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Real hypersurfaces in a complex projective space with pseudo-$\mathbb{D}$-parallel structure Jacobi operator

Hyunjin Lee
Department of Mathematics, Kyungpook National University,
Taegu, 702-701, KOREA
e-mail : lhjibis@hanmail.net

Juan de Dios Pérez
Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada,
18071-Granada, Spain
e-mail : jdperez@ugr.es

Young Jin Suh
Department of Mathematics, Kyungpook National University,
Taegu, 702-701, KOREA
e-mail : yjsuh@knu.ac.kr

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Abstract. We introduce the new notion of pseudo-$\mathbb{D}$-parallelness of the structure Jacobi operator for real hypersurfaces M in complex projective space as real hypersurfaces satisfying a condition about the covariant derivative of the structure Jacobi operator in any direction of the maximal holomorphic distribution. This condition generalizes parallelness of the structure Jacobi operator. We classify this type of real hypersurfaces.

1 Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a complex projective space endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a connected real hypersurfaces of $\mathbb{C}P^m$ without boundary. Let J denote the complex structure of $\mathbb{C}P^m$ and N a locally defined unit normal vector field on M. Then $-JN = \xi$ is a tangent

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vector field to M called the *structure vector field* on M. We also call $\mathbb{D}$ the *maximal holomorphic distribution* on M, that is, the distribution on M given by all vectors orthogonal to ξ at any point of M. The structure vector field ξ of M in $\mathbb{C}P^m$ is invariant by the shape operator A, M is said to be a *Hopf hypersurface*.

The study of real hypersurfaces in nonflat complex space forms is a classical topic in differential geometry. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [14], [15], [16], and is given by the following list :

(1) A_1 : Geodesic hyperspheres.

(2) A_2 : Tubes over totally geodesic complex projective spaces.

(3) B : Tubes over complex quadrics and $\mathbb{R}P^m$.

(4) C : Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n+1=m$ and $m \geq 5$.

(5) D : Tubes over the Plücker embedding of the complex Grassmannian manifold $G(2,5)$ in $\mathbb{C}P^9$.

(6) E : Tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$.

Other examples of real hypersurfaces are ruled real ones, that were introduced by Kimura, [5] : Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique complex projective hyperplane cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a *ruled real hypersurface*. It will be an embedded hypersurface locally although globally it will in general have self-intersections and singularities. Equivalently a ruled real hypersurface is such that $\mathbb{D}$ is integrable or, equivalently, $g(A\mathbb{D},\mathbb{D})=0$, where A denotes the shape operator of the immersion, see [5]. For further examples of ruled real hypersurfaces see [6].

Except these real hypersurfaces there are very few examples of real hypersurfaces in $\mathbb{C}P^m$. So we present a result about non-existence of a certain family of real hypersurfaces in complex projective space.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M},\tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called *Jacobi operator*. That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$, and X is any tangent vector field to $\tilde{M}$, the Jacobi operator (with respect to X) at $p \in M$, $\tilde{R}_X \in \mathrm{End}(T_pM)$, is defined as $(\tilde{R}_X Y)(p) = (\tilde{R}(Y,X)X)(p)$ for all $Y \in T_p\tilde{M}$, being a self-adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X. In particular, we will call the Jacobi operator on M with respect to ξ the *structure Jacobi operator* on M.

Among the results related to the structure Jacobi operator R_ξ we mention the following ones. In [3], the authors classify real hypersurfaces in $\mathbb{C}P^m$ whose structure Jacobi operator commutes both with the shape operator and with the restriction of the complex structure to M.

Theorem 1.1. ([3]) *Let M be a real hypersurfce of $\mathbb{C}P^m$. Suppose M satisfies $R_\xi A = AR_\xi$ and the same time $R_\xi \phi = \phi R_\xi$. Then ξ is a principal curvature vector field on M. Further, if $A\xi \neq 0$, then M is locally congruent to one of the following spaces:*

(1) A geodesic hypersphere of radius $0 < r < \pi/2$, $r \neq \pi/4$.

(2) A tube of radius r over a totally geodesic $\mathbb{C}P^k$, $0 < r < \pi/2$, $r \neq \pi/4$.

Also, Cho and Ki classify under certain additional conditions, real hypersurfaces of $\mathbb{C}P^m$ whose structure Jacobi operator is parallel. Here, $R'_\xi = (\nabla_\xi R)(\,\cdot\,, \xi, \xi)$.

Theorem 1.2. ([2]) *Let M be a real hypersurface of $\mathbb{C}P^m$. Suppose that ξ is a geodesic vector field on M and M satisfies $R'_\xi = 0$. Then ξ is a principal curvature vector field on M. Further assume that $\alpha = 2\cot(2r)$ and the rank of the focal map ψ_r is constant, then M is locally congruent to one of the homogeneous real hypersurfaces of type A_1, A_2 or a non-homogeneous tube of radius $\pi/4$ over $\psi_{\pi/4}(M)$ with nonzero principal curvatures $\neq 1, -1$.*

We will say that R_ξ is *parallel* if $\nabla_X R_\xi = 0$ for any X tangent to M. Using this notion Ortega, the second author and Santos gave the following theorem.

Theorem 1.3. ([11]) *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, whose structure Jacobi operator is parallel.*

Thereafter, many researchers have studied the generalized parallel structure Jacobi operator. For instance, in [12], the authors defined the $\mathbb{D}$-*parallel* structure Jacobi operator of real hypersurface in $\mathbb{C}P^m$, that is, $\nabla_X R_\xi = 0$ for any tangent vector $X \in \mathbb{D}$. And they gave the following theorem.

Theorem 1.4. ([12]) *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, with $\mathbb{D}$-parallel structure Jacobi operator.*

Moreover, in [7], the authors have defined new notion as like. The structure Jacobi operator R_ξ of M in $\mathbb{C}P^m$ is said to be *pseudo-parallel* if it satisfies

$$(\nabla_X R_\xi)Y = c\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}$$

for any X, Y tangent to M, where c is a nonzero constant. Using this notion, they classify real hypersurfaces of $\mathbb{C}P^m$.

Theorem 1.5. ([7]) *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 3$ with pseudo-parallel structure Jacobi operator. Then $c < 0$, $c \neq -1$ and M is locally congruent to a geodesic hypersphere of radius $0 < r < \pi/4$ or $\pi/4 < r < \pi/2$ such that $\cot^2(2r) = -\frac{(c+1)^2}{4c}$.*

In this talk, we introduce the notion of pseudo-$\mathbb{D}$-parallelness of the structure Jacobi operator for real hypersurfaces in $\mathbb{C}P^m$. It generalizes $\mathbb{D}$-parallelness of the structure Jacobi operator. The structure Jacobi operator of a real hypersurface of $\mathbb{C}P^m$ is *pseudo-$\mathbb{D}$-parallel* if it satisfies

$$(\nabla_X R_\xi)Y = c\{\,\eta(Y)\phi AX + g(\phi AX, Y)\xi\,\} \tag{1.1}$$

where c is a nonzero constant, $X \in \mathbb{D}$ and $Y \in TM$. Using this notion we obtain the following Main Theorem:

Main Theorem. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, with pseudo-$\mathbb{D}$-parallel structure Jacobi operator. If $c \neq -1$ then $c < 0$. Moreover, M is locally congruent to a geodesic hypersphere of radius r such that $\cot^2(r) = -c$.*

2 Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M. That is, we have

$$\phi^2 X = -X + \eta(X)\xi\,, \quad \eta(\xi) = 1\,, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0\,, \quad \eta(X) = g(X, \xi)\,. \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(\phi Y,X)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY\,, \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M.

In the sequel we need the following results:

Lemma 2.1. *[10] Let M be a connected real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. If M satisfies $\phi A = A\phi$ then M is locally congruent to one of the following spaces:*

(1) A_1: a geodesic hypersphere, that is, a tube of radius r over a hyperplane CP^{m-1}, where $0 < r < \frac{\pi}{2}$,

(2) A_2: a tube of radius r over a totally geodesic CP^k $(1 \leq k \leq m-2)$, where $0 < r < \frac{\pi}{2}$.

Lemma 2.2. *[10] If ξ is a principal vector with corresponding principal curvature α and $X \in \mathbb{D}$ is principal with principal curvature λ, then ϕX is principal with principal curvature $(\alpha\lambda + 2)/(2\lambda - \alpha)$.*

Lemma 2.3. *[13] Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, satisfying $t(p) \leq 2$ for any point p in M. Then M is a ruled real hypersurface. Here, $t(p)$ is the type number at p.*

3 Main Theorem

In this section, we will prove our Main Theorem in the introduction. In order to do this, we have the following Propositions.

Proposition 3.1. *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 4$, whose shape operator is given by*

$$A\xi = \alpha\xi + \beta U\,, \quad AU = \beta\xi\,, \quad A\phi U = 0$$

and there exist two nonnull holomorphic distributions $\mathbb{D}_0$ *and* $\mathbb{D}_c$ *such that* $\mathbb{D}_0 \oplus \mathbb{D}_c = \mathrm{Span}\{\xi\,, \mathrm{U}\,, \phi\mathrm{U}\,\}^{\perp}$,

$$AZ = 0, \quad \textit{for any } Z \in \mathbb{D}_0$$
$$AW = -\tfrac{c+1}{\alpha} W, \quad \textit{for any } W \in \mathbb{D}_c\,,$$

where U *is a unit vector field in* $\mathbb{D}$*,* α *and* β *are nonvanishing smooth functions defined on* M*,* $(\phi U)\beta = 0$ *and* c *is a constant* $c \neq 0\,, -1$.

Proposition 3.2. *Let* M *be a ruled real hypersurfaces in* $\mathbb{C}P^m$*,* $m \geq 2$*. Then* M *does not satisfy the condition*

$$(\nabla_X R_\xi)Y = c\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}\,,$$

for any $X \in \mathbb{D}$*,* $Y \in TM$*, where* c *is a nonzero constant.*

Proposition 3.3. *There exist no real hypersurfaces* M *in* $\mathbb{C}P^m$*,* $m \geq 3$*, whose shape operator is given by*

$$A\xi = (c+1)\xi + \beta U, \quad AU = \beta\xi + \big(\tfrac{\beta^2}{c+1} - 1\big)U,$$
$$A\phi U = -\phi U, \quad AZ = -Z$$

for any tangent vector Z *orthogonal to Span*$\{\xi, U, \phi U\}$*, where* U *is a unit vector field in* $\mathbb{D}$*,* β *is a nonvanishing smooth function defined on* M *and* c *is a constant* $c \neq 0, -1$.

By Proposition 3.1, 3.2 and 3.3, we assert the following

Proposition 3.4. *Let* M *be a real hypersurface in* $\mathbb{C}P^m$ *with pseudo-*$\mathbb{D}$*-parallel structure Jacobi operator. Then* M *is a Hopf. Moreover, the structure tensor* ϕ *of* M *commutes with the shape operator* A *of* M*.*

By Proposition 3.4 and Lemma 2.1, we know that any real hypersurface in $\mathbb{C}P^m$ with pseudo-$\mathbb{D}$-parallel structure Jacobi operator are locally congruent to a real hypersurface of type either A_1 or A_2. It can be easily checked that any real hypersurfaces of type A_1 satisfies the pseudo-$\mathbb{D}$-parallel structure Jacobi operator but in case of one of type A_2 does not like that. From this, we complete the proof of Main Theorem in the introduction.

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