( (\nabla_T X)_Y = [T_X, T_Y] - T_X T_Y \),

for \( X, Y, Z \in \mathfrak{X}(M) \). Here \( R \) denotes the Riemannian curvature tensor of \( M \) and \( \mathfrak{X}(M) \) is the Lie algebra of all \( C^\infty \) vector fields over \( M \).

In the above theorem, if we put \( \tilde{\nabla} := \nabla - T \), then the conditions (i), (ii) and (iii) are equivalent to \( \tilde{\nabla} g = 0, \tilde{\nabla} R = 0 \) and \( \tilde{\nabla} T = 0 \), respectively. We call \( T \) a homogeneous structure on \( M \).

If \( T \) satisfies in addition the condition (iv) \( T_X X = 0 \), then \( M \) is a naturally reductive homogeneous space and we call \( T \) a naturally reductive homogeneous structure on \( M \).

For real hypersurfaces of type (A) in complex space forms, we have the following theorem:

Theorem 3.2. ([14]) Let \( M \) be a real hypersurface of type (A) in a non-flat complex space form \( \overline{M}_n(c) \) (\( n \geq 2, c \neq 0 \)). Then

\[
T_X Y = \eta(Y)\phi AX - \eta(X)\phi AY - g(\phi AX, Y)\xi
\]

defines a naturally reductive homogeneous structure on \( M \).

Further, we have

Theorem 3.3. ([15]) In homogeneous real hypersurfaces of non-flat complex space forms, only the real hypersurfaces of type (A) are naturally reductive.
About weakly symmetric real hypersurfaces in non-flat complex space forms, we know the following theorem:

**Theorem 3.4.** ([3]) Let $M$ be a real hypersurface of type (A) in a non-flat complex space form. Then $M$ is weakly symmetric space.

In general a naturally reductive homogeneous space is a D’Atri space. And a weakly symmetric space is a D’Atri space. So, according to above theorems, we deduce that real hypersurfaces of type (A) are D’Atri space.

For real hypersurfaces of type (B), we have the following theorem:

**Theorem 3.5.** ([16]) The following tensor $T^{(B)}$ defines a homogeneous structure on a real hypersurface of type (B):

$$T^{(B)}_X Y = \frac{\alpha}{2} \eta(X) \phi Y + \eta(Y) \phi A X - g(\phi A X, Y) \xi,$$

where $\alpha$ is the principal curvature in the direction of $\xi$.

For a D’Atri space, we have the following:

**Proposition 3.1.** ([9]) Let $M$ be a D’Atri space. Then the curvature tensor $R$ and the Ricci tensor $S$ of $M$ satisfy the following Ledger conditions of order three and five:

$L_3 : (\nabla_X S)(X, X) = 0,$

$L_5 : \sum_{i,j} g(R(e_i, X)X, e_j)g((\nabla X R)(e_i, X)X, e_j) = 0, \quad X \in T_p M,$

where $\{e_i\}$ is an orthonormal basis of $T_p M, p \in M$.

The condition $L_3$ is equivalent to

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

This means that $S$ is cyclic-parallel.

Hopf hypersurfaces in $\mathbb{M}_n(c)$ with cyclic-parallel Ricci tensor are completely classified by J. H. Kwon and H. Nakagawa:

**Theorem 3.6.** ([10], [11]) Let $M$ be a Hopf hypersurface of $\mathbb{M}_n(c)$ ($n \geq 2, c = \pm 4$). Then $S$ is cyclic-parallel if and only if

(a) $\mathbb{M}_n(c) = \mathbb{C}P_n$ and $M$ is locally congruent to a real hypersurface of type (A) or to one of type (B) with $\alpha = 2\sqrt{3}\sqrt{n-1};$

(b) $\mathbb{M}_n(c) = \mathbb{C}H_n$ and $M$ is locally congruent to a real hypersurface of type (A).

As an application of the homogeneous structure $T^{(B)}$, we simplify the proof of the following theorem of Cho and Vanhecke:
Theorem 3.7. ([5]) A Hopf hypersurface in a non-flat complex space form \( \overline{M}_n(c) \) \((n \geq 2, c \neq 0)\) is a D’Atri space if and only if it is locally congruent to a real hypersurface of type \((A)\).

Proof. According to Theorem 3.6, we only have to prove that the real hypersurface of type \((B)\) in \( \mathbb{C}P_n \) with \( \alpha = 2\sqrt{3}/\sqrt{n - 1} \) does not satisfy the Ledger condition of order five \( L_5 \). Owing to Theorem 3.1, (3.1) and the symmetric properties of \( R, L_5 \) can be written in the form

\[
\sum_{i,j} g(R(e_i, X)X, e_j)g(R(e_i, T_X(B)X)X, e_j) = 0. \tag{3.2}
\]

For unit tangent vectors \( u \in T_x, v \in T_{-1/2} \), we put \( X = \xi + u + v \) and substituting this \( X \) in the right-hand side of (3.2), we are led to the following by long and straightforward calculation:

\[
\begin{align*}
\sum_{i,j} g(R(e_i, X)X, e_j)g(R(e_i, T_X(B)X)X, e_j) &= \frac{8}{(n-1)}(-18n^3 + 16n^2 - 79n + 29). \\
\end{align*} \tag{3.3}
\]

For any integer \( n \geq 2 \), the right-hand side of (3.3) does not vanish, because this is equivalent to

\[
n(18n^2 - 16n + 79) = 29
\]

and \( n = 29 \) is not a solution of (3.4). So the real hypersurface of type \((B)\) with cyclic-parallel Ricci tensor does not satisfy \( L_5 \). This completes the proof of Theorem 3.7.

References


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