Fibred Riemannian spaces with Sasaki-Einstein metric

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Abstract. In this talk, we consider the fibred Riemannian space with Sasaki-Einstein metric and summarize recent results for such a space. Moreover we investigate the construction of a space with Sasaki-Einstein metric and the relation of Sasaki-Einstein manifold and Calabi-Yau manifold.

1 Introduction

Recently many papers for the study of Sasaki-Einstein manifold were published. It is well known that the Sasaki-Einstein metric is deeply related to the theory of black holes. A Sasakian manifold is a odd-dimensional Riemannian manifold with normal contact structure, and Sasaki-Einstein manifold is a Riemannian manifold that is both Sasakian and Einstein. Sasakian manifold is the odd-dimensional cousin of Kaehler manifold.

Sasaki-Einstein metric on odd-dimensional Riemannian manifold is deeply related to the Kaehler Ricci flat metric, that is Calabi-Yau metric on an even-dimensional Riemannian manifold. More precisely Sasaki-Einstein manifold may be defined as an Einstein manifold whose metric cone is Ricci flat and Kaehler, that is Calabi-Yau manifold. Such manifolds provide interesting examples of the string theory [3,4,8].

The canonical example of a Sasaki-Einstein manifold is the odd-dimensional sphere equipped with its standard Einstein metric. In this case the Kaehler cone is $\mathbb{C}^n - \{0\}$ equipped with its flat metric [12].

In fact any complex surface whose metric is Kaehler-Einstein and of positive scalar curvature admits a unique simply connected circle bundle which is canonically Sasaki-Einstein.

A classification of Riemannian manifolds admitting real Killing spinors on $M$ correspond to the parallel spinors on $C(M) = (R^+ \times M, dr^2 + r^2 g)$ the metric cone on $M$. In this point of a view, there are many results about Sasaki-Einstein geometry using this cone manifold with the Kaehler structure.

Key words and phrases: Sasaki-Einstein, Fibred Riemannian space.

For the purpose of the construction of Sasaki- Einstein manifold, we use the Riemannian submersion theory.

From this point of a view, we review recent results and developments of fibred Riemannian space and Sasaki-Einstein geometry as well as other results not mentioned above. Moreover, we consider the relation of Sasaki-Einstein manifold and Calabi-Yau manifold.

2 Warped products and fibred Riemannian space

Let \( \{M, B, G, \pi\} \) be a fibred Riemannian space, that is \( M \) an \( m \)-dimensional total space with projectable Riemannian metric \( G \), \( (B, g) \) an \( n \)-dimensional base space, and \( \pi : M \to B \) a projection with a maximal rank \( n \). The fibre passing through a point \( q \) in \( M \) is denoted by \( F(q) \) or generally \( F \), which is a \( p \)-dimensional submanifold of \( M \), where \( p = m - n \).

The quantities \( h \) and \( L \) are the components of the second fundamental tensor and normal connection of each fibre respectively. If the horizontal mapping covering curve in \( M \) is an isometry (resp. conformal mapping) of fibres, then it is called a fibred Riemannian space with isometric (resp. conformal) fibres. It is well known that a necessary and sufficient condition for \( M \) to have isometric (resp. conformal) fibres is \( h = 0 \) (resp. \( h = \lambda \hat{g} \), where \( \hat{g} \) is an induced Riemannian metric on each fibre).

The following Theorem is well known.

**Theorem 2.1.** [6] If the components of \( L \) and \( h \) vanish identically in a fibred Riemannian space, then the fibred space is locally the Riemannian product of the base space and a fibre.

The warped product space is a special case of Riemannian submersion. Besse [2] introduced the relation of warped product space and Riemannian submersion as follows.

**Theorem 2.2.** [2] Let \( M = B \times_f F \) be the warped products of \( (B, g) \) and \( (F, \hat{g}) \). Then the projection \( \pi : M \to B \) onto the first factor is a Riemannian submersion.
Moreover the tensorial invariants of \( \pi \) satisfy

\[(2.1) \quad h = \lambda \bar{g}, \quad L = 0 \quad \text{and} \quad N \text{ is basic},\]

where \( N \) is the mean curvature vector along each fibre. In this case \( N \) is \( \pi \)– related to the vector field \(-\frac{1}{2}Df\) on \( B \), where \( Df \) is the gradient of \( f \) for \( g \) on \( B \).

Conversely, the conditions (2.1) characterize locally warped products among Riemannain submersions.

3 Sasaki-Einstein manifold

A compact Riemannian manifold \((M, g)\) is Sasakian if and only if its metric cone

\[(3.1) \quad C(M) = R^+ \times M, \quad G = dr^2 + r^2 g\]

is Kaehler manifold.

It follows that \( M \) has odd-dimension \( 2n - 1 \), where \( n \) denotes the complex dimension of the Kaehler cone.

Notice that the Sasakian manifold \((M, g)\) is naturally isometrically embedded into the cone via the inclusion

\[M = \{r = 1\} = \{1\} \times M \subset C(M).\]

The Kaehler structure of \((C(M), G)\), combined with its cone structure, induce the Sasakian structure on \( M = \{1\} \times M \subset C(M) \).

The following theorem is well known.

**Theorem 3.1.** [12] Let \((M, g)\) be a Sasakian manifold of dimension \( 2n - 1 \). Then the following are equivalent

1. \((M, g)\) is Sasaki-Einstein with \( \text{Ric}_g = 2(n - 1)g \)
2. The Kaehler cone \((C(S), G)\) is Ricci flat, \( \text{Ric}_G = 0 \)
3. The transverse Kaehler structure to the Reeb foliation is Kaehler-Einstein

\[\text{Ric}^T = 2ng^T\]

There are another method of the definition of Sasakian manifold. For an odd-dimensional manifold \( M^{2n+1} \), J. Gray [5] defined an almost contact structure as a reduction of the structural group to \( U(n) \times 1 \). In terms of structure tensors we say
$M^{2n+1}$ has an almost contact structure or sometimes $(\phi, \xi, \eta)$-structure if admits a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$  

It is well known that (3.2) reduces $\phi \xi = 0$ and $\eta \circ \phi = 0$. If a manifold $M^{2n+1}$ with $(\phi, \xi, \eta)$-structure admits a Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then we say $M^{2n+1}$ has an almost contact metric structure and $g$ is called a compatible metric [3,6].

An almost contact structure $(\phi, \xi, \eta)$ on $M$ is normal if the almost complex structure $J$ on $M \times \mathbb{R}$ given by

$$J(X, f d\tau) = (\phi X - f \xi, \eta(X) d\tau),$$

$f$ being a $C^\infty$-function on $M \times \mathbb{R}$ is integrable.

An almost contact metric manifold $(M, g)$ with $(\phi, \xi, \eta)$ is said to be [7,13]

(i) contact if $\Phi = d\eta$

(ii) K-contact if $\Phi = d\eta$ and $\xi$ is a Killing vector

(iii) Sasakian if $\Phi = d\eta$ and $(\phi, \xi, \eta)$ is normal,

where $\Phi(X, Y) = g(\phi X, Y)$.

Y. Tashiro and B.H. Kim [13] have studied the fibred almost contact metric space with invariant fibres tangent to the structure vector and deal with various almost contact structure. They considered the fibred Riemannian space $M$ with base space $(B, g)$ with almost complex manifold with almost complex structure $J$ and fibre $F$ with almost contact structure $(\phi, \xi, \eta, \bar{g})$.

If we put

$$\tilde{\phi} = J_b \ ^a E^b \otimes E_a + \bar{\phi}_\beta \ ^\alpha C^\beta \otimes C_\alpha,$$

$$\tilde{\eta} = \bar{\eta}_\alpha C^\alpha,$$

$$\tilde{\xi} = \bar{\xi}_\alpha C_\alpha,$$

$$G = \begin{pmatrix} g & 0 & \bar{g} \\ 0 & 0 & \bar{g} \end{pmatrix},$$

then we can easily see that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, G)$ defines an almost contact metric structure on $M$ where $(E_a, C_\alpha)$ is a local flame, $\{E_a\}$ are in the base space and $\{C_\alpha\}$ are in the fibre.

Conversely, if there is in $M$ an almost contact structure $(\phi, \xi, \eta, \bar{g})$, $G$ and $\tilde{\phi}$ are projectable and $\tilde{\xi}$ is always vertical, then the structure induces an almost Hermitian structure $(J, g)$ in the base space and almost contact metric structure $(\phi, \xi, \eta, \bar{g})$ in each fibre. In this case we have
Theorem 3.2. [7] If a fibred almost contact metric structure is Sasakian, then
the base space is Kaelerian and each fibre is Sasakian. In this case, each fibre is
minimal, and \( L = J \otimes \xi \), where \( J \) is a almost complex structure on the base space
and \( \xi \) is a structure vector of the fibre.

Theorem 3.3. [7] Let \( M \) be fibred Sasakian space with conformal fibres, then \( M \)
is Sasaki-Einstein if and only if \( B \) is Kaeher-Einstein, \( \bar{S} = \lambda \bar{g} - n \bar{\eta} \otimes \bar{\eta} \) and \( K = n(n + 2p + \bar{K})/p \), where \( \bar{S} \) is a Ricci curvature tensor of the fibre and \( \bar{K} \) is a scalar
curvature of the fibre.

In this case, each fibre is a totally geodesic submanifold of the total space and
\( S = (\alpha + 2)g \), where \( \alpha = \bar{K}/n \) and \( \bar{K} \) is a scalar curvature of the total space. Hence
the base space is Ricci flat if \( \bar{K} = -2m \). In this case, \( K = 0 \) and \( \bar{K} = -2p - n \).

Hence if we consider the 5-dimensional fibre Sasaki-Einstein space with conformal
fibres, then we can clarify the geometric structure of the base space and each
fibre in two cases, that is \( n = 4, p = 1 \) and \( n = 2, p = 3 \). In this case we see that

Theorem 3.4. Let \( M \) be a 5-dimensional fibred Sasakian space with conformal
fibre. Then \( M \) is Einstein (that is \( M \) is Sasaki-Einstein) if and only if

1. \( B \) is Kaeher-Einstein
2. \( F \) is \( \eta \)-Einstein
3. \( \bar{K} = K + 2 \).

From Theorems 2.2 and 3.3, we see that

Theorem 3.5. Let \( M = B \times_f F \) be the warped product of \( (B, g) \) and \( (F, \bar{g}) \). If
we consider \( M \) as a special case of fibred Riemannian space, then \( M \) with Sasakian
structure does not exist.

For the relation of the Sasaki-Einstein manifold and Calabi-Yau manifold, the
following theorems are well known ([3]).

Theorem 3.6. Let \( (M, g) \) be a Riemannian manifold. Then the metric \( g \) is Sasaki-
Einstein if and only if the cone metric \( G \) is Calabi-Yau, i.e., \( (C(M), G) \) is Kaehler-
Ricci flat.
Theorem 3.7. Any totally geodesic hypersurface of a nearly Kaehler 6-manifold admits a Sasaki-Einstein structure.

References