Hopf hypersurfaces in nonflat complex space forms

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Abstract. We study Hopf hypersurfaces in non-flat complex space forms by using some Gauss maps from a real hypersurface to complex (either positive definite or indefinite) 2-plane Grassmannian and quaternionic (para-)Kähler structures.

1 Introduction
Among real hypersurfaces in complex space forms, Hopf hypersurfaces have good geometric properties and studied widely. A real hypersurface $M$ in a complex space form $\tilde{M}(c)$ (of constant holomorphic sectional curvature $4c$) is called Hopf if the structure vector $\xi = -JN$ is an eigenvector of the shape operator $A$ of $M$. We say that the (constant) eigenvalue $\mu$ with $A\xi = \mu \xi$ a Hopf curvature of Hopf hypersurface $M$. When $c > 0$, Cecil and Ryan [4] proved that a Hopf hypersurface $M$ with Hopf curvature $\mu = 2\cot 2r$ $(0 < r < \pi/2)$ lies on the tube of radius $r$ over a complex submanifold in $\mathbb{C}P^n$, provided that the focal map $\phi_r : M \to \mathbb{C}P^n$, $\phi_r(x) = \exp_x(rN_x)$ $(x \in M, N_x \in T_x^\perp M, |N_x| = 1)$ has constant rank on $M$. After that Montiel [15] proved similar result for Hopf hypersurfaces with large Hopf curvature, (i.e., $|\mu| > 2$) in complex hyperbolic space $\mathbb{C}H^n$ (with $c = -1$). On the other hand, Ivey [7] and Ivey-Ryan [8] proved that a Hopf hypersurface with $|\mu| < 2$ in $\mathbb{C}H^n$ may be constructed from an arbitrary pair of Legendrian submanifolds in $S^{2n-1}$.

Here we will study (Hopf) hypersurfaces $M$ in $\mathbb{C}P^n$ (resp. $\mathbb{C}H^n$) by using some Gauss map $G$ (resp. $\tilde{G}$) from $M$ to complex 2-plane Grassmannian $G_2(\mathbb{C}^{n+1})$ (resp. Grassmannian $G_{1,1}(\mathbb{C}^{n+1})$ of 2-planes with signature (1,1) in $\mathbb{C}^{n+1}$), which maps each point of $M$ to the 2-plane spanned by position vector and structure vector.

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at the point in $M$. We see that: If $M$ is a Hopf hypersurface in $\mathbb{CP}^n$, then the Gauss image $G(M)$ is a real $(2n-2)$-dimensional totally complex submanifold with respect to quaternionic Kähler structure in $G_2(\mathbb{C}^{n+1})$ (Theorem 4.1). If $M$ is a Hopf hypersurface in $\mathbb{CP}^1$ with (constant) Hopf curvature $\mu$, then the Gauss image $\tilde{G}(M)$ is a real $(2n-2)$-dimensional totally complex submanifold with respect to quaternionic Kähler structure $I$ of $G_{1,1}(\mathbb{C}^{n+1})$ satisfying

1. $I^2 = -1$ provided $|\mu| > 2$,
2. $I^2 = 1$ provided $|\mu| < 2$,
3. $I^2 = 0$ provided $|\mu| = 2$

(Theorem 6.1). Detailed descriptions will appear elsewhere.

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2 Real hypersurfaces in $\mathbb{CP}^n$ and $\mathbb{CH}^n$

First we recall the Fubini-Study metric on the complex projective space $\mathbb{CP}^n$ (cf. [4, 11]). The Euclidean metric $\langle , \rangle$ on $\mathbb{C}^{n+1}$ is given by $\langle z, w \rangle = \text{Re}(z\overline{w})$ for $z, w \in \mathbb{C}^{n+1}$. The unit sphere $S^{2n+1}$ in $\mathbb{C}^{n+1}$ is the principal fiber bundle over $\mathbb{CP}^n$ with the structure group $S^1$ and the Hopf fibration $\pi: S^{2n+1} \rightarrow \mathbb{CP}^n$. The tangent space of $S^{2n+1}$ at a point $z$ is

$$T_z S^{2n+1} = \{ w \in \mathbb{C}^{n+1} | \langle z, w \rangle = 0 \}.$$

Let

$$T'_z S^{2n+1} = \{ w \in \mathbb{C}^{n+1} | \langle z, w \rangle = \langle iz, w \rangle = 0 \}.$$

With respect to the principal fiber bundle $S^{2n+1}(\mathbb{CP}^n, S^1)$, there is a connection such that $T'_z$ is the horizontal subspace at $z$. Then the Fubini-Study metric $g$ of constant holomorphic sectional curvature 4 is given by $g(X, Y) = \langle X^*, Y^* \rangle$, where $X, Y \in T_z \mathbb{CP}^n$, and $X^*, Y^*$ are respectively their horizontal lifts at a point $z$ with $\pi(z) = x$. The complex structure on $T'$ defined by multiplication by $\sqrt{-1}$ induces a canonical complex structure $J$ on $\mathbb{CP}^n$ through $\pi_*$.

Next we recall the metric of negative constant holomorphic sectional curvature on the complex hyperbolic space $\mathbb{CH}^n$ (cf. [15, 11]). The indefinite metric $\langle , \rangle$ of index 2 on $\mathbb{C}^{n+1}$ is given by

$$\langle z, w \rangle = \text{Re} \left( -z_0 \overline{w}_0 + \sum_{k=1}^{n} z_k \overline{w}_k \right)$$

for $z = (z_0, z_1, \ldots, z_n)$, $w = (w_0, w_1, \ldots, w_n) \in \mathbb{C}^{n+1}$. The anti de Sitter space is defined by

$$H^{2n+1}_1 = \{ z \in \mathbb{C}^{n+1} | \langle z, z \rangle = -1 \}.$$
$H^2_{n+1}$ is the principal fiber bundle over $\mathbb{C}H^n$ with the structure group $S^1$ and the fibration $\pi : H^2_{n+1} \to \mathbb{C}H^n$. The tangent space of $H^2_{n+1}$ at a point $z$ is

$$T_z H^2_{n+1} = \{ w \in \mathbb{C}^{n+1} | (z, w) = 0 \}.$$  

Let

$$T'_z = \{ w \in \mathbb{C}^{n+1} | (z, w) = (iz, w) = 0 \}.$$  

We observe that the restriction of $\langle , \rangle$ to $T'_z$ is positive-definite. The distribution $T'_z$ defines a connection in the principal fiber bundle $H^2_{n+1}(\mathbb{C}H^n, S^1)$, because $T'_z$ is complementary to the subspace $\{iz\}$ tangent to the fibre through $z$, and invariant under the $S^1$-action. Then the metric $g$ of constant holomorphic sectional curvature $-4$ is given by $g(X, Y) = \langle X^*, Y^* \rangle$, where $X, Y \in T_z \mathbb{C}H^n$, and $X^*, Y^*$ are respectively their horizontal lifts at a point $z$ with $\pi(z) = x$. The complex structure on $T'$ defined by multiplication by $\sqrt{-1}$ induces a canonical complex structure $J$ on $\mathbb{C}H^n$ through $\pi_*$.  

We recall some facts about real hypersurfaces in complex space forms. Let $M^n(c)$ be a space of constant holomorphic sectional curvature $4c$ with real dimension $2n$. Let $J : TM \to TM$ be the complex structure with properties $J^2 = -1$, $\nabla J = 0$, and $\langle JX, JY \rangle = \langle X, Y \rangle$, where $\nabla$ is the Levi-Civita connection of $M$. Let $M^{2n-1}$ be a real hypersurface in $M(c)$ and let $N$ be a local unit normal vector field of $M$ in $\tilde{M}$. We define structure vector of $M$ as $\xi = -JN$.  

If $\xi$ is a principal vector, i.e., $A_\xi = \mu \xi$ holds for the shape operator of $M$ in $\tilde{M}$, $M$ is called a Hopf hypersurface and $\mu$ is called Hopf curvature. It is well known ([13], [12]) that Hopf curvature $\mu$ of Hopf hypersurface $M$ is constant for non flat complex space forms. We note that $\mu$ is not constant for Hopf hypersurfaces in $\mathbb{C}H^n$ in general ([16], Theorem 2.1).  

Fundamental result for Hopf hypersurfaces in complex projective space $\mathbb{C}P^n$ was proved by Cecil-Ryan [4]:

**Theorem 2.1.** Let $M$ be a connected, orientable Hopf hypersurface of $\mathbb{C}P^n$ with Hopf curvature $\mu = 2\cot 2r$ $(0 < r < \pi/2)$. Suppose the focal map $\phi_r(x) = \exp_x(rN_x)$ $(x \in M, \ N_x \in T^*_{x}M, \ |N_x| = 1)$ has constant rank $q$ on $M$. Then:

1. $q$ is even and every point $x_0 \in M$ has a neighborhood $U$ such that $\phi_rU$ is an embedded complex $(q/2)$-dimensional submanifold of $\mathbb{C}P^n$.

2. For each point $x$ in such a neighborhood $U$, the leaf of the foliation $T_0$ through $x$ intersects $U$ in an open subset of a geodesic hypersphere in the totally geodesic $\mathbb{C}P^{n-q/2}$ orthogonal to $T_p(\phi_rU)$ at $p = \phi_r(x)$. Thus $U$ lies on the tube of radius $r$ over $\phi_rU$.

Similar result for Hopf hypersurfaces with large Hopf curvature in complex hyperbolic space $\mathbb{C}H^n$ was proved by Montiel [15]:
Theorem 2.2. Let $M$ be an orientable Hopf hypersurface of $\mathbb{CH}^n$ and we assume that $\phi_r$ ($r > 0$) has constant rank $q$ on $M$. Then, if $\mu = 2 \coth 2r$ (hence $\mu > 2$), for every $x_0 \in M$ there exists an open neighbourhood $W$ of $x_0$ such that $\phi_rW$ is a $(q/2)$-dimensional complex submanifold embedded in $\mathbb{CH}^n$. Moreover $W$ lies in a tube of radius $r$ over $\phi_rW$.

On the other hand, with respect to Hopf hypersurfaces with small Hopf curvature in $\mathbb{CH}^n$, Ivey [7] and Ivey-Ryan [8] proved that a Hopf hypersurface with $|\mu| < 2$ in $\mathbb{CH}^n$ may be constructed from an arbitrary pair of Legendrian submanifolds in $S^{2n-1}$.

3 Totally complex submanifolds in Quaternionic Kähler manifolds

We recall the basic definitions and facts on totally complex submanifolds of a quaternionic Kähler manifold (cf. [18]) Let $(\tilde{M}^4, \tilde{g}, Q)$ be a quaternionic Kähler manifold with the quaternionic Kähler structure $(\tilde{g}, Q)$, that is, $\tilde{g}$ is the Riemannian metric on $\tilde{M}$ and $Q$ is a rank 3 subbundle of $\text{End} T\tilde{M}$ which satisfies the following conditions:

1. For each $p \in \tilde{M}$, there is a neighborhood $U$ of $p$ over which there exists a local frame field $\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ of $Q$ satisfying
   \[
   \tilde{I}_1^2 = \tilde{I}_2^2 = \tilde{I}_3^2 = -1, \quad \tilde{I}_1\tilde{I}_2 = -\tilde{I}_2\tilde{I}_1 = \tilde{I}_3, \\
   \tilde{I}_2\tilde{I}_3 = -\tilde{I}_3\tilde{I}_2 = \tilde{I}_1, \quad \tilde{I}_3\tilde{I}_1 = -\tilde{I}_1\tilde{I}_3 = \tilde{I}_2.
   \]

2. For any element $L \in Q_p$, $\tilde{g}_p$ is invariant by $L$, i.e., $\tilde{g}_p(LX,Y) + \tilde{g}_p(X,LY) = 0$ for $X,Y \in T_p\tilde{M}, p \in \tilde{M}$.

3. The vector bundle $Q$ is parallel in $\text{End} T\tilde{M}$ with respect to the Riemannian connection $\nabla$ associated with $\tilde{g}$.

A submanifold $M^{2m}$ of $\tilde{M}$ is said to be almost Hermitian if there exists a section $\tilde{I}$ of the bundle $Q|_M$ such that (1) $\tilde{I}^2 = -1$, (2) $\tilde{I}TM = TM$ (cf. D.V. Alekseevsky and S. Marchiafava [1]). We denote by $I$ the almost complex structure on $M$ induced from $\tilde{I}$. Evidently $(M, I)$ with the induced metric $g$ is an almost Hermitian manifold. If $(M, g,I)$ is Kähler, we call it a Kähler submanifold of a quaternionic Kähler manifold $\tilde{M}$. An almost Hermitian submanifold $M$ together with a section $\tilde{I}$ of $Q|_M$ is said to be totally complex if at each point $p \in M$ we have $LT_pM \perp T_pM$, for each $L \in Q_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$ (cf. S. Funabashi [5]). It is known that a $2m$ ($m \geq 2$)-dimensional almost Hermitian submanifold $M^{2m}$ is Kähler if and only if it is totally complex ([1] Theorem 1.12).
4 Gauss map of real hypersurfaces in $\mathbb{CP}^n$

We recall complex Grassmann manifold $G_2(\mathbb{C}^{n+1})$ according to [11], Example 10.8. Let $M' = M'(n + 1, 2; \mathbb{C})$ be the space of complex matrices $Z$ with $n + 1$ rows and 2 columns such that $i^tZZ = E_2$ (or, equivalently, the 2 column vectors are orthonormal with respect to the standard inner product in $\mathbb{C}^{n+1}$). $M'$ is identified with complex Stiefel manifold $V_2(\mathbb{C}^{n+1})$. The group $U(2)$ acts freely on $M'$ on the right: $Z \mapsto ZB$, where $B \in U(2)$. We may consider complex 2-plane Grassmannian $G_2(\mathbb{C}^{n+1})$ as the base space of the principal fiber bundle $M'$ with group $U(2)$.

For each $Z \in M'$, the tangent space $T_Z(M')$ is

$$T_Z(M') = \{W \in M(n + 1, 2; \mathbb{C})| i^tWZ + i^tZW = 0\}.$$  

In $T_Z(M')$ we have an inner product $g(W_1, W_2) = \text{Re} \text{trace} (i^tW_1W_2)$. Let $T'_Z = \{ZA | A \in u(2)\} \subset T_Z(M')$ and let $T''_Z$ be the orthogonal complement of $T'_Z$ in $T_Z(M')$ with respect to $g$. The subspace $T'_Z$ admits a complex structure $W \mapsto iW$. We see that $\pi : M' \rightarrow G_2(\mathbb{C}^{n+1})$ induces a linear isomorphism of $T'_Z$ onto $T_{\pi(Z)}(G_2(\mathbb{C}^{n+1}))$. By transferring the complex structure $i$ and the inner product $g$ on $T'_Z$ by $\tilde{\pi}$, we get the usual complex structure $J$ on $G_2(\mathbb{C}^{n+1})$ and the Hermitian metric on $G_2(\mathbb{C}^{n+1})$, respectively. Note that $G_2(\mathbb{C}^{n+1})$ is a complex $2n - 2$-dimensional Hermitian symmetric space.

Let $U$ be an open subset in $G_2(\mathbb{C}^{n+1})$ and let $\tilde{Z} : U \rightarrow \tilde{\pi}^{-1}(U)$ be a cross section with respect to the fibration $\tilde{\pi} : M' \rightarrow G_2(\mathbb{C}^{n+1})$. Then for each $p \in U$, the subspace $T'_{\tilde{Z}(p)}$ admits 3 linear endomorphisms $I_1, I_2, I_3$:

$$I_1 : W \mapsto W \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad I_2 : W \mapsto W \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$  

$$I_3 : W \mapsto W \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$  

By transferring $I_1, I_2, I_3$ on $T'_{\tilde{Z}(p)} (p \in U)$ by $\tilde{\pi}$, we get local basis $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ of quaternionic Kähler structure $Q_p$ on $U \subset G_2(\mathbb{C}^{n+1})$ (cf. [2, 19]).

Let $M^{2n-1}$ be a real hypersurface in complex projective space $\mathbb{CP}^n$ and let $w \in S^{2n+1}$ with $\pi(w) = x$ for $x \in M$. We denote $\xi_w$ as the horizontal lift of the structure vector $\xi$ at $x$ to $T'_w$ with respect to the Hopf fibration $\pi : S^{2n+1} \rightarrow \mathbb{CP}^n$.

Then the Gauss map $G : M^{2n-1} \rightarrow G_2(\mathbb{C}^{n+1})$ of $M$ is defined by

$$G(x) = \tilde{\pi}(w, \xi_w),$$

(4.1)

where $\tilde{\pi} : M' = V_2(\mathbb{C}^{n+1}) \rightarrow G_2(\mathbb{C}^{n+1})$ given as above. It is seen that $G$ is well defined.

**Theorem 4.1.** Let $M$ be a real hypersurface in $\mathbb{CP}^n$ and let $G : M^{2n-1} \rightarrow G_2(\mathbb{C}^{n+1})$ be the Gauss map defined by (4.1). Suppose $M$ is a Hopf hypersurface. Then the image $G(M)$ is a real $(2n - 2)$-dimensional totally complex submanifold of $G_2(\mathbb{C}^{n+1})$. 
5 Totally (para-)complex submanifolds in Quaternionic Para-Kähler manifolds

We recall real Clifford algebras (cf. [6, 14]) and (quaternionic) para-complex structure (cf. [14]). Let \((V = \mathbb{R}(p, q), (\ , \ ))\) be a real symmetric inner product space of signature \(p, q\). The Clifford algebra \(C(p, q)\) is the quotient \(\otimes V/I(V)\), where \(I(V)\) is the two-sided ideal in \(\otimes V\) generated by all elements:

\[
x \otimes x + \{x, x\} \quad \text{with } x \in V.
\]

Examples of Clifford algebras are:

1. \(\mathbb{C} = C(0, 1)\), **Complex numbers**: \(z = x + iy, i^2 = -1, |z|^2 = x^2 + y^2\), there is no zero divisors.
2. \(\mathbb{C} = C(1, 0)\), **Para-complex numbers**: \(z = x + jy, j^2 = 1, |z|^2 = x^2 - y^2\), there exists zero divisors.
3. \(\mathbb{H} = C(0, 2)\), **Quaternions**: \(q = q_0 + iq_1 + jq_2 + kq_3, i^2 = j^2 = k^2 = -1, |q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2\), there is no zero divisors, \(\mathbb{H} \cong \mathbb{C}^2\), \(q = z_1(q) + jz_2(q)\).
4. \(\mathbb{H} = C(2, 0) = C(1, 1)\), **Para-quaternions**: \(q = q_0 + iq_1 + jq_2 + kq_3, i^2 = -1, j^2 = k^2 = 1, |q|^2 = q_0^2 + q_1^2 - q_2^2 - q_3^2\), there exists zero divisors, \(\mathbb{H} \cong \mathbb{C}^2\), \(q = z_1(q) + jz_2(q)\).

There is a natural correspondence between complex numbers \(\mathbb{C}\) and Euclidean plane \(\mathbb{R}^2\), and para-complex numbers \(\mathbb{C}\) are naturally identified with real symmetric inner product space \(\mathbb{R}(1, 1)\) of signature \((1, 1)\). Also there is a natural correspondence between quaternions \(\mathbb{H}\) and Euclidean 4-space \(\mathbb{R}^4\), and para-quaternions \(\mathbb{H}\) are naturally identified with real symmetric inner product space \(\mathbb{R}(2, 2)\) of signature \((2, 2)\).

Let \(M\) be a differentiable manifold. We consider the following structures on a real vector space \(V = T_x M, x \in M\), where one assumes that \(I, J, K\) are given endomorphisms:

1. **Complex**: \(J^2 = -1, V \otimes \mathbb{C} = V_J^+ \oplus V_J^-, V_J^\pm = \{u \in V \otimes \mathbb{C} \mid Ju = \pm iu\}\),
2. **Para-complex**: \(J^2 = 1, V \otimes \mathbb{C} = V_J^+ \oplus V_J^-, V_J^\pm = \{u \in V \otimes \mathbb{C} \mid Ju = \pm iu\}\), \(\dim V_J^+ = \dim V_J^-\),
3. **Hypercomplex**: \(H = (I, J, K), I^2 = J^2 = K^2 = -1, IJ = -JI = K (JK = -KJ = I, KI = -IK = J)\),
4. **Para-hypercomplex**: \(\tilde{H} = (I, J, K), I^2 = J^2 = K^2 = 1, IJ = -JI = K (JK = -KJ = I, KI = -IK = J)\).

It is known that the space of (oriented) geodesics (i.e., great circles \(S^1\)) in round sphere \(S^n\) is naturally identified with (oriented) real 2-plane Grassmannian \(G_2(\mathbb{R}^{n+1})\), and it has complex structure. On the other hand, the space of (oriented) geodesics (i.e., hyperbolic lines \(\mathbb{H}^1\)) in hyperbolic space \(\mathbb{H}^n\) is naturally identified with (oriented) Grassmannian \(G_{1,1}(\mathbb{R}^{n+1})\) of 2-plane with signature \((1, 1)\) in Minkowski space \(\mathbb{R}^{n+1}_1\), and it has para-complex structure \([3, 9, 10]\).

A (para-)hypercomplex structure on \(V\) generates respectively a structure:
1. **Quaternionic**, generated by \( H = (I, J, K) \): \( Q = \langle H \rangle = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K \), 
\( (I, J, K) \) determined up to \( A \in SO(3) \), \( S(Q) = \{ L \in Q \mid \| L \|^2 = 1 \} \).

2. **Para-quaternionic**, generated by \( \tilde{H} = (I, J, K) \): \( \tilde{Q} = \langle \tilde{H} \rangle = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K \), 
\( (I, J, K) \) determined up to \( A \in SO(2,1) \), \( S(\tilde{Q}) = S^+(\tilde{Q}) \cup S^-(\tilde{Q}) \), \( S^+(Q) = \{ L \in \tilde{Q} \mid L^2 = 1 \} \), \( S^-(Q) = \{ L \in \tilde{Q} \mid L^2 = -1 \} \).

To treat the quaternionic and para-quaternionic case simultaneously, we set \( \eta = 1 \) or \( \eta = -1 \) and \( (\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, -1) \) or \( (\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1) \) respectively for hypercomplex or para-hypercomplex structure \( (I_1, I_2, I_3) \): then the multiplication table looks

\[
I_{\alpha}^2 = \epsilon_\alpha 1, \quad I_{\alpha}I_{\beta} = -I_{\beta}I_{\alpha} = \eta\epsilon_{\gamma}I_{\gamma} \quad (\alpha = 1, 2, 3)
\]

where \((\alpha, \beta, \gamma)\) is any circular permutation of \((1, 2, 3)\).

An almost (para-)quaternionic structure on a differentiable manifold \( M^{4n} \) is a rank 3 subbundle \( Q \subset \text{End}(TM) \), which is locally spanned by a field \( H = (I_1, I_2, I_3) \) of (para-)hypercomplex structures. Such a locally defined triple \( H = (I_\alpha), \alpha = 1, 2, 3 \), will be called a (local) admissible basis of \( Q \).

An almost (para-)quaternionic connection is a linear connection \( \nabla \) which preserves \( Q \); equivalently, for any local admissible basis \( H = (I_\alpha) \) of \( Q \), one has

\[
\nabla I_\alpha = -\epsilon_\beta \omega_\gamma \otimes I_\beta + \epsilon_\gamma \omega_\beta \otimes I_\gamma \quad \text{(P.C.)}
\]

where (P.C.) means that \((\alpha, \beta, \gamma)\) is any circular permutation of \((1, 2, 3)\).

An almost (para-)quaternionic structure \( Q \) is called a (para-)quaternionic structure if \( M \) admits a (para-)quaternionic connection, i.e. a torsion-free almost (para-)quaternionic connection. An (almost) (para-)quaternionic manifold \((M^{4n}, Q)\) is a manifold \( M^{4n} \) endowed with an (almost) (para-)quaternionic structure \( Q \).

An almost (para-)quaternionic Hermitian manifold \((M, Q, g)\) is an almost (para-)quaternionic manifold \((M, Q)\) endowed with a pseudo Riemannian metric \( g \) which is \( Q \)-Hermitian, i.e. any endomorphism of \( Q \) is \( g \)-skew-symmetric. \((M, Q, g), (n > 1)\), is called a (para-)quaternionic Kähler manifold if the Levi-Civita connection of \( g \) preserves \( Q \), i.e. \( \nabla^g \) is a (para-)quaternionic connection.

Let \((\tilde{M}^{4n}, \tilde{g}, Q)\) be a para-quaternionic Kähler manifold with the para-quaternionic Kähler structure \((\tilde{g}, Q)\). A submanifold \( M^{2m} \) of \( \tilde{M} \) is said to be almost Hermitian (resp. almost para-Hermitian) if there exists a section \( \tilde{I} \) of the bundle \( Q|_{\tilde{M}} \) such that (1) \( \tilde{I}^2 = -1 \) (resp. \( \tilde{I}^2 = 1 \)), (2) \( \tilde{I}TM = TM \). An almost (para-)Hermitian submanifold \( M \) together with a section \( \tilde{I} \) of \( Q|_{\tilde{M}} \) is said to be totally (para-)complex if at each point \( p \in M \) we have \( LT_pM \perp T_pM \), for each \( L \in Q_p \) with \( \tilde{g}(L, \tilde{I}p) = 0 \).
6 Gauss map of real hypersurfaces in $\mathbb{CH}^n$

We consider Grassmann manifold $G_{1,1}(\mathbb{C}_1^{n+1})$ of 2-planes with signature $(1,1)$ in $\mathbb{C}_1^{n+1}$. We put $n \times n$ matrix $J_n$ as

$$J_n = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$ 

Let $\tilde{M}' = \tilde{M}'(n + 1, 2; \mathbb{C})$ be the space of complex matrices $Z$ with $n + 1$ rows and 2 columns such that $\mathcal{I}'Z_{n+1}Z = J_2$ (or, equivalently, the 2 column vectors are orthonormal with respect to the inner product in $\mathbb{C}_1^{n+1}$). $\tilde{M}'$ is identified with complex (indefinite) Stiefel manifold $V_{1,1}(\mathbb{C}_1^{n+1})$. The group $U(1, 1)$ acts freely on $\tilde{M}'$ on the right: $Z \mapsto ZB$, where $B \in U(1, 1)$. We may consider complex $(1,1)$-plane Grassmannian $G_{1,1}(\mathbb{C}_1^{n+1})$ as the base space of the principal fiber bundle $\tilde{M}'$ with group $U(1, 1)$.

For each $Z \in \tilde{M}'$, the tangent space $T_Z(\tilde{M}')$ is

$$T_Z(\tilde{M}') = \{W \in M(n + 1, 2; \mathbb{C}) | ^tWJ_{n+1}Z + ^tZJ_{n+1}W = 0\}.$$ 

In $T_Z(\tilde{M}')$ we have an inner product $\tilde{g}(W_1, W_2) = \text{Re} \text{trace} \ ^tW_2J_{n+1}W_1$. Let $T''_Z = \{ZA | A \in u(1,1)\} \subset T_Z(\tilde{M}')$ and let $T''_Z$ be the orthogonal complement of $T''_Z$ in $T_Z(\tilde{M}')$ with respect to $\tilde{g}$. The subspace $T''_Z$ admits a complex structure $\tilde{W} \mapsto i\tilde{W}$. We see that $\tilde{\pi} : \tilde{M}' \to G_{1,1}(\mathbb{C}_1^{n+1})$ induces a linear isomorphism of $T''_Z$ onto $T_{\tilde{\pi}(Z)}(G_{1,1}(\mathbb{C}_1^{n+1}))$. By transferring the complex structure $i$ and the inner product $g$ on $T''_Z$ by $\tilde{\pi}$, we get the complex structure $J$ on $G_{1,1}(\mathbb{C}_1^{n+1})$ and the (indefinite) Hermitian metric on $G_{1,1}(\mathbb{C}_1^{n+1})$, respectively.

Let $U$ be an open subset in $G_{1,1}(\mathbb{C}_1^{n+1})$ and let $\tilde{Z} : U \to \tilde{\pi}^{-1}(U)$ be a cross section with respect to the fibration $\tilde{\pi} : \tilde{M}' \to G_{1,1}(\mathbb{C}_1^{n+1})$. Then for each $p \in U$, the subspace $T''_{\tilde{Z}(p)}$ admits 3 linear endomorphisms $I_1, I_2, I_3$:

$$I_1 : W \mapsto W \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad I_2 : W \mapsto W \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_3 : W \mapsto W \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$ 

By transferring $I_1, I_2, I_3$ on $T''_{\tilde{Z}(p)} (p \in U)$ by $\tilde{\pi}$, we get local basis $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ of quaternionic para-Kähler structure $Q_p$ on $U \subset G_{1,1}(\mathbb{C}_1^{n+1})$.

Let $M^{2n-1}$ be a real hypersurface in complex hyperbolic space $\mathbb{CH}^n$ and let $w \in H_1^{2n+1}$ with $\pi(w) = x$ for $x \in M$. We denote $\xi'_w$ as the horizontal lift of the structure vector $\xi$ at $x$ to $T'_x$ with respect to the fibration $\pi : H_1^{2n+1} \to \mathbb{CH}^n$. Then the Gauss map $\tilde{G} : M^{2n-1} \to G_{1,1}(\mathbb{C}_1^{n+1})$ of $M$ is defined by

$$(6.1) \quad \tilde{G}(x) = \tilde{\pi}(w, \xi'_w).$$
where $\tilde{\pi} : \tilde{M}' = V_{1,1}(\mathbb{C}^{n+1}) \to G_{1,1}(\mathbb{C}^{n+1})$ given as above. It is seen that $\tilde{G}$ is well defined.

**Theorem 6.1.** Let $M^{2n-1}$ be a real hypersurface in $\mathbb{C}H^n$ and let $\tilde{G} : M \to G_{1,1}(\mathbb{C}^{n+1})$ be the Gauss map defined by (6.1). Suppose $M$ is a Hopf hypersurface with Hopf curvature $\mu$. Then the image $G(M)$ is a real $(2n-2)$-dimensional submanifold of $G_{1,1}(\mathbb{C}^{n+1})$ such that

1. $G(M)$ is totally complex provided $|\mu| > 2$,
2. $G(M)$ is totally para-complex provided $|\mu| < 2$,
3. There exists a section $\tilde{I}$ of the bundle $Q|_{G(M)}$ satisfying $\tilde{I}^2 = 0$ and $\tilde{I}TG(M) = TG(M)$ provided $|\mu| = 2$.

**References**


