Topological properties of symmetric products

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Abstract. In this article we investigate the topological properties of symmetric products of symplectic manifolds. For these we study the moduli space of stable $J$-holomorphic maps with marked points of product space and symmetric group action on it, Gromov-Witten invariant and relative Gromov-Witten invariant. Also we investigate the relations between symmetric invariant properties on the products space and the corresponding ones on the symmetric product, and compute a generating series for Gromov-Witten invariants. As an example we examine the symmetric product of $n$ copies complex projective line $\mathbb{C}P^1$.

1 Introduction

A $J$-holomorphic map is a $(j, J)$-holomorphic map $u : \Sigma \rightarrow M$ from a Riemann sphere $(\Sigma, j)$ to a closed symplectic manifold $(M, J)$. Let $A$ be a homology class in $H_2(M, \mathbb{Z})$ and let $\mathfrak{M}(M, A)$ be the moduli space consists of the stable $J$-holomorphic map $u : (\Sigma; z_1, z_2, z_3) \rightarrow M$ from a reduced rational curve $\Sigma$ with 3 marked points to a symplectic manifold $M$ of dimension $2n$ representing the class $A$. Then the moduli space is a compact space with virtual dimension $2n + 2c_1(TM)(A)$. The evaluation map

$$ev : \mathfrak{M}(M, A) \rightarrow M^3$$

is defined by $ev(u; z_1, z_2, z_3) = (u(z_1), u(z_2), u(z_3))$. Let $\alpha = \alpha_1 \otimes \alpha_2 \otimes \alpha_3$ be an element of $H_2(M^3)$, here $d + \dim \mathfrak{M}(M, A) = 6n$. We define the Gromov-Witten

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invariant

\[ \Phi_A : H_d(M^3) \to \mathbb{Q} \]

\[ \Phi_A(\alpha) = ev \cdot \alpha = \int_{\mathfrak{M}(M,A)} ev^*(PD\alpha), \]

to be the number of these intersection points counted with signs of their orientations. Roughly speaking, the Gromov-Witten invariant \( \Phi_A(\alpha) \) is the number of \( (j,J) \)-holomorphic maps

\[ u : (\Sigma; z_1, z_2, z_3) \to M \]

counting the class \( A \) such that \( u(z_1) \in \alpha_1, u(z_2) \in \alpha_2, u(z_3) \in \alpha_3 \). We define the quantum multiplication \( a \star b \) of \( a \in H^k(M) \) and \( b \in H^l(M) \) as the formal sum

\[ a \star b = \sum_{A \in H^2(M)} (a \star b)_A q^{c_1(A)}/N, \]

where \( q \) is an auxiliary variable of degree \( 2N \), \( N \) is called the minimal Chern number defined by \( < c_1, H^2(M) >= N \mathbb{Z} \), and the cohomology class \( (a \star b)_A \in H^{k+l-2c_1(A)}(M) \) is defined by the Gromov-Witten invariant by

\[ \int_{\gamma} (a \star b)_A = \Phi_A(\alpha \otimes \beta \otimes \gamma), \]

for \( \gamma \in H^{k+l-2c_1(A)}(M) \) where \( \alpha \) and \( \beta \) are Poincaré dual of \( a \) and \( b \), respectively.

Let \( QH^*(M) = H^*(M) \otimes \mathbb{Q}[q] \), where \( \mathbb{Q}[q] \) is the Laurent polynomial ring in the variable \( q \) of degree \( 2N \). The element \( a \star b \in QH^*(M) \). Extending by linearity, we have a multiplication

\[ \star : QH^*(M) \otimes QH^*(M) \to QH^*(M). \]

We call \( QH^*(M) \) the quantum cohomology ring of \( M \). In fact, if the minimal Chern number \( N \geq 2 \), then the quantum product on the quantum cohomology group \( QH^*(M) \) is associative.

In section 2, we introduce the symmetric product of a closed symplectic manifold and the Macdonald formula for the generating series of the Euler characteristic, arithmetic genus, signature and Hirzebruch genus. Also we consider the symmetric group action on product space and its invariant homology and cohomology. In section 3 we have a relation between the invariant moduli space relative to fixed point set and the relative moduli space on its orbit space. We investigate the symmetric invariant relative Gromov-Witten invariant and the relative Gromov-Witten invariant on its orbit space, and investigate small quantum cohomology of product spaces, symmetric invariant quantum cohomologies on product space and its orbit space [5]. In section 4, as an example we examine the Gromov-Witten invariants of the symmetric product of \( n \) copies of the complex projective line \( \mathbb{P}^1 \), and compute a generating series for Gromov-Witten invariants [17].
2 Generating Series of Symmetric Products

(2.1) Let \((M, \omega_M, J_M)\) be a \(2n\)-dimensional closed symplectic manifold with symplectic form \(\omega_M\) and its compatible complex structure \(J_M\). And let \(X = M \times \cdots \times M\) be the product space of \(k\) copies of the manifold \(M\). Then \(X\) is a \(2nk\)-dimensional manifold with symplectic form \(\omega\) and compatible complex structure \(J\) induced by \(\omega_M\) and \(J_M\), respectively. The symmetric group \(S_k\) of \(k\) letters acts canonically on the product space \(X\) by

\[
\sigma \cdot (x_1, \ldots, x_k) = (x_{\sigma_1}, \ldots, x_{\sigma_k}),
\]

for each \(\sigma \in S_k\) and \((x_1, \ldots, x_k) \in X\).

The action of symmetric group \(S_k\) on \(X\) preserves the symplectic and \(J\)-holomorphic structures on \(X\), i.e., \(\sigma \ast \omega = \omega\) and \(J \circ \sigma = \sigma \circ J\) for each \(\sigma \in S_k\). Each element of the group \(S_k\) has its fixed point set which is a kind of diagonal in \(X\) as follows.

Lemma 2.1. (1) If a cyclic element \(\sigma \in S_k\) has order \(k_1\), then the fixed point set is the set of form

\[
X^\sigma \cong \triangle_{k_1} \times M \times \cdots \times M,
\]

where the \(\triangle_{k_1}\) is the diagonal of the product of \(k_1\) copies of \(M\), and \(M \times \cdots \times M\) is the product of \((k - k_1)\) copies of \(M\).

(2) If \(\sigma = \sigma_1 \sigma_2\) is a product of disjoint cyclic elements \(\sigma_1\) and \(\sigma_2\) of orders \(k_1\) and \(k_2\), respectively, then the fixed point set \(X^\sigma \simeq \triangle_{k_1} \times \triangle_{k_2} \times M \times \cdots \times M\) where \(\triangle_{k_i}\) is the diagonal of the product of \(k_i\) copies of \(M\), which is diffeomorphic to \(M\) and \(M \times \cdots \times M\) is the product of \(k - k_1 - k_2 + 2\) copies of \(M\).

Let \(X\) be the product of \(k\) copies of \(M\) and the symmetric group \(S_k\) act canonically on \(X\). The orbit space \(X' := X/S_k\) is called the \(k\)-th order symmetric product of \(M\).

Let us give some examples of symmetric product spaces.

(1) The \(n\)-th symmetric product space of the complex projective line \(\mathbb{CP}^1\) is diffeomorphic to the complex projective space \(\mathbb{CP}^n\) of complex dimension \(n\).

(2) Let \(M\) be the torus \(S^1 \times S^1\). The \(S^1 \times S^1\) is a Kähler manifold. The \(n\)-th symmetric product space of the \(S^1 \times S^1\) is known to be homotopic to \(S^1 \times S^1 \times \mathbb{CP}^{n-1}\).

(3) Let \(M\) is the circle \(S^1\). The circle \(S^1\) is not a symplectic manifold. To get the \(n\)-th symmetric product space of the \(S^1\) we cannot follow the argument (1) since the real polynomial of degree \(n\) may not have real \(n\) roots.
In fact, the \( n \)-th symmetric product space of the circle \( S^1 \) is homeomorphic to the closure of a tubular neighbourhood of a real projective line in the real projective space \( \mathbb{RP}^n \) of real dimension \( n \). Thus the \( n \)-th symmetric product space of the \( S^1 \) is homotopically equivalent to the circle \( S^1 \) itself.

\[ \text{(2.2)} \]

For simplicity we denote \( M^n \) the \( n \)-th product space of \( n \) copies of \( M \), \( M^{(n)} \) the \( n \)-th symmetric product space of \( n \) copies of \( M \). If \( x_0 \in M \) is a preferred point, then there are canonical inclusions

\[ M^{(n)} \subset M^{(n+1)} \]

and whose union \( M^{(\infty)} \). There is a natural question which is “how do we calculate invariants \( I(M^{(n)}) \) of the symmetric products of \( M \)?”. The standard approach is to encode the invariants of all symmetric products in a generating series

\[ SI(M) := \sum_{n \geq 0} I(M^{(n)}) t^n, \]

if the invariants \( I(M^{(n)}) \) are defined for all \( n \), and to calculate \( SI(M) \) in terms of invariants of \( M \). Then \( I(M^{(n)}) \) is the coefficient of \( t^n \) in the expression in the invariants of \( M \).

There are some results:

\[ \text{(2.2.1) Euler-Poincaré characteristic.} \]

The Euler-Poincaré characteristic of \( M \) is defined by

\[ \chi(M) = \sum_{k \geq 0} (-1)^k b_k(M), \]

where the \( b_k(M) \) is the \( k \)-th Betti number of \( M \), \( k \geq 0 \). The generating series, due to Macdonald [14], is

\[ \sum_{n \geq 0} \chi(M^{(n)}) t^n = (1 - t)^{-\chi(M)}. \]

\[ \text{(2.2.2) Arithmatic genus.} \]

Let \( M \) be a compact projective variety. The arithmatic genus of \( M \) is defined by

\[ \chi_a(M) = \sum_{n \geq 0} (-1)^n \dim H^k(M, \mathcal{O}_M). \]

The generating series of \( M \), due to Moonen [12], is

\[ \sum_{n \geq 0} \chi_a(M^{(n)}) t^n = (1 - t)^{-\chi_a(M)}. \]

\[ \text{(2.2.3) Signature.} \]

Let \( \sigma(M) \) is the signature of a compact, oriented manifold \( M \). The signature \( \sigma(M) \)
is defined as follows for $\dim M = 4k$:
choose a basis $\alpha_1, \ldots, \alpha_r$ for $H^{2k}(M; \mathbb{Q})$ so that the symmetric $r \times r$ matrix
\[
(\langle \alpha_i \cup \alpha_j, M \rangle)
\]
is diagonal. Then $\sigma(M)$ is the number of positive diagonal entries minus the number of negative ones. The generating series, due to Hirzebruch and Zagier [16], is
\[
\sum_{n \geq 0} \sigma(M^{(n)}) t^n = \frac{(1 + t)^\frac{1}{2} (\sigma(M) - \chi(M))}{(1 - t)^\frac{1}{2} (\sigma(M) + \chi(M))}.
\]
(2.2.4) Hirzebruch $\chi_y$-genus.
Let $M$ be a compact complex algebraic manifold. There is a decomposition of complex vector spaces
\[
H^k(M; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q},
\]
where $H^k(M; \mathbb{C}) = H^k(M; \mathbb{R}) \otimes \mathbb{C}$, $H^{p,q} = H^q(M, \Omega^p_M)$ and $H^{p,q} = \overline{T}^{q,p}$. The Hirzebruch $\chi_y$-genus of $M$ is defined by
\[
\chi_y(M) = \sum_{p,q} (-1)^q h^{p,q}(M) y^p,
\]
where $h^{p,q}(M) = \dim H^q(X, \Omega^p_M)$ is the Hodge number of $M$.
The generating series of $M$, due to Borisou-Libgober and Zhou [1],
\[
\sum_{n \geq 0} \chi_y(M^{(n)}) t^n = \exp\left(\sum_{n \geq 1} \chi_y^n(M) \frac{t^n}{n}\right).
\]
In particular, for a compact complex algebraic manifold $\chi_{-1}(M) = \chi(M)$, $\chi_0(M) = \chi_a(M)$, and $\chi_1(M) = \sigma(M)$.
If $M$ is a Riemann surface of genus $g$, then the generating series is
\[
\sum_{n \geq 0} \chi_y(M^{(n)}) t^n = [(1 - t)(1 - yt)]^{g-1}.
\]
Therefore,
\[
h^{p,q}(M^{(n)}) = \sum_{0 \leq k \leq p} \binom{g}{p-k} \binom{g}{q-k}, \quad 0 \leq p \leq q, \quad p + q \leq n,
\]
and $M^{(n)} = \mathbb{C}P^n$ if $g = 0$.
(2.3) Let $(M, \omega, J)$ be a closed symplectic manifold of dimension $2k$ with an almost complex structure $J$ compatible with symplectic structure $\omega$ on $M$. Let $\pi : M^n \rightarrow M^{(n)} = M^n/S_n$ be the projection from the product space $M^n$ onto the $n$-th symmetric product space of $M$. 

Theorem 2.2 ([17]). The following diagram is commutative.

\[
\begin{array}{ccc}
H^{2kn-2}(M^n; \mathbb{Z}) & \to & H^{2kn-2}(M^{(n)}; \mathbb{Z}) \\
\downarrow \cong & & \downarrow \cong \\
H^{2n-2}(M^n; \mathbb{Z})_{S_n} & \to & H^{2n-2}(M^{(n)}; \mathbb{Z})_{S_n} \\
\downarrow PD & & \downarrow PD \\
H_2(M^n; \mathbb{Z})_{S_n} & \to & H_2(M^{(n)}; \mathbb{Z})_{S_n} \\
\downarrow p' & & \downarrow p' \\
H_2(M^n; \mathbb{Z}) & \to & H_2(M^{(n)}; \mathbb{Z})
\end{array}
\]

where PD is the Poincaré dual, $|S_n|$ is the order of the symmetric group $S_n$, that is, $n!$, and $p$ is the projection given by $p(\alpha) = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma^*(\alpha)$, for each $\alpha \in H^{2n-2}(M^n; \mathbb{Z})$, and $p'$ is the projection defined similarly.

3 Quantum Cohomologies

(3.1) Consider the symmetric group $S_k$ of $k$ letters action on the product $X = M \times \cdots \times M$ of $k$ copies of $2n$-dimensional symplectic manifold $M$. We may define Gromov-Witten invariant on $X$, $S_k$-invariant Gromov-Witten invariant, and relative Gromov-Witten invariant related by a subspace of $X$, and also Gromov-Witten invariant on the orbit space $X' = X/S_k$.

That is, for $A \in H_2(X; \mathbb{Z})$, Gromov-Witten invariant on $X$ is defined by

\[
\Phi^X_A : H^*(X^3) \to \mathbb{Q},
\]

\[
\Phi^X_A (\alpha \otimes \beta \otimes \gamma) = \int_{M(X,A)} ev^*(\alpha \otimes \beta \otimes \gamma),
\]

where $ev : \mathcal{M}(X,A) \to X^3$ is the evaluation map defined by

\[
ev(u; z_1, z_2, z_3) = (u(z_1), u(z_2), u(z_3)), \text{ and } \alpha, \beta, \gamma \in H^*(X).
\]

For element $A \in H_2(X; \mathbb{Z})^{S_k}$, the $S_k$-invariant GW-invariant is defined by

\[
\Phi^{X,S_k}_A : H^*(X^3)^{S_k} \to \mathbb{Q},
\]

\[
\Phi^{X,S_k}_A (\alpha \otimes \beta \otimes \gamma) = \int_{\mathcal{M}(X,A)^{S_k}} ev^*(\alpha \otimes \beta \otimes \gamma),
\]

where $\alpha, \beta, \gamma \in H^*(X)^{S_k}$.
Similarly the $S_k$-invariant GW-invariant relative to the diagonal $\Delta$ is defined by

$$\Phi^X_{A, S_k} (\alpha \otimes \beta \otimes \gamma) = \int_{\mathfrak{M}(X; \Delta; A)^{S_k}} ev^* (\alpha \otimes \beta \otimes \gamma),$$

where $\alpha, \beta, \gamma \in H^*(X)^{S_k}$.

Consider the following commutative diagram:

$$\begin{array}{ccc}
H^*(X)^{S_k} & \xrightarrow{\mu} & H_*(X)^{S_k} \\
\downarrow{\pi^*} & & \downarrow{\pi_*} \\
H^*(X') & \xrightarrow{\mu'} & H_*(X'),
\end{array}$$

where $\mu$ and $\mu'$ are Kronecker products.

**Theorem 3.1** ([5]). Let $\pi : X \to X'$ be the projection. If $\pi^* (\alpha_i) = k \alpha_i$, $i = 1, 2, 3$, and $A \in H_2(X)^{S_k}$, $\pi_*(A) = kB$, then the Gromov-Witten invariants,

$$\Phi^X_{A, S_k} (\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \Phi^{X'}_{B, (\alpha_1' \otimes \alpha_2' \otimes \alpha_3')},$$

are the same.

(3.2) Suppose that $(X, \omega, J)$ is the product of $k$ copies of symplectic manifold $(M, \omega_M, J_M)$. The minimal chern number $N$ of $X$ is defined by

$$(c_1(TX), H_2X) = NZ,$$

where $\omega = \lambda c_1$ for some $\lambda > 0$.

The quantum multiplication $\alpha \ast \beta$ of classes $\alpha \in H^k(X)$ and $\beta \in H^l(X)$ as follows

$$\alpha \ast \beta = \sum_{A \in H_2(X)} (\alpha \ast \beta)_A q^{\frac{c_1(A)}{2N}},$$

where $q$ is a degree $2N$ auxiliary variable, the cohomology class $(\alpha \ast \beta)_A \in H^{k+l-2\ell_1(A)}(X)$ is defined by the Gromov-Witten invariant

$$\int_c (\alpha \ast \beta)_A = \Phi^X_A (\alpha \otimes \beta \otimes \gamma),$$

where $c \in H_{k+l-2\ell_1(A)}(X)$ and $\gamma = PD(c)$.

The tensor product

$$QH^*(X) = H^*(X) \otimes Q(q)$$

is called the small quantum cohomology of $X$, where $Q(q)$ is the ring of Laurent polynomials.

Extending the multiplication $\ast$ by linearity on the quantum cohomology group $QH^*(X)$, we have
Theorem 3.2 ([11]). The quantum multiplication

\[ \ast : QH^*(X) \otimes QH^*(X) \to QH^*(X) \]

is distributive over addition, skew-commutative and associative.

The \( S_k \)-invariant quantum cohomology is defined by

\[ QH^*(X)_{S_k} = H^*(X)^{S_k} \otimes \mathbb{Q}, \]

where the degree of \( q = 2N' \), \( N' \mathbb{Z} = \langle c_1(X'), H_2(X') \rangle \).

Let us introduce some theorems without proof. For details see [5].

Theorem 3.3 ([5]). The quantum cohomologies \( QH^*(X)_{S_k} \) and \( QH^*(X') \) are isomorphic.

Here the degree of \( q' = 2N', N' \mathbb{Z} = \langle c_1(X'), H_2(X') \rangle = \langle c_1(V), H_2(X)^{S_k} \rangle + \langle c_1(\Delta), H_2(X) \rangle \), \( V \) is the normal bundle of \( \Delta \) in \( X \). Consider the canonical inclusions and projections,

\[ M \xrightarrow{I_i} X \xrightarrow{\pi_i} M, \quad i = 1, \ldots, k. \]

For \( A_i \in H_2(M), a_i, b_i, c_i \in H^*(M) \), \( i = 1, \ldots, k \), we denote \( A = \sum_{i=1}^k I_i(A_i), a = \prod_{i=1}^k \pi_i^*(a_i), b = \prod_{i=1}^k \pi_i^*(b_i), c = \prod_{i=1}^k \pi_i^*(c_i). \)

Theorem 3.4 ([5]). The dimensions of the moduli space \( \mathfrak{M}(X, A) \) and the product of moduli spaces \( \prod_{i=1}^k \mathfrak{M}(M, A_i) \) are the same.

Theorem 3.5 ([5]). The moduli space \( \mathfrak{M}(X, A) \) and the product \( \prod_{i=1}^k \mathfrak{M}(M, A_i) \) are homeomorphic.

Let the virtual dimensions of moduli space be \( \dim \mathfrak{M}(X, A) = d = \sum_{i=1}^k d_i \), where \( \dim \mathfrak{M}(X, A_i) = d_i, i = 1 \cdots k \). Suppose that \( \deg(a) + \deg(b) + \deg(c) = d \), and \( \deg(a_i) + \deg(b_i) + \deg(c_i) = d_i, i = 1, \cdots, k \).

Theorem 3.6 ([5]). The Gromov-Witten invariants

\[ \Phi_A^X(a \otimes b \otimes c) = \prod_{i=1}^k \Phi_M^M(a_i \otimes b_i \otimes c_i) \]

are the same.

Let \( \Lambda = \mathbb{Q}(q) \) be the Laurent polynomials in \( q \).

Theorem 3.7 ([5]). The quantum cohomologies

\[ QH^*(X; \Lambda) \simeq \bigotimes_{i=1}^k QH^*(M; \Lambda) \]

are isomorphic as \( \Lambda \)-modules.
4 Example

(4.1) Let $X$ be the $n$-th product space of $n$ copies of the complex projective line $\mathbb{CP}^1 = S^2$. The $X$ has a natural Kähler structure induced by the $\mathbb{CP}^1$. Recall a $k$-pointed stable rational map representing a 2-dimensional integral homology class $A \in H_2(X; \mathbb{Z})$ consists of a connected reduced rational curve $(C; z_1, \cdots, z_k)$ with $k$ marked points and $u : C \to X$ is a pseudo-holomorphic map on each component of $C$ such that

(i) $u_*(\lbrack C \rbrack) = A$,

(ii) the only singularities of $C$ are ordinary double points,

(iii) $z_1, \cdots, z_k$ are distinct ordered smooth points of $C$,

(iv) If $C_i$ is a component of $C$ such that $u$ is constant on $C_i$, then $C_i$ contains at least 3 special (double or marked) points.

Every stable map $(C; z_1, \cdots, z_k; u)$ has only finitely many automorphisms, which mean that the maps are identical on marked points and image.

(4.2) Let $\mathcal{M}_{0,k}(X; A, J)$ be the moduli space of the equivalence classes of $J$-holomorphic stable maps with $k$ marked points representing the homology class $A \in H_2(X; \mathbb{Z})$. Then for a generic almost complex structure $J$ on $X$, the moduli space $\mathcal{M}_{0,k}(X; A, J)$ is a compact smooth manifold with dimension

$$d := 2c_1(X)[A] + 2\dim \mathbb{C}X - 6 + 2k.$$

There is a natural evaluation map

$$ev : \mathcal{M}_{0,k}(X; A, J) \to X^k,$$

$$ev([C; z_1, \cdots, z_k; u]) = (u(z_1), \cdots, u(z_k))$$

for each element $[C; z_1, \cdots, z_k; u] \in \mathcal{M}_{0,k}(X; A, J)$.

The Gromov-Witten invariant is defined by

$$I_n^k : H^*(X^k; \mathbb{Q}) \to \mathbb{Q}$$

$$I_n^k(\alpha) := \int_{\mathcal{M}_{0,k}(X; A, J)} ev^* (\alpha), \quad \text{for } \alpha \in H^d(X^k; \mathbb{Q}).$$

Let $X' = X/S_n$ be the $n$-th symmetric product space which is the complex projective space $\mathbb{CP}^n$ of complex dimension. Especially, if the number of marked points $k = n + 3$, then the dimension of moduli space

$$\dim \mathcal{M}_{0,k}(X'; n[S^2]) = 2(n + 1)n + 2n - 6 + 2k$$

$$= 2n(n + 3)$$

$$= 2nk$$

$$= \dim(X')^k = \dim(\mathbb{CP}^n)^k.$$
Theorem 4.1 ([17]). (1) If $k = n + 3$, then the Gromov-Witten invariant

$$I_n^k : H^{2nk}(X'; \mathbb{Q}) \longrightarrow \mathbb{Q}$$

is given by

$$I_n^k (\otimes_{i=1}^k c_1^n(\mathcal{O}(1)_i)) = \int_{\overline{M}_{0,k}(X', n[S^2])} \Pi_{i=1}^k ev^*(c_1^n(\mathcal{O}(1)_i))$$

= the degree of $ev : \overline{M}_{0,k}(X'; n[S^2]) \longrightarrow (X')^k$,

where $\mathcal{O}(1)_i$ is the canonical pullback of the line bundle $\mathcal{O}(1)$ from the $i$-th factor $X'$ of $(X')^k$.

(2) The generating series of $\mathbb{P}^1$ for the invariant $I_n$

$$\sum_{n \geq 1} I_n^{n+3}(\mathbb{P}^n)t^n = t + \sum_{n \geq 2} C_5 4^{n-2}(n-2)! t^n.$$  

Note. The results of this article will appear in papers [5] and [17].

References


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