On some class of hypersurfaces in spheres

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Abstract. This paper is a survey on our results of the construction of certain families of submanifolds, such as hypersurfaces with constant $m^{th}$ mean curvature, Willmore submanifolds, minimal Lagrangian submanifolds in complex hyperquadrics and so on.

1 Introduction

In this section, we review the construction of rotational hypersurfaces in the unit sphere, then we give some preliminary definitions and introduce some elementary properties of rotational hypersurfaces.

Let $M$ be a rotational hypersurface of $S^{n+1}(1)$, that is, invariant by the orthogonal group $O(n)$ considered as a subgroup of isometries of $S^{n+1}(1)$. Let us parametrize the profile curve $\alpha$ in $S^2(1)$ by $y_1 = y_1(s) \geq 0$, $y_{n+1} = y_{n+1}(s)$ and $y_{n+2} = y_{n+2}(s)$. We take $\varphi(t_1, \cdots, t_{n-1}) = (\varphi_1, \cdots, \varphi_n)$ as an orthogonal parametrization of the unit sphere $S^{n-1}(1)$. It follows that the rotational hypersurface (see [4])

$$x : M^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2},$$

$$(s, t_1, \cdots, t_{n-1}) \mapsto (y_1(s)\varphi_1, \cdots, y_1(s)\varphi_n, y_{n+1}(s), y_{n+2}(s)).$$

$\varphi_1 = \varphi_1(t_1, \cdots, t_{n-1})$, $\varphi_1^2 + \cdots + \varphi_n^2 = 1$

is a parametrization of a rotational hypersurface generated by a curve $y_1 = y_1(s)$, $y_{n+1} = y_{n+1}(s)$ and $y_{n+2} = y_{n+2}(s)$, where the parameter $s$ can be chosen as its arc length.

Put $f(s) = y_1(s)$, do Carmo and Dajczer proved the following

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Lemma 1.1. ([4]) Let $M^n$ be a rotational hypersurface of $S^{n+1}(1)$. Then the principal curvatures $\lambda_i$ of $M^n$ are

$$\lambda_i = \lambda = -\frac{\sqrt{1-f^2-f^2}}{f}$$

for $i = 1, \ldots, n-1$, and

$$\lambda_n = \mu = \frac{\bar{f} + f}{\sqrt{1-f^2-f^2}}$$

On the other hand, the $m$th mean curvature $H_m$ of the hypersurface $M$ can be given in such a way that

$$C^m_n H_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m},$$

where $C^m_n = \frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots 4}$, $\lambda_i$'s are the principal curvatures of $M$.

Definition 1.1. A hypersurface $M$ is called $m$-minimal if $H_m = 0$, for some integer $m$ ($1 \leq m \leq n-1$).

Example 1.1. $M_{m,n-m} = S^{n-1}(\sqrt{\frac{n-m}{n}}) \times S^1(\sqrt{\frac{m}{n}})$, $1 \leq m \leq n-1$.

By a direct calculation, we know that $M_{m,n-m}$ has two distinct constant principal curvatures

$$\lambda_1 = \cdots = \lambda_{n-1} = \sqrt{\frac{m}{n-m}}, \quad \lambda_n = -\sqrt{\frac{n-m}{m}},$$

then $H_m \equiv 0$, and $M_{m,n-m}$ is $m$-minimal.

The paper is organized as follows. In section 2, we discuss $n$-dimensional compact nontrivial embedded hypersurfaces with constant $m$th mean curvature $H_m > 0$ in a unit sphere $S^{n+1}(1)$, for $1 \leq m \leq n-1$. In section 3, we consider Willmore hypersurfaces in the unit sphere $S^{n+1}(1)$. In section 4, we investigate a class of compact minimal Lagrangian submanifolds in complex hyperquadrics by studying Gauss maps of compact rotational hypersurfaces in the unit sphere.

2 Embedded constant $m$th mean curvature hypersurfaces

It is well known that Alexandrov [1] and Montiel-Ros [13] proved that the standard round spheres are the only possible oriented compact embedded hypersurfaces
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with constant $m^{th}$ mean curvature $H_m$ in a Euclidean space $\mathbb{R}^{n+1}$, for \( m \geq 1 \). For hypersurfaces in a unit sphere $S^{n+1}(1)$, standard round spheres and Clifford hypersurfaces $S^k(a) \times S^{n-k}(\sqrt{1-a^2})$, $1 \leq k \leq n-1$ are compact embedded hypersurfaces in $S^{n+1}(1)$. Hence the following problem is interesting (also see [8], [21]):

**Problem 2.1.** Do there exist compact embedded hypersurfaces with constant $m^{th}$ mean curvature $H_m$ in $S^{n+1}(1)$ other than the standard round spheres and Clifford hypersurfaces?

When $m = 1$, namely, when the mean curvature is constant, Ripoll [18] has proved the existence of compact embedded hypersurfaces of $S^3(1)$ with constant mean curvature ($H \neq 0$, $\pm \sqrt[3]{3}$) other than the standard round spheres and the Clifford hypersurfaces. For general $n$, Perdomo [17] has proved

**Theorem 2.1** (Main Theorem of [17]). For any $n \geq 2$ and any integer $k \geq 2$, if mean curvature $H$ takes value between \( \frac{1}{(\tan \frac{k}{2})^2} \) and \( \frac{k^2 - 2}{\sqrt{k^2 - 1}} \), then there exists an $n$-dimensional compact nontrivial embedded hypersurface with constant mean curvature $H > 0$ in $S^{n+1}(1)$.

For $m = 2$, that is, when the scalar curvature is constant, Cheng, Li and Wei [21] has proved

**Theorem 2.2** ([21]). For any $n \geq 3$ and any integer $k \geq 2$, if $H_2 = \frac{R_n(n-1)}{n(n-1)}$ takes value between \( \frac{1}{(\tan \frac{k}{2})^4} \) and \( \frac{k^4 - 2}{n} \), then there exists an $n$-dimensional compact nontrivial embedded hypersurface $M$ with constant 2-th mean curvature $H_2 > 0$ (i.e. scalar curvature $R > n(n-1)$) in $S^{n+1}(1)$, where $R$ is the scalar curvature of $M$.

For $m = 4$, Cheng, Li and Wei [21] has proved

**Theorem 2.3** ([21]). For any $n \geq 5$ and any integer $k \geq 3$, if 4-th mean curvature $H_4$ takes value between \( \frac{1}{(\tan \frac{k}{2})^m} \) and \( \frac{k^4 - 2}{n(n-4)} \), then there exists an $n$-dimensional compact nontrivial embedded hypersurface with constant $H_4 > 0$ in $S^{n+1}(1)$.

For general $1 \leq m \leq n-1$, Wei and Wen [22] proved

**Theorem 2.4** ([22]). For $1 \leq m \leq n-1$ and any integer $k \geq 2$, if $m^{th}$ mean curvature $H_m$ takes value between \( \frac{1}{(\tan \frac{k}{2})^m} \) and \( \frac{k^2 - 2}{n} \left( \frac{k^2 + m - 2}{n - m} \right)^{\frac{m-2}{2}} \), then there exists at least one $n$-dimensional compact nontrivial embedded hypersurface with constant $H_m > 0$ in $S^{n+1}(1)$. 
Corollary 2.1. For \( n \geq 3 \) and any positive number \( C \), there exists at least one \( n \)-dimensional compact nontrivial embedded hypersurface with constant \( H_{n-1} = C \) in \( S^{n+1}(1) \).

Remark 2.1. The embedded hypersurfaces of above theorems are nothing but rotational hypersurfaces.

3 Willmore hypersurfaces

A hypersurface \( x : M^n \to S^{n+1}(1) \) is called a Willmore hypersurface if it is a critical hypersurface of the Willmore functional \( \int_M (S - nH^2) \frac{\partial}{\partial n} dv \), where \( H \) is the mean curvature and \( S \) is the square of the length of the second fundamental form. We define the following non-negative function on \( M \)

\[
\rho^2 = S - nH^2,
\]

which vanishes exactly at the umbilical points of \( M \). Willmore functional is the following non-negative functional (see [19])

\[
\int_M \rho^n dv = \int_M (S - nH^2) \frac{\partial}{\partial n} dv.
\]

It was shown in [19] that this functional is an invariant under conformal transformations of \( S^{n+1} \).

Let \( M \) be a hypersurface in \( S^{n+1}(1) \), it was proved by Li [9] and Wang [19] that \( M \) is a Willmore hypersurface if and only if

\[
-\rho^{n-2}(2HS - nH^3 - \sum_{i,j,k} h_{ij}h_{jk}h_{ki})
+ (n - 1) \Delta(\rho^{n-2}H) - \sum_{i,j} (\rho^{n-2})_{,ij}(nH\delta_{ij} - h_{ij}) = 0,
\]

where \( \Delta \) is the Laplacian, \( (,)_{,ij} \) is the covariant derivative relative to the induced metric.

Guo, Li and Wang [5] first gave the following example:

Example 3.1 ([5]). The tori

\[
W_{k,n-k} = S^k \left( \sqrt{\frac{n-k}{n}} \right) \times S^{n-k} \left( \sqrt{\frac{k}{n}} \right), \quad 1 \leq k \leq n - 1
\]

are Willmore hypersurfaces in \( S^{n+1}(1) \).

Equation (*) is a complicated equation to deal with except in some special case, such as isoparametric hypersurfaces such that people know few examples of Willmore hypersurfaces in \( S^{n+1}(1) \). But for rotational hypersurfaces, Wei [20] obtained a lot of examples of Willmore hypersurfaces. In [20], Wei proved
Theorem 3.1 ([20]). For \( n \geq 3 \), let \( M \) be an \( n \)-dimensional compact \((n-1)\)-minimal rotational hypersurface in \( S^{n+1}(1) \). Then \( M \) is a Willmore hypersurface.

Theorem 3.2 ([20]). For \( n \geq 3 \) and \( 1 \leq j \leq n-2 \), there are no compact \( j \)-minimal rotational Willmore hypersurfaces of \( S^{n+1}(1) \) other than round geodesic spheres.

Remark 3.1. From [17], we know that there exist many compact immersed \( k \)-minimal \((1 \leq k \leq n-1)\) rotational hypersurfaces of \( S^{n+1}(1) \).

Remark 3.2. Only the hypersurface in Theorem 3.1 conformally equivalent to the hypersurface \( S^{n-1} \left( \sqrt{\frac{1}{n}} \right) \times S^1 \left( \sqrt{\frac{n-1}{n}} \right) \) is that hypersurface itself.

4 A class of minimal Lagrangian submanifolds in complex hyperquadrics

There is an interesting link between Lagrangian geometry in the complex hyperquadrics and hypersurface geometry in the unit spheres ([2], [16], [12]). A fundamental fact is that the Gauss map of any oriented hypersurface in the unit sphere \( S^{n+1}(1) \) is always a Lagrangian immersion into the complex hyperquadric \( Q_n(\mathbb{C}) \). In [2], Castro and Urbano studied minimal Lagrangian surfaces of \( Q_2(\mathbb{C}) \), which is isometric to \( S^2 \times S^2 \), and showed that minimal Lagrangian surfaces of \( Q_2(\mathbb{C}) \) can be locally described as Gauss maps of minimal surfaces in \( S^3 \). On the other hand, Palmer gave a nice formula for the mean curvature form of the Gauss map in terms of the principal curvatures of the oriented hypersurface in the unit sphere ([16]). From this formula, it is easy to see that the Gauss map of any minimal surface in the unit 3-sphere is a minimal Lagrangian immersion in \( Q_2(\mathbb{C}) \). And the Gauss map of an oriented \emph{austere hypersurface} or an isoparametric hypersurface in \( S^{n+1}(1) \) is also a minimal Lagrangian immersion in the complex hyperquadric \( Q_n(\mathbb{C}) \). In [12], Ma and Ohnita concentrated on the relation between Lagrangian submanifolds in complex hyperquadrics and isoparametric hypersurfaces in spheres. About non-isoparametric hypersurfaces in the sphere, it is natural to ask the following problem:

Problem 4.1. Does there exist any non-isoparametric hypersurface in the sphere \( S^{n+1}(1) \) such that their Gauss maps are minimal Lagrangian immersions in the complex hyperquadric \( Q_n(\mathbb{C}) \)?

About the above problem, Li, Ma and Wei [10] gave an affirmative answer. In fact, we proved

Theorem 4.1 ([10]). There exist a lot of compact non-isoparametric hypersurfaces in the sphere \( S^{n+1}(1) \) such that their Gauss maps are minimal Lagrangian immersions in the complex hyperquadric \( Q_n(\mathbb{C}) \).
Theorem 4.2 ([10]). There exists at least one compact non-isoparametric embedded hypersurface in the sphere $S^{n+1}(1)$ for $n \geq 3$ such that its Gauss map is a minimal Lagrangian immersion in the complex hyperquadric $Q_n(\mathbb{C})$.

Our main idea is to look for compact non-isoparametric rotational hypersurfaces satisfying a weaker condition than austerity, based on Palmer’s formula. In fact, we construct compact rotational non-isoparametric hypersurfaces in the unit sphere satisfying some property, whose Gauss maps provide minimal Lagrangian immersions in the complex hyperquadric.

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References


