Conformal diffeomorphisms and curvatures

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Abstract. In a study of Riemannian manifolds admitting conformal transformation, the Riemannian and Ricci curvatures play an important role for characterizations and classify such a manifold. In this note, we summarize known results about Riemannian manifold admitting conformal transformations and related topics.

1. Introduction

Let $M$ and $M^*$ be $m$-dimensional connected Riemannian manifolds with metric tensor fields $g$ and $g^*$ respectively, and consider a conformal diffeomorphism $\phi$ of $M^*$ into $M$. Then the metric tensor fields are related by

$$(1.1) \quad g^* = \frac{1}{\rho^2} g,$$

where $\rho$ is a positive valued scalar field on $M$ and said to be associated with $\phi$. In case $M = M^*$, $\phi$ is called a conformal transformation. If $\rho$ is constant, then $\phi$ is called a homothety [1, 2, 4]. A classical theorem of Liouville determines all possible conformal diffeomorphisms between Euclidean metrics. As a generalization, a conformal transformation $\phi$ is called a Liouville transformation if $\text{Ric}(g) = \text{Ric}(g^*)$ [4].

A concircular transformation is by definition a conformal transformation preserving geodesic circles. It is well known that a conformal transformation between Einstein spaces is a concircular transformation [7, 8]. Hence it is natural to investigate conformal transformations between spaces of constant curvature. Recently we proved the following Theorem :

**Theorem 1.** A conformal transformation between spaces of constant curvature with $\dim M > 2$ is a Liouville transformation. Conversely if $M$ is a space of constant curvature and $\phi$ is a Liouville transformation, then $M^*$ is also a space of constant curvature.

Hence if the complete space of constant curvature $M$ with $\dim M > 2$ admitting a global conformal transformation, then we can see that $K^* = \rho^2 K$ from (2.3) and (3.5). Since the scalar curvatures $K$ and $K^*$ are constant, $\rho$ becomes constant. Therefore we have
Corollary 2. Let $M$ be a complete space of constant curvature with $\dim M > 2$, admitting a global conformal transformation $\phi$. Then $\phi$ is a homothety.

In this talk, we are to give a survey report on the Riemannian manifold admitting conformal diffeomorphisms and related topics.

2. Conformal transformation

Let $\phi : (M^*, g^*) \to (M, g)$ be a conformal transformation. The geometric objects $\{R, S, K\}$ are the Riemannian curvature, Ricci curvature and scalar curvature respectively. $\{R^*, S^*, K^*\}$ are the corresponding objects of $(M^*, g^*)$. Then we have [1, 2]

(2.1) \[ R^*_{kji}^h = R_{kji}^h + \rho^{-1}(\delta^h_k \nabla_j \rho_i - \delta^h_j \nabla_k \rho_i + g_{ji} \nabla^h_p - g_{ki} \nabla^p_j) - \rho^{-2} \rho l^i \rho^j (\delta^h_k g_{ji} - \delta^h_j g_{ki}), \]

(2.2) \[ S^*_i = S_i + (m - 2) \rho^{-1} \nabla_j \rho_i + \rho^{-1} g_{ji} \nabla^l - (m - 1) \rho^{-2} \rho l^i g_{ji}, \]

(2.3) \[ K^* = \rho^2 K + 2m^{-1} \rho \nabla_l \rho^l - \rho l^l, \]

where $\rho_i = \partial_i \rho$ and the range of indices $i, j, k, l$ is $1, 2, 3, \ldots, m$. Assume that $M$ and $M^*$ are spaces of constant curvature. Then, by use of (2.1) and (2.3), we get

(2.4) \[ \frac{2m^{-1} \rho \nabla_l \rho^l - \rho l^l}{m(m - 1) \rho^2} (g_{ji} \delta^h_k - g_{ki} \delta^h_j) = \rho^{-1}(\delta^h_k \nabla_j \rho_i - \delta^h_j \nabla_k \rho_i + g_{ji} \nabla^h_p - g_{ki} \nabla^p_j) - \rho^{-2} \rho l^i \rho^j (\delta^h_k g_{ji} - \delta^h_j g_{ki}). \]

Contracting $i$ and $j$ in (2.4), we obtain

(2.5) \[ \frac{2m^{-1} \rho \nabla_l \rho^l - \rho l^l}{m \rho^2} \delta^h_k = \rho^{-1}(\delta^h_k \nabla_l \rho^l + (m - 2) \nabla^h \rho^l - \rho^{-2} (m - 1) \rho l^l \delta^h_k). \]

If we contract $k$ and $h$ again in (2.5), we have

(2.6) \[ m(1 + m - m^2) \{2 \rho \nabla_l \rho^l - m \rho l^l \} = 0, \]
that is

(2.7) \[ 2\rho \nabla_l \rho^l = m \rho_i \rho^i \]

because \( m \) is a positive integer.

Substituting (2.7) into (2.5), we obtain

(2.8) \[ \nabla_j \rho_i = \alpha g_{ji}, \]

if \( m > 2 \), where \( \alpha = \frac{1}{2} \rho^{-1} \rho_i \rho^i \).

3. Proof of Theorem 1

If we consider (2.1), (2.2) and (2.8), then we can see that \( R = R^* \) and that \( S = S^* \). That is, a conformal transformation between spaces of constant curvature is a Liouville transformation if \( m > 2 \).

Conversely, assume that \( \phi \) is a Liouville transformation, that is \( S = S^* \). In [1], present authors proved that a conformal transformation with \( S = S^* \) preserves Riemannian curvature tensor, that is \( R = R^* \). Hence (2.1) induces

(3.1) \[ \rho(\delta^h_k \nabla_j \rho_i - \delta^h_j \nabla_k \rho_i + g_{ji} \nabla_k \rho^h - g_{ki} \nabla_j \rho^h) - \rho_i \rho^i (\delta^h_k g_{ji} - \delta^h_j g_{ki}) = 0. \]

If we contract (3.1) with respect to \( k \) and \( h \), we obtain

(3.2) \[ \rho((m - 2) \nabla_j \rho_i + g_{ji} \nabla_l \rho^l) - (m - 1) \rho_i \rho^i = 0, \]

and that

(3.3) \[ 2(m - 1) \rho(\nabla_l \rho^l) - m(m - 1) \rho_i \rho^i = 0, \]

that is

(3.4) \[ 2\rho \nabla_l \rho^l = m \rho_i \rho^i. \]

Substituting (3.4) into (3.2), we get

(3.5) \[ \nabla_j \rho_i = \alpha g_{ji}, \]
if $m > 2$. Then we see that $K^* = \rho^2 K$ from (2.3) and (3.5). Hence if we assume that $M$ is a space of constant curvature, then we can see that

$$R_{kji}^* = R_{kji} = \frac{K^*}{m(m-1)} (g_{ji}^* \delta_k^h - g_{ki}^* \delta_j^h),$$

that is $M^*$ is a space of constant curvature. Thus we complete the proof of Theorem 1.

### 4. Conformal transformation and Ricci curvature

As to the Riemannian manifold admitting conformal transformation, consideration of the behavior of the Ricci curvature is an effective to study characterization or classify such a manifold. Related to these facts, W. Kuhnel and H. B. Rademacher [4] proved

**Theorem 3.** Two conformally equivalent metrics $g$ and $g^* = \frac{1}{\rho^2} g$ satisfy the relation

$$\left[S^* - S\right] = [g] = [g^*]$$

if and only if the function $\rho$ satisfy the equation

$$\nabla^2 \rho = \frac{\Delta \rho}{m} g.$$ 

A Riemannian manifold $(M, g)$ is called (geodesically) complete if every geodesic can be defined over $R$.

**Theorem 4.** ([4]) Let $(M, g)$ be complete and admitting a global conformal transformation $g^* = \frac{1}{\rho^2} g$ satisfying

$$S^* - S = c(m - 1)g$$

for some constant $c$, then one of the following three cases occurs:

1. $\rho$ is constant
2. $(M, g)$ and $(M^*, g^*)$ are simple connected Riemannian spaces of constant sectional curvature
3. $(M, g)$ is a warped product $R \times e^t N$, where $N$ is a complete Ricci-flat $(m-1)$-dimensional Riemannian manifold.
On the other hand H. W. Brinkmann\cite{3} studied the problem in the case where $M$ itself is an Einstein space. As a matter of course, his research was of local theory and in that age the relativity was greatly interested in. He gave local structures of conformally related Einstein spaces not only of general dimension with positively definite metric but also of dimension 4 with indefinite metric. It is noticed the metric considered in this talk is assumed to be positive definite.

On the other hand, K. Yano\cite{8} introduced in 1940 the notion of concircular transformation. This is a conformal transformation $\phi : M \to M^*$ carrying geodesic circles of $M$ into geodesic circles of $M^*$, and it is characterized by the equation

\begin{equation}
\nabla\nabla \rho = \varphi g : \nabla_\mu \rho_\lambda = \varphi g_{\mu\lambda}
\end{equation}

on the associated scalar $\rho$, where $\nabla$ indicates the covariant differentiation with respect to the metric $g$ and $\varphi$ is a scalar field on $M$. Here and hereafter we put $\rho_\lambda = \nabla_\lambda \rho$. This is the reason why long after Y. Tashiro called a scalar field $\rho$ satisfying the equation (4.3) a concircular one. If in particular the function $\varphi$ is linear in $\rho$, say

\begin{equation}
\nabla\nabla \rho = (-k\rho + b)g,
\end{equation}

$k$ and $b$ being constants, and the concircular scalar field $\rho$ to be special. Applying Ricci’s identity to (4.3), it is easily seen that a concircular scalar field $\rho$ on an Einstein space $M$ is special with constant $k$ equal to the scalar curvature defined by

\begin{equation}
k = \frac{1}{n(n-1)} K_{\mu\lambda} g^{\mu\lambda}.
\end{equation}

Brinkmann’s arguments were to solve the equation (4.4) in a suitable local coordinate system, for instance, a normal coordinate system. The first result concerning global conformal transformations was a theorem due to K. Yano and T. Nagano\cite{9} in 1959.

**Theorem 5.** If a complete Einstein space $M$ admits a global one-parameter group of non-isometric conformal transformations, then $M$ is isometric to an $n$-dimensional sphere $S^n$.

Let $v$ be a vector field generating a one-parameter group of conformal transformations. It is also called a conformal Killing vector and characterized by

\begin{equation}
L_v g_{\mu\lambda} = \nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 2\rho g_{\mu\lambda},
\end{equation}
indicating Lie derivative. In an Einstein space, the associated scalar field \( \rho \) is a special concircular one. If moreover \( M \) is complete and the one-parameter group is global, then it is proved the constant scalar curvature \( k \) is positive and, by means of Meyer’s theorem, the space \( M \) is compact, and they proved the theorem by use of integral formula. At the almost same time, N. Tanaka[6] gave the following

**Theorem 6.** Let \( M \) and \( M^* \) be complete Riemannian spaces with parallel Ricci tensor, and related by a conformal transformation \( \phi : M \to M^* \). If the eigenvalues of the Ricci tensors of \( M \) and \( M^* \) satisfy certain equalities respectively, then the transformation \( \phi \) is homothetic.

Though the case considered by Tanaka was essential for conformally related spaces with parallel Ricci tensor, T. Nagano[5] deal with the cases excepted from Tanaka’s consideration, and stated the following theorem by removing the condition on eigenvalues.

**Theorem 7.** If \( M \) and \( M^* \) are complete Riemannian spaces with parallel Ricci tensor, and related by a non-homothetic conformal transformation \( \phi \), then \( M \) and \( M^* \) are irreducible and isometric to a sphere.

From these and other reasons, Y. Tashiro had conjectured for a long time that there were no global non-isometric conformal transformation between complete product Riemannian spaces \((M, g)\) and \((M^*, g^*)\) which are not Euclidean. In other words, is a conformal transformation \( \phi : (M, g) \to (M^*, g^*) \) between the above spaces necessarily isometric?

**References**


