Moment maps and isoparametric hypersurfaces in spheres - an introduction

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(2010 Mathematics Subject Classification : 53C40, 53D20, 53C35.)

Abstract. Our expectation is that all isoparametric hypersurfaces in spheres with four distinct principal curvatures are related to the moment maps for certain Hamiltonian actions. In this expository paper, as an introduction to our project, we describe some necessary materials such as isoparametric hypersurfaces in spheres, Cartan-Münzner polynomials, moment maps, and linear isotropy representations. We also explain our results that our expectation is true for Hermitian case, that is, the case of isoparametric hypersurfaces obtained from the orbits of the linear isotropy representations of Hermitian symmetric spaces.

1 Introduction

1.1 Motivation

Isoparametric hypersurfaces in spheres have been studied by many mathematicians for a long time. In particular, a classification of such hypersurfaces with four distinct principal curvatures is one of the central problems in differential geometry and submanifold theory. Recently this classification problem has been studied very actively, for example in [4], [9], [2], [3], and some big progresses have been made. But, at the moment, a complete classification seems to be still open.
1.2 Our Approach

In [6] and [7], we have developed a moment map approach for studying isoparametric hypersurfaces in spheres with four distinct principal curvatures. It is known that such hypersurfaces are corresponding to Cartan-Münzner polynomials $F$ of degree four. Our approach is to use the moment maps $\mu$ for certain Hamiltonian actions. Our expectation is that all Cartan-Münzner polynomials of degree four can be obtained as $||\mu||^2$, squared norms of certain moment maps. Note that both $F$ and $||\mu||^2$ are polynomials of degree four. This would be a new strategy for studying isoparametric hypersurfaces, which has a possibility for applying to the classification problem.

1.3 Contents of this paper

The aim of this expository paper is to give an introduction to our project. In the next three sections, we describe some necessary materials, such as

- isoparametric hypersurfaces and Cartan-Münzner polynomials in Section 2,
- Hamiltonian actions and moment maps in Section 3,
- homogeneous hypersurfaces and linear isotropy representations in Section 4.

These are necessary materials. We also give some very easy examples in each of these sections. After that, we mention

- our results ([6] and [7]) in Section 5,
- some open problems in Section 6.

2 Isoparametric hypersurfaces in spheres

In this section, we recall isoparametric hypersurfaces in spheres and Cartan-Münzner polynomials. We also see some easy examples of Cartan-Münzner polynomials.

2.1 Isoparametric hypersurfaces and Cartan-Münzner polynomials

Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$. In this paper, we always assume that a hypersurface $N$ in $S^n$ is complete and connected.

Definition. A hypersurface $N$ in $S^n$ is said to be isoparametric if the principal curvatures are constant.

For an isoparametric hypersurface $N$ in $S^n$, we denote by $g$ the number of distinct principal curvatures, by $\lambda_1 > \lambda_2 > \cdots > \lambda_g$ the principal curvatures, and
by $m_1, \ldots, m_g$ their multiplicities. There are some restrictions on these invariants. Münzner ([11], [12]) proved that
\begin{equation}
(2.1) \quad g \in \{1, 2, 3, 4, 6\}, \quad m_i = m_{i+2},
\end{equation}
where the indices are considered mod $g$. The second condition yields that, if $g$ is odd, then all multiplicities are same. If $g$ is even, then the pair $(m_1, m_2)$ determines all multiplicities. The following theorem is one of the most important facts on isoparametric hypersurfaces.

**Theorem 2.1** ([11], [12]). Let $N$ be a hypersurface in $S^n$. Then, $N$ is isoparametric with $g$ distinct principal curvatures if and only if $N$ is a level set of $F : \mathbb{R}^{n+1} \to \mathbb{R}$, the homogeneous polynomial of degree $g$ satisfying
\begin{equation}
(2.2) \quad ||\text{grad} F(x)||^2 = g^2 ||x||^{g-2}, \quad \Delta F(x) = ((m_2 - m_1)/2)g^2 ||x||^{g-2}.
\end{equation}

The above polynomial is called a Cartan-Münzner polynomial. This plays a very fundamental role for studying isoparametric hypersurfaces. Our study is also based on this fact; in particular, our approach is to construct Cartan-Münzner polynomials from certain moment maps.

### 2.2 Examples of Cartan-Münzner polynomials

We see some easy examples. For $F : \mathbb{R}^{n+1} \to \mathbb{R}$, recall that the gradient $\text{grad}$ and the Laplacian $\Delta$ are defined by
\begin{equation}
(2.3) \quad \text{grad} F(x) = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n+1}} \right), \quad \Delta F(x) = \sum_{i=1}^{n+1} \frac{\partial^2 F}{\partial x_i^2},
\end{equation}
where $\{x_1, \ldots, x_{n+1}\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$.

**Example 1.** The following polynomials are Cartan-Münzner polynomials:
\begin{align*}
(2.4) & \quad F_1 : S^n \to \mathbb{R} : x = (x_1, \ldots, x_{n+1}) \mapsto x_{n+1}, \\
(2.5) & \quad F_2 : S^{p+q+1} \to \mathbb{R} : x \mapsto (x_1^2 + \cdots + x_{p+1}^2) - (x_{p+2}^2 + \cdots + x_{p+q+2}^2).
\end{align*}

They can be easily checked by definition. In fact, we have, for (2.4),
\begin{equation}
(2.6) \quad ||\text{grad} F_1(x)||^2 = 1, \quad \Delta F_1(x) = 0.
\end{equation}

For (2.5), we have
\begin{equation}
(2.7) \quad ||\text{grad} F_2(x)||^2 = 4||x||^2, \quad \Delta F_2(x) = 2p - 2q.
\end{equation}

One can also see that, as level sets, (2.4) gives the hyperspheres $S^{n-1} \subset S^n$, and (2.5) gives the generalized tori $S^p \times S^q \subset S^{p+q+1}$.

Other explicit descriptions of Cartan-Münzner polynomials can be found in [13] and [15].
3 Moment Maps

In this section, we recall the moment maps for Hamiltonian actions. We also see an easy example of moment maps.

3.1 Hamiltonian actions and moment maps

Let \( p \) be a \( 2n \)-dimensional smooth manifold and \( \omega \) be a two-form on \( p \). Recall that \((p^{2n}, \omega)\) is said to be \textit{symplectic} if \( d\omega = 0 \) and \( \wedge^n \omega \neq 0 \). Let \( K \) be a Lie group, \( \mathfrak{k} \) be the Lie algebra of \( K \), and \( \mathfrak{t}^* \) be the dual space of \( \mathfrak{t} \).

**Definition.** For an action of \( K \) on \((p, \omega)\), a map \( \mu : p \rightarrow \mathfrak{t}^* \) is called a \textit{moment map} if (i) \( \mu \) is \( K \)-equivariant, and (ii) \( d\mu \) is compatible with \( \omega \).

Roughly speaking, a moment map is a map compatible with both of the \( K \)-action and the symplectic structure \( \omega \). The condition (i) means that

\[
\forall g \in K, \mu \circ g = \text{Ad}^*_g \circ \mu,
\]

where \( \text{Ad}^*_g \) is the coadjoint representation of \( K \) on \( \mathfrak{t}^* \). The condition (ii) means that

\[
\forall p \in p, \forall X \in T_p p, \forall \xi \in \mathfrak{t}, (d\mu)_p(X)(\xi) = \omega_p(\tilde{\xi}_p, X).
\]

Note that \( \tilde{\xi} \in \mathfrak{X}(p) \) is the fundamental vector field of \( \xi \in \mathfrak{t} \), defined by

\[
\tilde{\xi}_p := \frac{d}{dt} (\exp(t\xi).p)|_{t=0}.
\]

Recall that an action of \( K \) on \((p, \omega)\) is said to be \textit{symplectic} if \( K \) preserves \( \omega \), and is said to be \textit{Hamiltonian} if it is symplectic and there exists a moment map.

3.2 An example of moment maps

As an example, let us determine the moment map of the standard action of \( K := \text{SO}(2) \) on \((\mathbb{R}^2, \omega)\), where \( \omega = dx \wedge dy \). Let

\[
p := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2, \quad j := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{t}, \quad \xi := \theta j \in \mathfrak{t}.
\]

First of all, under the identification \( T_p \mathbb{R}^2 = \mathbb{R}^2 \), we have

\[
\tilde{\xi}_p = \frac{d}{dt} (\exp(t\xi).p)|_{t=0} = \frac{d}{dt} \begin{pmatrix} \cos(t\theta) & -\sin(t\theta) \\ \sin(t\theta) & \cos(t\theta) \end{pmatrix} p|_{t=0} = \theta \begin{pmatrix} -p_2 \\ p_1 \end{pmatrix}.
\]

Take \( X \in T_p \mathbb{R}^2 = \mathbb{R}^2 \). Then, by definition of \( \omega \), we have

\[
\omega_p(\tilde{\xi}_p, X) = \omega_p(\theta \begin{pmatrix} -p_2 \\ p_1 \end{pmatrix}, X) = -\theta(p, X).
\]
Assume that there exists a moment map $\mu : \mathbb{R}^2 \to \mathfrak{k}^*$. Consider the differential map
\begin{equation}
(d\mu)_p : T_p \mathbb{R}^2 \to T_{\mu(p)} \mathfrak{k}^* : X \mapsto (d\mu)_p(X).
\end{equation}
Under the identification $T_{\mu(p)} \mathfrak{k}^* = \mathfrak{k}^*$, we have
\begin{equation}
(d\mu)_p(X)(j) = \omega_{\mu(p)}(j, X) = -(p, X).
\end{equation}
Let $j^*$ be the dual basis of $j$ of $\mathfrak{k}^* = \mathfrak{so}(2)^*$. Then, we have
\begin{equation}
(d\mu)_p(X) = -(p, X) j^*.
\end{equation}
By integrating this, one can see that
\begin{equation}
\mu(p) = -(1/2)||p||^2 j^* + \text{const}.
\end{equation}
This is a candidate for the moment map. By checking the $K$-equivariance, we conclude that

**Example 2.** The moment map of the standard action of $K := \text{SO}(2)$ on $(\mathbb{R}^2, \omega)$, where $\omega = dx \wedge dy$, is given by $\mu(p) = -(1/2)||p||^2 j^* + \text{const}$.

Intuitively, the moment maps give a procedure for constructing equivariant maps in terms of the symplectic structures.

## 4 Homogeneous hypersurfaces in spheres

In this section, we recall homogeneous hypersurfaces in $S^n$ and the linear isotropy representations of symmetric spaces. We also see an easy example of linear isotropy representations, and the classification list of homogeneous hypersurfaces with $g = 4$.

### 4.1 Homogeneous hypersurfaces and linear isotropy representations

A hypersurface $N$ in $S^n$ is said to be homogeneous if $N$ is an orbit of a Lie subgroup of the isometry group of $S^n$. By definition, it is easy to see that

**Proposition 4.1.** Every homogeneous hypersurface in $S^n$ is isoparametric.

To describe the classification of homogeneous hypersurfaces in $S^n$, we need the linear isotropy representations of Riemannian symmetric spaces. Let $M = G/K$ be a compact Riemannian symmetric space. Denote the origin by $o \in M$. For $g \in K$, since the isometry $g : M \to M$ satisfies $g.o = o$ by definition, one has
\begin{equation}
(dg)_o : T_o M \to T_o M,
\end{equation}
the differential map of $g$ at $o$. This gives a linear representation of $K$ on $T_o M$, which is called the linear isotropy representation of $M = G/K$. Note that the action of $K$ preserves the inner product on $T_o M$, defined by the Riemannian metric on $M$. Thus, $K$ acts on the unit sphere $S^n$ in $T_o M$. 
Example 3. The linear isotropy representation of the sphere $S^n = \text{SO}(n+1)/\text{SO}(n)$ coincides with the natural action of $\text{SO}(n)$ on $\mathbb{R}^n$.

To see examples, it is good to use the Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In fact, under the natural identification $T_o M = \mathfrak{p}$, the linear isotropy representation is equivariant to the action of $K$ on $\mathfrak{p}$ by the adjoint representation $\text{Ad}$. Consider the sphere $S^n = \text{SO}(n+1)/\text{SO}(n)$. The Cartan decomposition $\mathfrak{so}(n+1) = \mathfrak{so}(n) \oplus \mathfrak{p}$ is given by

\[\mathfrak{so}(n) \cong \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ \end{bmatrix} \mid X \in \mathfrak{so}(n) \right\}, \quad \mathfrak{p} = \left\{ \begin{bmatrix} 0 & -t & 0 \\ v & 0 & \end{bmatrix} \mid v \in \mathbb{R}^n \right\}.\]

Thus, a direct calculation of $\text{Ad}$, one can show the above example. In this case, one obtains the action of $\text{SO}(n)$ on $S^{n-1}$, which is transitive.

Similarly, if the Lie groups involved are of classical type, then one can have a matrix expression of the isotropy representation. In fact, hyperspheres (2.4) are obtained by the isotropy representation of $S^1 \times S^n$, and generalized tori (2.5) are obtained by that of $S^{p+1} \times S^{q+1}$.

4.2 Classification of homogeneous hypersurfaces in spheres

Consider the linear isotropy representation of $K$ on $\mathfrak{p}$, and the actions on the unit sphere $S^n \subset \mathfrak{p}$. If rank $(M) = 1$, then this action is transitive on $S^n$. If rank $(M) = 2$, then this action has an orbit of codimension one in $S^n$, that is, an orbit which is a homogeneous hypersurface. This is a general construction. Now we can state the classification of homogeneous hypersurfaces in $S^n$.

**Theorem 4.2** ([8]). *All homogeneous hypersurfaces in spheres can be obtained as orbits of the linear isotropy representations of symmetric spaces of rank two.*

The principal curvatures of homogeneous hypersurfaces in spheres have been completely determined by Takagi and Takahashi ([16]). We are interested in the case $g = 4$, that is, ones with four distinct principal curvatures. By [16], a homogeneous hypersurface in sphere satisfies $g = 4$ if and only if it is given by the isotropy representation of

\begin{align*}
(4.3) & & G_2(\mathbb{R}^{k+2}) = \text{SO}(2 + k)/\text{SO}(2) \times \text{O}(k)), \\
(4.4) & & G_2(\mathbb{C}^{k+2}) = \text{SU}(2 + k)/\text{SU}(2) \times \text{U}(k)), \\
(4.5) & & \text{SO}(10)/\text{U}(5), \\
(4.6) & & E_6/\text{U}(1) \times \text{Spin}(10), \\
(4.7) & & G_2(\mathbb{H}^{k+2}) = \text{Sp}(2 + k)/\text{Sp}(2) \times \text{Sp}(k), \\
(4.8) & & \text{SO}(5) = \text{SO}(5) \times \text{SO}(5)/\text{SO}(5).
\end{align*}

Note that these are precisely the symmetric spaces with the (restricted) root systems of type $C_2$ and $BC_2$. We also note that the symmetric spaces (4.3) – (4.6) are Hermitian symmetric spaces, and (4.7), (4.8) are not Hermitian.
5 Main result on Hermitian case

In this section, we describe our result ([6], [7]), that is, the Cartan-Münzner polynomial for Hermitian case (ones obtained by (4.3) – (4.6)) can be expressed as a squared norm of certain moment map.

5.1 Moment maps

Let \( M = G/K \) be a compact symmetric space, and \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the Cartan decomposition. Let \( B \) be the Killing form of \( \mathfrak{g} \), and define an inner product by \( (\cdot, \cdot) := -B|_{\mathfrak{p} \times \mathfrak{p}} \).

From now on assume that \( M = G/K \) is Hermitian. Then, it is known that \( \mathfrak{k} \) has a non-trivial center, namely \( \mathfrak{u}(1) \), and certain element \( Z \in \mathfrak{u}(1) \) defines a complex structure \( J := \text{ad}_Z \) on \( \mathfrak{p} \). Furthermore,

\[
\omega(X,Y) := \langle JX, Y \rangle
\]

is a symplectic structure on \( \mathfrak{p} \). With respect to this symplectic structure, the linear isotropy representation of \( K \) on \( (\mathfrak{p}, \omega) \) is Hamiltonian, and the moment maps can be described explicitly.

Proposition 5.1 ([14], see also [6]). The moment map of the linear isotropy representation of \( K \) on \( (\mathfrak{p}, \omega) \) is given by, for \( P \in \mathfrak{p} \),

\[
\mu(P) = (1/2)\langle [P,[P,Z]], \cdot \rangle.
\]

5.2 Main result

Now we state our result obtained in [6], [7]. Let \( M = G/K \) be a compact irreducible Hermitian symmetric space of rank two, that is, one of (4.3) – (4.6). Then, we have two objects.

- A Cartan-Münzner polynomial \( F : \mathfrak{p} \to \mathbb{R} \). Since \( M = G/K \) is of rank two, the linear isotropy representation has a hypersurface orbit in the unit sphere \( S^n \subseteq \mathfrak{p} \). This is a homogeneous hypersurface, and hence isoparametric. Thus it is a level set of a Cartan-Münzner polynomial \( F : \mathfrak{p} \to \mathbb{R} \) of degree four. Note that \( F \) is \( K \)-invariant.

- A moment map \( \mu : \mathfrak{p} \to \mathfrak{t}^* \). Since \( M = G/K \) is Hermitian, there exists a moment map \( \mu : \mathfrak{p} \to \mathfrak{t}^* \) as described above. Note that \( \mu \) is \( K \)-equivariant.

Our result states that they are related in the following sense.

Theorem 5.2 ([6], [7]). Let \( M = G/K \) be a compact irreducible Hermitian symmetric space of rank two. Then, there exists a \( K \)-invariant norm \( \| \cdot \| \) on \( \mathfrak{t}^* \) such that \( F = \| \mu \|^2 \).
5.3 Sketch of proof

We here mention a sketch of the proof. We identify $\mathfrak{t}^\ast$ with $\mathfrak{t}$ by $\langle \cdot, \cdot \rangle$. Recall that there exists $Z \in \mathfrak{t}$ which defines a complex structure on $\mathfrak{p}$. Since $\text{span}\{Z\} = \mathfrak{u}(1)$ is central in $\mathfrak{t}$, we have a direct sum decomposition $\mathfrak{t} = \mathfrak{u}(1) \oplus \mathfrak{t}'$ as Lie algebra. Now let us consider norms on $\mathfrak{t}$,

\begin{equation}
||X||_{a,b}^2 := aB(X_{\mathfrak{u}(1)}, X_{\mathfrak{u}(1)}) + bB(X_{\mathfrak{t}'}, X_{\mathfrak{t}'}) \tag{5.3}
\end{equation}

where $X_{\mathfrak{u}(1)}$ and $X_{\mathfrak{t}'}$ denote the corresponding components of $X \in \mathfrak{t}$. Note that these norms are obviously $K$-invariant, and are candidates for the one we desire. We define $K$-invariant polynomials on $\mathfrak{p}$ by

\begin{equation}
f_{a,b}(P) := ||\mu(P)||_{a,b}^2 \tag{5.4}
\end{equation}

We show that, there exists a pair $(a, b)$ such that $f_{a,b}$ is a Cartan-Münzner polynomial of degree four. For the proof, we need to calculate

\begin{equation}
||| \text{grad} f_{a,b} |||^2, \quad \Delta f_{a,b} \tag{5.5}
\end{equation}

and compare with the definition of Cartan-Münzner polynomials. In [6], the first author calculated them for classical case (4.3) – (4.5), in terms of matrix representations of classical Lie groups. The calculations in [7] are based on the structure theory (root systems) of symmetric spaces, which is applicable for both of classical case (4.3) – (4.5) and exceptional case (4.6).

6 Future Plan and Open Problems

Our expectation is that every isoparametric hypersurfaces in spheres with $g = 4$ can be obtained by some moment maps. We already have a result for Hermitian case. The natural remaining problems are the followings.

- Study homogeneous but non-Hermitian cases. They are exactly the ones obtained by the isotropy representations of $G_2(\mathbb{K}^{k+2})$ and $SO(5)$. We expect that their Cartan-Münzner polynomials can also be expressed as squared norms of certain moment maps.

- Study isoparametric hypersurfaces of OT-FKM type (which contains inhomogeneous examples). From the construction they are related to representations of Clifford algebras. Hence it is natural to expect that they are related to spin actions.

- Study the group of automorphisms of Cartan-Münzner polynomials. For studying above problems, we need to find a correct group acting in a Hamiltonian way. The automorphism groups would help to find the correct groups.

Note that Miyaoka ([10]) recently proved that all Cartan-Münzner polynomials of degree four are related to some moment maps, but her formulation is different from ours. We expect that Cartan-Münzner polynomials of degree four are related to moment maps in several ways.
References


