Abstract. We study on the geometry of CR-manifolds of contact type. This is a survey article whose contents are mainly based on the recent results in [14], [15] and [17].

1. Introduction

Let $TM$ be the tangent bundle of a $(2n + k)$-manifold $M$. A CR-structure of type $(n, k)$ on $M$ is a complex rank $n$ subbundle $\mathcal{H} \subset CTM = TM \otimes \mathbb{C}$ satisfying

(i) $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$,

(ii) $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$ (integrability).

We say $(M, \mathcal{H})$ the CR manifold of type $(n, k)$ and $n$ the CR dimension and $k$ the CR codimension of $(M, \mathcal{H})$. In particular, a CR manifold of type $(n, 1)$ is contact type (or hypersurface type). Hereafter, we deal with only contact type CR manifold. Then there exists a unique subbundle $D = \text{Re}\{\mathcal{H} \oplus \bar{\mathcal{H}}\}$, called the Levi distribution (maximally holomorphic distribution) of $(M, \mathcal{H})$, and a unique bundle map $J$ such that $J^2 = -I$ and $\mathcal{H} = \{X - iJX | X \in D\}$. We call $\{D, J\}$ the real representation of $\mathcal{H}$. Let $E \subset T^*M$ be the conormal bundle of $D$. If $M$ is an oriented CR-manifold then $E$ is a trivial bundle, hence $TM$ admits globally defined a nowhere zero section, i.e., a real one-form on $M$ such that $\text{Ker}(\eta) = D$. For $\{D, J\}$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on $M$. If the Levi form is hermitian, then the CR-structure is called pseudo-hermitian, in addition, if the Levi form is non-degenerate (positive or negative definite, resp.), then the CR-structure is called a non-degenerate (strongly pseudo-convex, resp.) pseudo-hermitian CR structure.

Lee ([24]) introduced the notion of pseudo-Einstein structure, namely, the pseudo-hermitian Ricci curvature tensor (of the Tanaka-Webster connection) is a scalar multiple of the Levi form, in a nondegenerate integrable CR manifold. This notion is a pseudo-homothetic (or $D$-homothetic) invariant. The pseudo-homothetic transformation is given by a special one of the gauge transformation (see section 2).
Also, in [14], [15] the present author defined another pseudo-homothetic invariant, i.e., the contact strongly pseudo-convex CR-space of constant pseudo-holomorphic sectional curvature (with respect to the Tanaka-Webster connection) (see section 3). In this article, we arrange the recent results on the geometry of CR-manifolds of contact type.

2. Preliminaries

We start by collecting some fundamental materials about contact Riemannian geometry and contact strongly pseudo-convex CR-manifold. We refer to [4], [5] for further details. All manifolds in the present paper are assumed to be connected and of class $C^\infty$.

A $(2n+1)$-dimensional manifold $M^{2n+1}$ is said to be a contact manifold if it admits a global one-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form $\eta$, there exists a unique vector field $\xi$, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field $X$. It is well-known that there also exists a Riemannian metric $g$ and a $(1,1)$-tensor field $\varphi$ such that

\begin{equation}
(1) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi,
\end{equation}

where $X$ and $Y$ are vector fields on $M$. From (1), it follows that

\begin{equation}
(2) \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi).
\end{equation}

A Riemannian manifold $M$ equipped with structure tensors $(\eta, g)$ satisfying (1) is said to be a contact Riemannian manifold or contact metric manifold and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold $M$, we define a $(1,1)$-tensor field $h$ by $h = \frac{1}{2}L_\xi \varphi$, where $L$ denotes Lie differentiation. Then we may observe that $h$ is symmetric and satisfies

\begin{equation}
(3) \quad h\xi = 0, \quad h\varphi = -\varphi h,
\end{equation}

\begin{equation}
(4) \quad \nabla_X \xi = -\varphi X - \varphi h X,
\end{equation}

where $\nabla$ is Levi-Civita connection. We denote by $R$ the Riemannian curvature tensor. Along a trajectory of $\xi$, the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field. We have

\begin{equation}
(5) \quad \nabla_\xi h = \varphi - \varphi R_\xi - \varphi h^2,
\end{equation}

\begin{equation}
(6) \quad g(R(X, Y)\xi, Z) = g((\nabla_Y \varphi)X - (\nabla_X \varphi)Y, Z) + g((\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y, Z)
\end{equation}

for all vector fields $X, Y, Z$ on $M$. A contact Riemannian manifold for which $\xi$ is a Killing vector field, is called a $K$-contact manifold. It is easy to see that a contact
Riemannian manifold is $K$-contact if and only if $h = 0$. For a contact Riemannian manifold $M$, one may define naturally an almost complex structure $J$ on $M \times \mathbb{R}$ by

$$J(X, f \frac{d}{dt}) = (\varphi X - f \xi, \eta(X) \frac{d}{dt}),$$

where $X$ is a vector field tangent to $M$, $t$ the coordinate of $\mathbb{R}$ and $f$ a function on $M \times \mathbb{R}$. If the almost complex structure $J$ is integrable, $M$ is said to be normal or Sasakian. It is known that $M$ is normal if and only if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$. A Sasakian manifold is also characterized by the condition

$$(7) \quad (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for all vector fields $X$ and $Y$ on the manifold and this is equivalent to

$$(8) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields $X$ and $Y$.

For a contact Riemannian manifold $M$, the tangent space $T_pM$ of $M$ at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM|\eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to $\xi$. The 2n-dimensional distribution (or subbundle) $D$ is called the contact distribution (or contact subbundle). For a given contact Riemannian manifold $M = (M; \eta, g)$ its associated almost CR-structure is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX : X \in D\}$$

of the complexification $TM^C$ of the tangent bundle $TM$, where $J = \varphi|D$, the restriction of $\varphi$ to $D$. Then we see that each fiber $\mathcal{H}_x$ ($x \in M$) is of complex dimension $n$ and $\mathcal{H} \cap \mathcal{H} = \{0\}$. Furthermore, we have $CD = \mathcal{H} \oplus \mathcal{H}$. We say that the associated CR-structure is integrable if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on $M$. Then we see that the Levi form is Hermitian and positive definite, that is, the CR-structure is strongly pseudo-convex, pseudo-Hermitian CR-structure. We call the pair $(\eta, L)$ a strongly pseudo-convex, pseudo-hermitian structure on $M$. Since $d\eta(\varphi X, \varphi Y) = d\eta(X, Y)$, we see that $[JX, JY] - [X, Y] \in D$ and $[JX, Y] + [X, JY] \in D$ for $X, Y \in D$, further if $M$ satisfies the condition

$$[J, J](X, Y) = 0$$

for $X, Y \in D$, then the pair $(\eta, L)$ is called a strongly pseudo-convex (integrable) CR-structure and $(M; \eta, L)$ is called a strongly pseudo-convex CR-manifold. For
a given contact strongly pseudo-convex almost CR manifold \( M \), the almost CR structure is integrable if and only if \( M \) satisfies the integrability condition \( \Omega = 0 \), where \( \Omega \) is a \((1,2)\)-tensor field on \( M \) defined by

\[
\Omega(X,Y) = (\nabla_X \varphi)Y - g(X + hX, Y)\xi - \eta(X)\eta(Y)\xi + \eta(Y)(X + hX)
\]

for all vector fields \( X, Y \) on \( M \) (see [30], Proposition 2.1). Taking account of (7) and (9) we see that for a Sasakian manifold the associated CR structure is strongly pseudo-convex integrable (cf. [19]). It is well known that for 3-dimensional contact Riemannian manifolds their associated CR structures are always integrable (cf. [30]). One more class of contact Riemannian manifolds whose associated CR structures are integrable is so-called a \((k, \mu)\)-space, which is determined by the condition

\[
R(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y)
\]

for \((k, \mu) \in \mathbb{R}^2\). A \((k, \mu)\)-space remains invariant under a pseudo-homothetic transformation (see [6]). A pseudo-homothetic transformation (or \( D \)-homothetic transformation) is defined by a change of structure tensors of the form:

\[
\bar{\eta} = a\eta, \quad \bar{\xi} = 1/\bar{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{\bar{g}} = ag + a(a - 1)\eta \otimes \eta,
\]

where \( a \) is a positive constant.

Now, we review the generalized Tanaka-Webster connection ([30]) on a contact strongly pseudo-convex almost CR-manifold \( M = (M; \eta, L) \). The generalized Tanaka-Webster connection \( \tilde{\nabla} \) is defined by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi
\]

for all vector fields \( X, Y \) on \( M \). Together with (4), \( \tilde{\nabla} \) may be rewritten as

\[
\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y),
\]

where we have put

\[
A(X,Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi.
\]

We see that the generalized Tanaka-Webster connection \( \tilde{\nabla} \) has the torsion

\[
\tilde{T}(X,Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY.
\]

In particular, for a \( K \)-contact Riemannian manifold we get

\[
A(X,Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi.
\]

Furthermore, it was proved that
Proposition 1. ([30]) The generalized Tanaka-Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M; \eta, g)$ is the unique linear connection satisfying the following conditions:

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$;
(iii - 1) $\hat{T}(X, Y) = 2L(X, JY)\xi$, $X, Y \in D$;
(iii - 2) $\hat{T}(\xi, \varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in D$;
(iv) $(\hat{\nabla}_X\varphi)Y = \Omega(X, Y)$, $X, Y \in TM$.

The Tanaka-Webster connection ([29], [34]) on a nondegenerate integrable CR-manifold is defined as the unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and $\Omega = 0$. The metric affine connection $\hat{\nabla}$ is a natural generalization of the Tanaka-Webster connection. In fact, in [5] the authors deal with the use of $\nabla$ in the non-integrable case. We refer to [5] for more details about pseudo-hermitian geometry in contact Riemannian manifolds.

Remark 1. The contact strongly pseudo-convex CR manifold is invariant under the pseudo-homothetic transformation. In fact, by direct computations we have

$$(\hat{\nabla}_X\varphi)Y = (\nabla_X\varphi)Y + (a - 1)\eta(Y)\varphi^2X - (a - 1)/ag(X, hY)\xi.$$ 

From this, we easily see that $\Omega = 0$ implies $\tilde{\Omega} = 0$.

Now, we review the gauge transformation of contact Riemannian structure. Given a contact form $\eta$, we consider a 1-form $\tilde{\eta} = \sigma\eta$ for a positive smooth function $\sigma$. It is natural that one may consider the transformation $(\eta, g) \rightarrow (\tilde{\eta}, \tilde{g})$. We call this a gauge transformation of contact Riemannian structure. By assuming $\varphi|D = \varphi|D$ the associated Riemannian structure $\tilde{g}$ of $\tilde{\eta}$ is determined in a natural way. Namely, we have

$$\tilde{\xi} = (1/\sigma)(\xi + \zeta), \quad \zeta = (1/2\sigma)\varphi\text{grad} \sigma, \quad \varphi = \varphi + (1/2\sigma)\eta \otimes \text{grad} \sigma \otimes \xi,$$

$$\tilde{g} = \sigma g - \sigma(\eta \otimes \nu + \nu \otimes \eta) + \sigma(\sigma - 1 + ||\zeta||^2)\eta \otimes \eta,$$

where $\nu$ is dual to $\zeta$ with respect to $g$. We remark that particularly when $\sigma$ is a constant then a gauge transformation reduces to a pseudo-homothetic transformation.

Let $\omega$ be a nowhere vanishing $(2n + 1)$-form on $M$ and fix it. Let $dM(g) = ((-1)^n/2^n n!)\eta \wedge (d\eta)^n$ denote the volume element of $(M, \eta, g)$. We define $\lambda$ by $dM(g) = \pm \epsilon^\lambda \omega$ and $\theta \in \Gamma(D^*)$ by $\theta(X) = X\lambda$ for $X \in D$. And [31] [32] defined
\[ B \in \Gamma(D \otimes D^{*3}) \text{ by} \]
\[
(2n + 4)g(B(X, Y)Z, W) = (2n + 4)g(R(X, Y)Z, W) - g(QY, Z)g(X, W) + g(QX, Z)g(Y, W)
\]
\[
- g(Y, Z)g(QX, W) + g(X, Z)g(QY, W) + g(QY, \varphi Z)g(\varphi X, W)
\]
\[
- g(QX, \varphi Z)g(\varphi Y, W) + g((\varphi Q + Q\varphi)X, Y)g(\varphi Z, W)
\]
\[
+ g(Y, \varphi Z)g(\varphi QX, W) - g(X, \varphi Z)g(\varphi QY, W)
\]
\[
- g(X, \varphi Y)g((\varphi Q + Q\varphi)Z, W)
\]
\[
+ [\tilde{r}/(2n + 2) - 4][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
\]
\[
+ [\tilde{r}/(2n + 2) + 2n][g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W)]
\]
\[
- 2g(\varphi X, Y)g(\varphi Z, W)] + (2n - 2)[g(\varphi Y, Z)g(\varphi hX, W)
\]
\[
- g(\varphi X, Z)g(\varphi hY, W) + g(\varphi hY, Z)g(\varphi X, W) - g(\varphi hX, Z)g(\varphi Y, W)]
\]
\[
- 6g(hX, Z)g(X, W) - g(hX, Z)g(Y, W) + g(Y, Z)g(hX, W)
\]
\[
- g(X, Z)g(hY, W)] + (2n + 4)[g(\varphi hY, Z)g(\varphi hX, W)
\]
\[
- g(\varphi hX, Z)g(\varphi hY, W)] + g(\varphi(\nabla_{\xi}h)Y, Z)g(X, W)
\]
\[
- g(\varphi(\nabla_{\xi}h)X, Z)g(Y, W) + g(Y, Z)g(\varphi(\nabla_{\xi}h)X, W)
\]
\[
- g(X, Z)g(\varphi(\nabla_{\xi}h)Y, W) - g(\varphi Y, Z)g((\nabla_{\xi}h)X, W)
\]
\[
+ g(\varphi X, Z)g((\nabla_{\xi}h)Y, W) - g((\nabla_{\xi}h)Y, Z)g(\varphi X, W)
\]
\[
+ g((\nabla_{\xi}h)X, Z)g(\varphi Y, W) - (n + 2)/(n + 1)g(U(X, Y, Z; \theta), W)),
\]

where \( \tilde{r} = r - g(Q\xi, \xi) + 4n \) is the generalized Tanaka-Webster scalar curvature and 
\( U \in \Gamma(D \otimes D^{*3}) \) defined by (7) in [32]. Moreover, the following was proved in [32].

**Theorem 2.** Let \( M = (M^{2n+1}, \eta, g) \) be a contact Riemannian manifold and let \( \omega \) be a fixed nowhere vanishing \((2n+1)\)-form on \( M \). Then \( B \) defined by (2.13) is a gauge invariant of type \((1, 3)\) of the contact Riemannian structure. Furthermore,

(i) If \( B = 0 \), then the associated CR structure \((\eta, J)\) is integrable.

(ii) If \( \Omega = 0 \), then \( B \) reduces to the Chern-Morser-Tanaka invariant.

**Remark 2.** (1) If \( \Omega = 0 \), then from the definition of \( U \) we easily see that \( U = 0 \) (see (7) in [32]).

(2) If \( n = 1 \) (dim \( M = 3 \)), then we have \( B = 0 \) (see Remark in [32]).

Moreover, we have

**Proposition 3.** ([14]) If \( M \) is a normal contact Riemannian (or Sasakian) manifold, then the CR-invariant \( B \) coincides with C-Bochner curvature tensor.
Here, we recall the notion of a \textit{pseudo-homothetic transformation} (or \textit{D-homothetic transformation}) of a contact metric manifold. This transformation means a change of structure tensors of the form

\begin{equation}
\bar{\eta} = a\eta, \quad \bar{\xi} = 1/a\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,
\end{equation}

where \(a\) is a positive constant. From (14), we have \(\bar{h} = (1/a)h\). By using the well-known formula

\[2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Y], Z) + g([X, Z], Y) - g([Y, Z], X)\]

We have

\begin{equation}
\nabla_{\bar{X}} Y = \nabla_X Y + C(X, Y),
\end{equation}

where \(C\) is the (1,2)-type tensor defined by

\[C(X, Y) = -(a - 1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - \frac{a - 1}{a}g(\varphi hX, Y)\xi.\]

\textbf{Remark 3.} (1) From (14) and the expression of the torsion tensor \(\hat{T}\) of the (generalized) Tanaka-Webster connection, we easily see \(\hat{T}\) is pseudo-homothetically invariant, namely, \(\bar{\hat{T}} = \hat{T}\).

(2) The contact strongly pseudo-convex CR-manifold is invariant under the pseudo-homothetic transformation. In fact, by direct computations we have

\[(\nabla_{\bar{X}} \varphi) Y = (\nabla_X \varphi) Y + (a - 1)\eta(Y)\varphi^2 X - \frac{a - 1}{a}g(\varphi hX, Y)\xi.\]

From this, we easily see that \(\Omega = 0\) implies \(\bar{\Omega} = 0\).

\textbf{3. Pseudo-Einstein contact strongly pseudo-convex CR-manifolds with \(B=0\)}

We define the pseudo-hermitian curvature tensor of \(\hat{R}\) (with respect to the Tanaka-Webster connection) \(\hat{\nabla}\) by

\[\hat{R}(X, Y) Z = \hat{\nabla}_X (\hat{\nabla}_Y Z) - \hat{\nabla}_Y (\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]} Z\]

for all vector fields \(X, Y, Z\) in \(M\). Then we have

\textbf{Proposition 4.} ([14], [15])

\[\hat{R}(X, Y) Z = -\hat{R}(Y, X) Z,\]

\[L(\hat{R}(X, Y) Z, W) = -L(\hat{R}(X, Y) W, Z).\]
The first identity follows trivially from the definition of $\hat{R}$. Since the connection is Levi-parallel, (i.e., $\hat{\nabla}L = 0$), we have also the second one by a similar way as the case of Riemannian curvature tensor. We remark that the Tanaka-Webster connection is not torsion-free, the Jacobi- or Bianchi-type identities do not hold, in general. Before we study the curvature tensor $\hat{R}$, from (3), (10) and (11) we have

$$\hat{\nabla}_X h Y = (\nabla_X h) Y + A(X, h Y) - h A(X, Y)$$

where $\nabla$ is the Levi-Civita connection. From the definition of $\hat{R}$, together with (10), taking account of $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$ and (16), the straightforward computations yield

$$\hat{R}(X, Y) Z = R(X, Y) Z + \eta(Z) \left( \varphi P(X, Y) + \varphi(A(X, h Y) - A(Y, h X)) \right.$$ 

$$- \varphi h(A(X, Y) - A(Y, X)) \bigg)$$

where

$$B(X, Y) Z = \eta(Z) \varphi P(X, Y) - g(\varphi P(X, Y), Z) \xi - \eta(Z) [\eta(Y)(X + h X)$$

$$- \eta(X)(Y + h Y)] + \eta(Y) g(X + h X, Z) \xi - \eta(X) g(Y + h Y, Z) \xi$$

$$+ g(\varphi Y + \varphi h Y, Z)(\varphi X + \varphi h X) - g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y)$$

$$- 2g(\varphi X, Y) \varphi Z$$

for all vector fields $X, Y, Z$ in $M$. The pseudohermitian Ricci (curvature) tensor $\hat{\rho}$ of $\hat{\nabla}$ is

$$\hat{\rho}(X, Y) = \text{trace of } \{ V \mapsto \hat{R}(V, X) Y \},$$

where $X, Y$ are vector fields in $M$. From (17), by using (3) and (4), for an local orthonormal frame field $\{ e_i \}$, $i = 1, 2, \ldots, 2n + 1$ we have

$$\hat{\rho}(X, Y) = \rho(X, Y) + \sum_{i=1}^{2n+1} L(B(e_i, X) Y, e_i)$$

$$= \rho(X, Y) + 2g(X, Y) - g(\varphi (\hat{\nabla}_h) X, Y)$$
where we have put $\rho(X,Y) = g(QX,Y)$ and $\hat{\rho}(X,Y) = L(\hat{Q}X,Y)$. Since $\nabla_\xi \varphi = 0$ (cf. [4] pp.67), from (3) we see that $\hat{Q}$ is the symmetric $(1,1)$-tensor field in $M$.

Now, we define the pseudo-Einstein structure in contact strongly pseudo-convex almost CR manifold (cf. [5], [14], [24]).

**Definition 1.** Let $(M;\eta,L)$ be a contact strongly pseudo-convex almost CR manifold. Then the pseudohermitian structure $(\eta,L)$ is said to be pseudo-Einstein if the pseudohermitian Ricci tensor is proportional to the Levi form, namely,

$$\hat{\rho}(X,Y) = \lambda L(X,Y)$$

or equivalently

$$\rho(X,Y) = (\hat{r}/2n - 2)g(X,Y) + g(\varphi(\nabla_\xi h)X,Y)$$

for $X,Y \in D$, where $\lambda = \hat{r}/2n$.

From the definition, in case that $M$ is K-contact, we know that $h = 0$, and

$$\rho(X^T,Y^T) = (r - 2n)/2ng(X^T,Y^T),$$

where $X^T$ denotes the component of $X$ orthogonal to $\xi$. Since $Q\xi = 2n\xi$, further we have

$$\rho(X,Y) = (r - 2n/2n)g(X,Y) + (2n - r - 2n/2n)\eta(X)\eta(Y)$$

for any vector fields $X,Y$ in $M$. Thus, we have

**Proposition 5.** In a K-contact manifold the pseudo-Einstein condition coincides with the $\eta$-Einstein condition.

We note that a K-contact manifold is called $\eta$-Einstein if the Ricci tensor is of the form $Q = aI + b\eta \otimes \xi$ where $a, b$ are constants (cf. pp. 285 in [36]). We also give

**Definition 2.** ([14], [15]) Let $(M;\eta,L)$ be a contact strongly pseudo-convex CR manifold. Then $M$ is said to be of constant pseudo-holomorphic sectional curvature $c$ with respect to the Tanaka-Webster connection or shortly, be of constant pseudo-holomorphic sectional curvature $c$ if $M$ satisfies

$$L(\hat{R}(X,\varphi X)\varphi X,X) = c$$

for any unit horizontal vector field $X$. If

$$L(\hat{R}(X,\varphi X)\varphi X,X) = f(p)$$
for any unit horizontal vector $X(p) \in D(p)$, then $M$ is said to have a pointwise constant pseudo-holomorphic sectional curvature.

From the definition of pseudo-homothetic transformation (or $D$-homothetic transformation) given in section 2, we have

\begin{equation}
\nabla_X Y = \nabla_X Y - (a-1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - (a-1)/ag(\varphi h X, Y)\xi.
\end{equation}

Furthermore, we have shown in [15] that

**Proposition 6.** The contact strongly pseudo-convex CR-space of (pointwise) constant pseudo-holomorphic sectional curvature is a pseudo-homothetic invariant.

**Proof.** From (19) and the definition of the curvature tensor, by long but tedious computations, we get

\begin{align*}
\bar{g}(\bar{R}(X, \varphi X)\varphi X, X) &= g(R(X, \varphi X)\varphi X, X) - (a-1)[3 + g(h X, X)]
\end{align*}

\begin{align*}
&- \frac{a-1}{a}[g(\varphi h X, X)^2 + g(h X, X)(g(h X, X) - 1)]
\end{align*}

\begin{align*}
&+ \frac{(a-1)^2}{a}g(h X, X)
\end{align*}

for any unit horizontal vector $X \in D(p), p \in M$. For any unit horizontal vector $X$, from (17), we get

\begin{align*}
\bar{L}(\bar{R}(X, \varphi X)\varphi X, X) &= 3 + \bar{g}(\bar{R}(X, \varphi X)\varphi X, X) - \bar{g}(\varphi h X, X)^2 - \bar{g}(h X, X)^2.
\end{align*}

Hence, we have

\begin{align*}
\bar{L}(\bar{R}(X, \varphi X)\varphi X, X) &= aL(R(X, \varphi X)\varphi X, X).
\end{align*}

For a unit horizontal vector $X$, if we denote by $\bar{H}(X) = L(R(X, \varphi X)\varphi X, X)$ the pseudo-holomorphic sectional curvature, then this is rewritten by

\begin{align*}
\bar{H}(X) &= a\bar{H}(X).
\end{align*}

Thus, we have proved.

We also have

**Proposition 7.** The generalized Tanaka-Webster connection is pseudo-homothetically invariant.

**Proof.** From (10), we have

\begin{align*}
\hat{\nabla}_X Y &= \nabla_X Y + A(X, Y)
\end{align*}
From this and (19), it follows that the generalized Tanaka-Webster connection is pseudo-homothetically invariant.

**Corollary 8.** The Tanaka-Webster curvature tensor $\hat{R}$ and pseudohermitian Ricci tensor $\hat{S}$ are pseudo-homothetically invariant.

**Remark 4.** Together with the Definition 1 and the Corollary 8, we see that the pseudo-Einstein structure is a pseudo-homothetic invariant.

Since every 3-dimensional contact strongly pseudo-convex CR manifold has automatically pseudo-Einstein structure (see, [24]), we assume that $n \geq 2$. Then we have ([14])

**Theorem 9.** A pseudo-Einstein contact strongly pseudo-convex CR manifold with $B = 0$ has a pointwise constant pseudo-holomorphic sectional curvature

$$\hat{H} = \frac{\hat{r}}{n(n+1)}$$.

**Proof.** Let $M$ be a pseudo-Einstein contact strongly pseudo-convex CR manifold. Then from Definition 1, we see that $Q = (\hat{r}/2n-2)I + \varphi(\nabla \xi h)$ on $D$. From this, we also get

$$\varphi Q + Q \varphi = (\hat{r}/n - 4)\varphi$$.

We suppose that $B = 0$. Then together with (14), for a unit vector $X$, we have

$$g(R(X, \varphi X)\varphi X, X) = \left(\frac{\hat{r}}{n(n+1)} - 3\right) + g(hX, \varphi X)^2 + g(hX, X)^2.$$

But, for any unit horizontal vector $X$, from (17), we obtain

$$g(\hat{R}(X, \varphi X)\varphi X, X) = 3 + g(R(X, \varphi X)\varphi X, X) - g(\varphi hX, X)^2 - g(hX, X)^2.$$

Hence, we have

$$g(\hat{R}(X, \varphi X)\varphi X, X) = \frac{\hat{r}}{n(n+1)}.$$
4. The Pseudo-Einstein unit tangent sphere bundles

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [18], [23], [35]). We only briefly review of notations and their definitions. Let $M = (M, G)$ be a $n$-dimensional Riemannian manifold $TM$ denote its tangent bundle with the projection $\pi: TM \to M$, $\pi(x, u) = x$. For a vector $X \in T_xM$, we denote by $X^H$ and $X^V$, the horizontal lift and the vertical lift, respectively. Then we can define a Riemannian metric $\bar{g}$, Sasaki metric on $TM$ in a natural way. That is,

$$\bar{g}(X^H, Y^H) = \bar{g}(X^V, Y^V) = G(X, Y) \circ \pi, \quad \bar{g}(X^H, Y^V) = 0$$

for all vector fields $X$ and $Y$ on $M$. Also, a natural almost complex structure tensor $J$ of $TM$ is defined by $JX^H = X^V$ and $JX^V = -X^H$. Then we easily see that $(TM; \bar{g}, J)$ is an almost Hermitian manifold. We note that $J$ is integrable if and only if $(M, G)$ is locally flat ([18]).

Now we consider the unit tangent sphere bundle $(T_1M, g')$, which is isometrically embedded hypersurface in $(TM, \bar{g})$ with unit normal vector field $N = u^V$. For $X \in T_xM$, we define the tangential lift of $X$ to $(x, u) \in T_1M$ by

$$X^T_{(x, u)} = X^V_{(x, u)} - G(X, u)N_{(x, u)}.$$ 

Clearly, the tangent space $T_{(x,u)}T_1M$ spanned by vectors of the form $X^H$ and $X^T$ where $X \in T_xM$. We put

$$\xi' = -JN, \quad \varphi' = J - \eta' \otimes N.$$

Then we find $g'(X, \varphi'Y) = 2dh'(X, Y)$. By taking $\xi = 2\xi'$, $\eta = \frac{1}{2}\eta'$, $\varphi = \varphi'$, and $g = \frac{1}{2}g'$, we get the standard contact Riemannian structure $(\varphi, \xi, \eta, g)$. Indeed, we easily check these tensors satisfy (1). The tensors $\xi$ and $\varphi$ are explicitly given by

$$\xi = 2u^H,$$

$$\varphi X^T = -X^H + \frac{1}{2}G(X, u)\xi,$$

$$\varphi X^H = X^T$$

where $X$ and $Y$ are vector fields on $M$. From now we consider $T_1M = (T_1M; \eta, g)$ with the standard contact Riemannian structure. We arrange fundamental formulas, which is needed for the proof of our Theorem, without proofs (cf. [9], [10], [32], [33]). We denote by $\nabla$ and $R$, the Levi-Civita connection and the Riemannian curvature tensor associated with $g$, respectively. We, also, denote by $D$ and $K$, the Levi-Civita connection and the Riemannian curvature tensor associated with
for all vector fields $X$, $Y$ and $Z$ on $M$. Next, we calculate the Ricci tensor $\rho$ of $(T_1M, g)$ at the point $(x, u) \in T_1M$. Let $\{E_1, \cdots, E_n = u\}$ be an orthonormal basis of $T_xM$. Then $\{2E_i^1, \cdots, 2E_n^{n-1}, 2E_i^h, \cdots, 2E_n^h = u\}$ is an orthonormal basis for $T_{(x,u)}T_1M$ and $\rho$ is given by

$$\rho(A, B) = \sum_{i=1}^{n-1} R(2E_i^1, A, B, 2E_i^1) + \sum_{i=1}^{n} R(2E_i^h, A, B, 2E_i^h).$$

$$R(X^T, Y^T)Z^T = -g(X^T, Z^T)Y^T + g(Z^T, Y^T)X^T,$$

$$R(X^T, Y^T)Z^H = \{K(X - G(X, u)u, Y - G(Y, u)u)Z\}^H + \frac{1}{4}\{[K(u, X), K(u, Y)]Z\}^H$$

$$R(X^H, Y^T)Z^T = -\frac{1}{2}\{K(Y - G(Y, u)u, Z - G(Z, u)u)X\}^H - \frac{1}{4}\{K(u, Y)K(u, Z)X\}^H$$

$$R(X^H, Y^T)Z^H = \frac{1}{2}\{K(X, Z)(Y - G(Y, u))\}^T$$

$$R(X^H, Y^H)Z^T = \{K(X, Y)(Z - G(Z, u)u)\}^T + \frac{1}{4}\{K(Y, K(u, Z)X)u - K(X, K(u, Z)Y)u\}^T + \frac{1}{2}\{(DXK)(u, Z)Y - (DYK)(u, Z)X\}^H,$$

$$R(X^H, Y^H)Z^H = (K(X, Y)Z)^H + \frac{1}{2}\{K(u, K(X, Y)u)Z\}^H - \frac{1}{4}\{K(u, K(Y, Z)u)X - K(u, K(X, Z)u)Y\}^H + \frac{1}{2}\{(DZK)(X, Y)u\}^T$$
From this, we obtain
\[
\rho(X^t, Y^t) = (n-2)(G(X, Y) - G(X, u)G(Y, u)) + \frac{1}{4} \sum_{i=1}^{n} G(K(u, X)E_i, K(u, Y)E_i),
\]
(23)
\[
\rho(X^t, Y^h) = \frac{1}{2} \left( (\nabla_u S)(X, Y) - (\nabla_X S)(u, Y) \right),
\]
\[
\rho(X^h, Y^h) = \rho(X, Y) - \frac{1}{2} \sum_{i=1}^{n} G(K(u, E_i)X, K(u, E_i)Y).
\]

Contraction of \( \rho \) with respect to the metric \( g \) gives the scalar curvature \( r \):

\[
r = \sum_{i=1}^{n-1} \rho(2E_i^t, 2E_i^t) + \sum_{i=1}^{n} \rho(2E_i^h, 2E_i^h)
= s + (n-1)(n-2) - \varpi(u, u)/4
\]
where \( \varpi(u, v) = \sum_{i,j=1}^{n} G(K(u, E_i)E_j, K(v, E_i)E_j) \). From the Definition 1 and (23) we have ([17])

**Theorem 10.** The standard contact strongly pseudo-convex CR-structure on the unit tangent sphere bundles \( T_1M \) of a space \( M \) with dimension \( \geq 3 \) and of constant curvature \( c \) admit the pseudo-Einstein structures if and only if \( c = 1 \).

In [2], the authors showed that the unit tangent sphere bundle of a space of constant curvature 1 has an pseudo-Einstein structure. The class of \((k, \mu)\)-spaces, mentioned in Preliminary, contains Sasakian spaces \((\kappa = 1 \text{ and } h = 0)\), further in [6] it was shown that the only unit tangent sphere bundles with this curvature property are precisely those of spaces of constant curvature \( c \) (with \( k = c(2-c) \) and \( \mu = -2c \)). Moreover, they ([6]) calculated the Ricci tensor for the non-Sasakian case, which gives
\[
\rho(X, Y) = [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y)
\]
for any vector fields \( X, Y \) on \( D \). We now suppose that a contact \((\kappa, \mu)\)-space \( M^{2n+1} \) is a pseudo-Einstein space, then together with the 2nd equation of Definition 1, we have
\[
(\lambda - 2)I - \mu h = [2(n-1) - n\mu]I + [2(n-1) + \mu]h
\]
on \( D \), where we have used \( \nabla \xi h = \mu h \varphi \) (see [6]). If we operates \( h \) again, and taking the trace of this equation, then we get
\[
(n-1)\text{trace of } h^2 = 0,
\]
which says that a non-Sasakian contact \((\kappa, \mu)\)-space is a pseudo-Einstein only in dimension three. Namely, we have
Theorem 11. ([17]) If contact \((\kappa, \mu)\)-space \(M^{2n+1}\) \(n \geq 2\) is a pseudo-Einstein space, then \(M\) is a Sasakian space.

5. Pseudo-Einstein Hopf-hypersurfaces in a complex space form

Let \(M\) be an oriented real hypersurface of a Kählerian manifold \(\tilde{M} = (\tilde{M}; J, \tilde{g})\) and \(N\) a global unit normal vector on \(M\). By \(\tilde{\nabla}\), \(A\) we denote the Levi-Civita connection in \(\tilde{M}\) and the shape operator with respect to \(N\), respectively. Then the Gauss and Weingarten formulas are given respectively by

\[
\tilde{\nabla}_XY = \nabla_XY + g(AX, Y)N, \quad \tilde{\nabla}_XN = -AX
\]

for any vector fields \(X\) and \(Y\) tangent to \(M\), where \(g\) denotes the Riemannian metric of \(M\) induced from \(\tilde{g}\). An eigenvector (resp. eigenvalue) of the shape operator \(A\) is called a principal curvature vector (resp. principal curvature). We denote by \(V_\lambda\) the eigenspace associated with an eigenvalue \(\lambda\). For any vector field \(X\) tangent to \(M\), we put

\[
JX = \varphi X + \eta(X)N, \quad JN = -\xi.
\]

We easily see that the structure \((\eta, \varphi, \xi, g)\) is an almost contact metric structure on \(M\). Indeed, the structure tensors satisfy

\[
\varphi^2X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

From the condition \(\tilde{\nabla}J = 0\), the relations (25) and by making use of the Gauss and Weingarten formulas, we have

\[
(\nabla_X\varphi)Y = \eta(Y)AX - g(AX, Y)\xi,
\]

(26)

\[
\nabla_X\xi = \varphi AX.
\]

(27)

By using (26) and (27), we see that a real hypersurface in a Kählerian manifold always has an integrable associated CR structure. Here, we note that the associated Levi form in a real hypersurface of a Kählerian manifold is \(L(X, Y) = 1/2g((\varphi A + A\varphi)X, \varphi Y)\), where we denote by \(A\) the restriction \(A\) to \(D\). Hence, we see that its associated CR-structure in general neither is pseudo-hermitian nor is non-degenerate. But, since a contact structure satisfies \(d\eta(X, Y) = g(X, \varphi Y)\) (the 2nd eq. in (1)), together with (27) we proved in [12], [16]

Proposition 12. Let \(M = (M; \eta, \varphi, g)\) be a real hypersurface of a Kählerian manifold. The almost contact metric structure of \(M\) is contact metric if and only if \(\varphi A + A\varphi = \pm 2\varphi\), where \(\pm\) is determined by the orientation. Further, its associated CR-structure is pseudo-hermitian, strongly pseudo-convex.
Moreover, Remark 1 in [12] says that a geodesic hypersphere of radius $\frac{\pi}{4}$ in $CP^n$, a horosphere in $CH^n$, a unit sphere $S^{2n-1}(1)$ or a generalized cylinder $S^{n-1}(1/2) \times E^n$ in $CE^n$ only has a contact metric structure. Now we define the pseudo-Einstein real hypersurface in a Kählerian manifold by the same condition as in Definition 1. Let $\tilde{M} = \tilde{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature $c$ and $M$ a real hypersurface of $\tilde{M}$. Then we have the following Gauss and Codazzi equations:

$$R(X, Y)Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z \} + g(AY, Z)AX - g(AX, Z)AY,$$

(28)

$$\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{ \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \}$$

(29)

for any tangent vector fields $X, Y, Z$ on $M$. From (28) we get for the Ricci tensor $Q$ of type $(1,1)$:

$$QX = \frac{c}{4} \{ (2n + 1)X - 3\eta(X)\xi \} + hAX - A^2 X,$$

(30)

where $h = \text{trace of } A$ denotes the mean curvature. From the definition of Lie derivative and using (26) and (27) we have

$$hX = 1/2(L_\xi \varphi)X = 1/2(L_\xi \varphi X - \varphi L_\xi X)$$

$$= 1/2([\xi, \varphi X] - \varphi[\xi, X])$$

$$= 1/2(\nabla_\xi \varphi X - \nabla_{\varphi X} \xi + \varphi \nabla_X \xi)$$

$$= 1/2(\eta(X)A\xi - \varphi A\varphi X - AX).$$

(31)

By using (30) and (31), we have

Theorem 13. ([17]) Let $M$ be a Hopf hypersurface of a complex space form $\tilde{M}_n(c)$ with constant holomorphic sectional curvature $c$ and $n \geq 3$. Suppose that $M$ admits a pseudo-Einstein structure. Then we have

(I) in case that $\tilde{M}_n(c) = P_nC$, $M$ is locally congruent to one of the following:

(A1) a geodesic hypersphere of radius $r$, where $0 < r < \frac{\pi}{2}$,

(A2) a tube of radius $r$ over a totally geodesic $P_kC(1 \leq k \leq n - 2)$, $0 < r < \frac{\pi}{2}$ and $\cot^2 r = q/p$, where $a_1 = \cot r$ of multiplicity $2p$ and $a_2 = -\tan r$ of multiplicity $2q$,

(II) in case that $\tilde{M}_n(c) = H_nC$, $M$ is locally congruent to one of the following:
(A_0) a horosphere,
(A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}C$,

(III) in case that $\tilde{M}_n(c) = E_nC$, $M$ is locally congruent to one of the following:

(A_1) a sphere $S^{2n-1}(r)$ of radius $r \in R_+$,
(A_2) a plane $E^{2n-1}$,
(B) $E^n \times S^{n-1}(n/2(n - 1))$

Remark 5. (1) In real hypersurfaces in a complex space form, Kon [22] defines the “pseudo-Einstein space” by the same condition of $\eta$-Einstein for (K-)contact geometry. We call the space “$\eta$-Einstein space” to avoid the confusion. As compare with two notions, they are properly distinguished each other. Indeed, homogeneous tubes of type (B) with special radii in $P_nC$ are $\eta$-Einsteinian, not pseudo-Einsteinian (cf. [22]). We also find that $E^n \times S^{n-1}(n/2(n - 1))$ in $E_nC$ are a pseudo-Einstein space that is not an $\eta$-Einstein space.

(2) From the observations of the above, it leads naturally the following problem: Is there a (non-Sasakian) contact pseudo-Einstein space of dimension $\geq 5$?

References


