Complete hypersurfaces in a unit sphere

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Abstract. This paper is a survey on results of eigenvalues of Jacobi operator $J_m, J_s$ and index of hypersurfaces in a unit sphere and on results of some special hypersurfaces in a unit sphere.

1 Hypersurfaces with constant mean curvature

Let $\phi : M^n \to S^{n+1}(1)$ be an $n$-dimensional compact orientable hypersurface of a unit sphere $S^{n+1}(1)$. Every smooth function $u \in C^\infty(M)$ induces a normal variation $\phi_t$ of the immersion $\phi$ with variational normal field $uN$ and the first variation of the volume functional $V(t)$ given by

$$\frac{d}{dt}|_{t=0} V(t) = -n \int_M uH dv.$$ 

As a consequence, minimal hypersurfaces ($H = 0$) are characterized as critical points of the volume functional, constant mean curvature hypersurfaces can be viewed as critical points of the volume functional restricted to variations that preserve a certain volume function, that is, $\int_M udv = 0$. For such critical points, the second variation of the volume functional is given by

$$\frac{d^2}{dt^2}|_{t=0} V(t) = \int_M uJ_m(u) dv,$$

where $J_m = -\triangle - S - n$, $\triangle$ is the Laplace operator and $S$ denotes the squared norm of the second fundamental form of $M$. $J_m$ is called a Jacobi operator or a stability operator. Its spectral behavior is directly related to the instability of such hypersurfaces.

The first eigenvalue of the Jacobi operator $J_m$ of minimal hypersurfaces in $S^{n+1}(1)$ was studied by Simons [15], Wu [18] and Perdomo [13]. A characterization of Clifford torus is given by the first eigenvalue of the Jacobi operator $J_m$, that is, they proved

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Theorem 1.1. If $M$ is an $n$-dimensional compact orientable minimal hypersurface in $S^{n+1}(1)$, then the first eigenvalue $\lambda^J_1$ of the Jacobi operator $J_m$ satisfies

1. $\lambda^J_1 = -n$ and $M$ is totally geodesic.
2. $\lambda^J_1 \leq -2n$ and $\lambda^J_1 = -2n$ if and only if $M$ is a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$, $1 \leq m \leq n - 1$.

Recently, Alías, Barros and Brasil [1] considered compact hypersurfaces with constant mean curvature $H$ in $S^{n+1}(1)$ and obtained

Theorem 1.2 (Alias, Barros and Brasil, [1]). If $M$ is an $n$-dimensional compact orientable hypersurface with constant mean curvature in $S^{n+1}(1)$, then the first eigenvalue $\lambda^J_1$ of the Jacobi operator $J_m$ satisfies

1. $\lambda^J_1 = -n(1 + H^2)$ and $M$ is totally umbilical.
2. $\lambda^J_1 \leq -2n(1 + H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} |H| \max \sqrt{S-nH^2}$ and $\lambda^J_1 = -2n(1 + H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} |H| \max \sqrt{S-nH^2}$ if and only if $M$ is a Clifford hypersurface $S^{n-1}(\sqrt{1-c^2}) \times S^1(\sqrt{1-c^2})$.

The above theorems are about the first eigenvalue of $J_m$. Now we introduce some results about the second eigenvalue of $J_m$. For a compact hypersurface $M$ with constant mean curvature in $S^{n+1}(1)$, Soufi and Ilias [10] proved

Theorem 1.3 (Soufi and Ilias, [10]). The second eigenvalue $\lambda^J_2$ of $J_m$ satisfies

$$\lambda^J_2 \leq 0$$

and $\lambda^J_2 = 0$ if and only if $M$ is totally umbilical.

The Jacobi operator $J_m = -\Delta - S - n$ induces the quadratic form $Q_m(u, u) = \int_M u J_m(u)dv$ acting on $C^\infty(M)$ (the space of smooth functions on $M$). In the case of constant mean curvature hypersurfaces, one has

Definition 1.1. The index of a constant mean curvature hypersurface $M$, $\text{Ind}(M)$ is defined as the maximum dimension of subspace of $C^\infty(M)$ on which $Q_m$ is negative definite.

Equivalently, $\text{Ind}(M)$ is the number of negative eigenvalues of $J_m$. From Definition 1.1, we have the following known results

Corollary 1.1 (Simons). If $M$ is a compact minimal hypersurface in $S^{n+1}(1)$, then

$$\text{Ind}(M) \geq 1$$
and $\text{Ind}(M) = 1$ if and only if $M$ is totally geodesic.

**Corollary 1.2** (Soufi and Ilias). If $M$ is a compact hypersurface with constant mean curvature, then

$$\text{Ind}(M) \geq 1$$

and $\text{Ind}(M) = 1$ if and only if $M$ is totally umbilical. If $M$ is a compact non-totally umbilical hypersurface with constant mean curvature, then

$$\text{Ind}(M) \geq 2.$$

**Theorem 1.4** (Perdomo). If $M$ is a compact non-totally geodesic minimal hypersurface in $S^{n+1}(1)$, then

$$\text{Ind}(M) \geq n + 3.$$

Since the minimal Clifford hypersurfaces satisfy $\text{ind}(M) = n + 3$, we propose the following

**Conjecture 1.1.** The only minimal hypersurfaces in $S^{n+1}(1)$ with $\text{Ind}(M) = n + 3$ are the minimal Clifford hypersurfaces.

**Conjecture 1.2.** If $M$ is a compact non-totally umbilical hypersurface with constant mean curvature, then $\text{Ind}(M) \geq n + 3$.

## 2 Hypersurfaces with constant scalar curvature

Let $\phi : M^n \to S^{n+1}(1)$ be an $n$-dimensional compact orientable hypersurface of a unit sphere $S^{n+1}(1)$. We consider the functional

$$F(t) = \int_M (nH_t)dv_t$$

and compute the first variation of the functional $F(t)$. For volume preserving variations ($\int_M u_t dv_t = 0$), the first derivative of $F(t)$ at $t = 0$ is given by

$$\frac{d}{dt}igg|_{t=0} F(t) = \int_M u(-R + k)dv,$$

where $k$ stands for a constant, $u$ is the normal projection of the variation vector. Let $\phi : M^n \to S^{n+1}(1)$ be an $n$-dimensional compact orientable hypersurface with
constant scalar curvature $R$. For volume preserving variations, the second variation of the functional $F(t)$ at $t = 0$ is given by

$$\frac{d^2}{dt^2} F(t) = 2 \int_M u J_s(u) dv,$$

where $J_s = -\Box - \{ n(n-1)H + nHS - f_3 \}$, $f_3 = \sum_{i=1}^{n} k_i^3$ and $k_i$’s are the principal curvatures of $M$, the differential operator $\Box$ defined by

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}) f_{ij},$$

where $h_{ij}$ denotes components of the second fundamental form of $M$. The differential operator $\Box$ was introduced and used by S. Y. Cheng and S. T. Yau in [9] to study compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. They proved that if $M$ is an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, and if the sectional curvature of $M$ is non-negative, then $M$ is a totally umbilical hypersurface $S^n(c)$ or a Clifford hypersurface $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, where $S^k(c)$ denotes a sphere of radius $c$. We should notice that the differential operator $\Box$ is self-adjoint. Cheng [5] and Li [11] also studied complete hypersurfaces with constant scalar curvature.

It is not difficult to prove that if $R > n(n-1)$, then $J_s$ is elliptic. The spectral behavior of $J_s$ is also directly related to the instability of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$.

Recently, Cheng [6] studied the first eigenvalue of the Jacobi operator $J_s$ of hypersurfaces with constant scalar curvature $n(n-1)r$, $r > 1$ in $S^{n+1}(1)$. He proved

**Theorem 2.1** (Cheng, [6]). *Let $M$ be an $n$-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then the Jacobi operator $J_s$ is elliptic and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator $J_s$ satisfies

$$\lambda_1^{J_s} \leq -n(n-1)r\sqrt{r-1}$$

and the equality holds if and only if $M$ is totally umbilical and non-totally geodesic.*

About the second eigenvalue of $J_s$, Cheng and Wei [8] proved

**Theorem 2.2** (Cheng and Wei, [8]). *For $n \geq 5$, let $M^n$ be an $n$-dimensional compact orientable hypersurface with scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then the mean curvature $H$ does not vanish on $M$ and the second eigenvalue $\lambda_2^{J_s}$ of $J_s = -\Box - \{ n(n-1)H + nHS - f_3 \}$ satisfies

$$\lambda_2^{J_s} \leq \frac{1}{V(M)} \int_M \frac{n(S - nH^2)(1-r)}{2|H|} dv \leq 0,$$*
and $\chi_2^f = 0$ if and only if $M$ is totally umbilical and non-totally geodesic. Here $V(M)$ denotes the volume of $M$.

The Jacobi operator $J_s = -\Box - \{n(n-1)H + nHS - f_3\}$ induces the quadratic form $Q_s(u,u) = \int_M uJ_s(u)dv$ acting on $C^\infty(M)$ and $C^\infty_T = \{u \in C^\infty(M) : \int_M udv = 0\}$. In the case of hypersurfaces with constant scalar curvature, the index $\text{Ind}(M)$ and the weak stability index $\text{Ind}_T(M)$ are defined by

**Definition 2.1.** Let $M$ be an $n$-dimensional compact orientable hypersurface with scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. The index of $M$, $\text{Ind}(M)$, is the index of the quadratic form $Q_s$, where $Q_s(u,u) = \int_M uJ_s(u)dv$ and $u \in C^\infty(M)$; the weak stability index of $M$, $\text{Ind}_T(M)$, is the maximal dimension of any subspace $V$ of $C^\infty_T(M)$ on which the quadratic form $Q_s$ is negative definite, where $C^\infty_T(M) = \{u \in C^\infty(M) : \int_M udv = 0\}$ and $Q_s(u,u) = \int_M uJ_s(u)dv$.

From Definition 2.1, we know that $\text{Ind}(M)$ is equal to the number of negative eigenvalues of the Jacobi operator $J_s$. Moreover, we have from Theorem 2.2 that

**Corollary 2.1** (Cheng-Wei, [8]). For $n \geq 5$, let $M^n$ be an $n$-dimensional compact orientable hypersurface with scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then

$$\text{Ind}(M) \geq 1$$

and $\text{Ind}(M) = 1$ if and only if $M$ is totally umbilical and non-totally geodesic.

We next study compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$ and estimate the weak stability index. Specifically, we get

**Theorem 2.3** (Cheng, Li and Wei, [7]). Let $M$ be a compact hypersurface in $S^{n+1}(1)$ with constant scalar curvature $R = n(n-1)r > n(n-1)$. If $H_1$ and $H_3$ are constants. Then, either

1. the weak stability index $\text{Ind}_T(M)$ of $M$ is equal to zero. In this case, $M$ is totally umbilical, or

2. the weak stability index $\text{Ind}_T(M)$ of $M$ is greater than or equal to $n+2$, and the equality holds if and only if $M$ is $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, where $c$ satisfies

$$\frac{nm + \sqrt{m((2-n)m + (n-1)(n+2))}}{(n-1)(n+2)} \leq c^2 \leq \frac{(nm + n - 2) + \sqrt{(n-m)(3n - 2m + nm - 2)}}{(n-1)(n+2)}.$$
Theorem 2.4 (Cheng, Li and Wei, [7]). Let $M$ be a compact hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ ($r > 1$ and
\[
r \neq 2\frac{(2k+m)n^2 - (2k^2 + 4k + 2km + m)n + 2k(m+k+1)}{n(2k+m)(2(n-1) - (2k+m))},
\]
for $0 \leq m \leq n - 2, 1 \leq k \leq n - 1 - m$). If $H_1$ and $H_3$ are constants, then either
1. $M$ is totally umbilical, or
2. $M$ is a Clifford hypersurface, or
3. the weak stability index of $M$ is greater than or equal to $2n + 4$.

3 Hypersurfaces with the proportional relationship between $l_v$ and $f_v$

Let $\phi : M \to S^{n+1}(1) \subset R^{n+2}$ be an isometric immersion of an $n$-dimensional complete Riemannian manifold. For any point $x \in M$, we will denote by $T_xM$ and $N_xM$ the tangent space and normal space of $M$ at $x$, respectively. Let us denote by $\nu : M \to S^{n+1}(1)$, a normal vector field along $M$. The shape operator $A_x : T_xM \to T_xM$ is given by $A_x(v) = -d\nu_x(v) = -\beta(0)$, where $\beta(t) = \nu(\alpha(t))$ and $\alpha(t)$ is any smooth curve in $M$ such that $\alpha(0) = x$ and $\alpha'(0) = v$. One knows that the linear map $A_x$ is symmetric and its eigenvalues $k_1(x), \ldots, k_n(x)$ are called principal curvatures of $M$ at $x$.

We consider elementary symmetric functions $S_m(x)$ of the principal curvatures of $M$ defined by
\[
\det(tI - A_x) = \sum_{m=0}^{n} (-1)^m S_m(x) t^{n-m}.
\]

$$H_m(x) = \frac{S_m(x)}{C_m^n}$$

is called the $m$-th mean curvature of $M$, namely,
\[
H_m(x) = \frac{1}{C_m^n} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} k_{i_1}(x) \cdots k_{i_m}(x), \quad C_m^n = \frac{n!}{m!(n-m)!}.
\]

Hence, the mean curvature $H(x)$ of $M$ satisfies $H(x) = \frac{k_1(x) + \ldots + k_n(x)}{n} = H_1(x)$, the scalar curvature $R(x) = n(n-1)r(x) = n(n-1) + 2S_2(x) = n(n-1) + n(n-1)H_2(x)$ and Gauss-Kronecker curvature $K(x)$ of $M$ is $K(x) = k_1(x) \cdots k_n(x) = H_n(x) = S_n(x)$.

On the other hand, let $v \in R^{n+2}$ be an arbitrary vector, we then define functions $l_v : M \to R$ and $f_v : M \to R$ by
\[
l_v(x) = \langle \phi(x), v \rangle, \quad f_v(x) = \langle \nu(x), v \rangle,
\]
where $\phi : M \to S^{n+1}(1)$ is an isometric immersion, $\nu : M \to S^{n+1}(1)$ is a normal vector field along $M$.

By investigating the proportional relationship between $l_v$ and $f_v$, we obtain
**Theorem 3.1** (Cheng, Li and Wei, [7]). Let $\phi : M \to S^{n+1}(1)$ be an isometric immersion of an $n$-dimensional complete Riemannian manifold $M$ with constant ratio of the Gauss-Kronecker curvature and the $(n-1)$-th mean curvature, that is, $S_n(x) = cs_{n-1}(x)$, where $c$ is a constant. If $l_v = \lambda f_v$, for some non-zero vector $v$ and some real number $\lambda$, then $\phi(M)$ is either a totally umbilical sphere or a Clifford hypersurface.

**Theorem 3.2** (Cheng, Li and Wei, [7]). Let $\phi : M \to S^{n+1}(1)$ be an isometric immersion with constant Gauss-Kronecker curvature $c$ ($c \neq \pm 1$) of an $n$-dimensional complete Riemannian manifold. If $l_v = \lambda f_v$, for some non-zero vector $v$ and some real number $\lambda$, then $\phi(M)$ is either a totally umbilical sphere or a Clifford hypersurface.

**Theorem 3.3** (Cheng, Li and Wei, [7]). Let $\phi : M \to S^{n+1}(1)$ be an isometric immersion of an $n$-dimensional complete Riemannian manifold $M$ with constant scalar curvature $n(n-1)r$, where $r$ satisfies

$$r \neq 2\frac{(2k+m)n^2 - (2k^2 + 4k + 2km + m)n + 2k(m + k + 1)}{n(2k+m)(2(n-1) - (2k + m))},$$

for $0 \leq m \leq n - 2$ and $1 \leq k \leq n - 1 - m$. If $l_v = \lambda f_v$, for some non-zero vector $v$ and some real number $\lambda$, then $\phi(M)$ is either a totally umbilical sphere or a Clifford hypersurface.

4 Examples

**Example 1.** $S^m(e) \times S^{n-m}(\sqrt{1 - e^2})$, $1 \leq m \leq n - 1$.

We will compute the weak stability index of the Clifford hypersurface $S^m(e) \times S^{n-m}(\sqrt{1 - e^2})$, $1 \leq m \leq n - 1$. Since $S^m(e) \times S^{n-m}(\sqrt{1 - e^2})$, $1 \leq m \leq n - 1$, is an isoparametric hypersurface in $S^{n+1}(1)$, its principal curvatures are given by

$$k_1 = \cdots = k_m = \frac{\sqrt{1 - e^2}}{c}, \quad k_{m+1} = \cdots = k_n = \frac{e}{\sqrt{1 - e^2}}.$$

Hence, its mean curvature $H$, the squared norm $S = |A|^2$ of the second fundamental form and $f_3$ are given by

$$H = \frac{ne^2 - m}{nc\sqrt{1 - e^2}}, \quad S = |A|^2 = \frac{ne^4 - 2ne^2 + m}{c^2(1 - e^2)}, \quad f_3 = \frac{-m(1 - e^2)^{3/2}}{c^3} + \frac{(n-m)e^3}{(1 - e^2)^{3/2}}.$$

From the Gauss equation, we have

$$R - n(n-1) = n(n-1)(r-1) = n^2H^2 - S = \frac{n(n-1)e^4 + 2m(1-n)e^2 + m(m-1)}{c^2(1 - e^2)},$$
where $R$ is the scalar curvature. Thus, we infer that $r > 1$ if and only if
\[ c^2 > \frac{m(n-1) + \sqrt{m(n-1)(n-m)}}{n(n-1)} \]
or
\[ c^2 < \frac{m(n-1) - \sqrt{m(n-1)(n-m)}}{n(n-1)}. \]

If the scalar curvature $R = n(n-1)r > n(n-1)$, we know from the Gauss equation $n^2H^2 = S + n(n-1)(r-1)$ that the mean curvature $H$ does not vanish. Without loss of generality, assume the mean curvature $H > 0$, that is,
\[ c^2 > \frac{m}{n}. \]

Then we have that
\[ 1 > c^2 > \frac{m(n-1) + \sqrt{m(n-1)(n-m)}}{n(n-1)}. \]

Therefore, we have
\[ n(n-1)H + nHS - f_3 = \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}, \]
and the Jacobi operator $J_s = -\Delta - \{n(n-1)H + nHS - f_3\}$ becomes
\[ J_s = -\Delta - \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}. \]

Thus, eigenvalues of $J_s$ are given by
\[ \lambda_{1s}^J = \lambda_2^D + \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}}, \]
where $\lambda_2^D$ denote the eigenvalues of the differential operator $\Box$.

Since the differential operator $\Box$ is self-adjoint and the Clifford hypersurface is closed, we have $\lambda_1^D = 0$ and its corresponding eigenfunctions are non-zero constant functions. Hence,
\[ \lambda_{1s}^J = \frac{(n-2m)(1-n)c^4 + 2m(1-m)c^2 + m(m-1)}{c^3(1-c^2)^{3/2}} \]
with multiplicity 1 and its corresponding eigenfunctions are non-zero constant functions. Hence, $\lambda_{1s}^J$ does not contribute to $\text{Ind}_T(M)$. Since the other eigenfunctions $u$ of $J_s$ other than the first eigenfunctions are orthogonal to the constant functions, namely, $\int_M u = 0$. We know that the other eigenvalues of $J_s$ contribute to $\text{Ind}_T(M)$ if they are negative.

Let $\Delta_1$ and $\Delta_2$ denote the Laplacians on $S^m(c)$ and on $S^{n-m}(\sqrt{1-c^2})$, respectively. We can derive
\[ \Box f = (nH\delta_{ij} - h_{ij})f_{,ij} = (nH - k_1)\Delta_1 f + (nH - k_2)\Delta_2 f. \]
Hence, eigenvalues $\lambda_1^\Delta$ are given by

$$\lambda_1^\Delta = (nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_1^{\Delta_2},$$

the multiplicity of $\lambda_1^\Delta$ is the sum of the products $m_{\lambda_1^{\Delta_1}} m_{\lambda_1^{\Delta_2}}$ for all possible values of $\lambda_1^{\Delta_1}$ and $\lambda_1^{\Delta_2}$ satisfying $\lambda_1^\Delta = (nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_1^{\Delta_2}$, where $m_{\lambda_1^{\Delta_i}}$ denote the multiplicity of $\lambda_1^{\Delta_i}$.

We recall that eigenvalues of the Laplacian $\Delta_1$ on $S^m(c)$ are given by

$$\lambda_1^{\Delta_1} = \frac{(i - 1)(m + i - 2)}{c^2}, \quad i = 1, 2, 3, \ldots,$$

with multiplicities

$$m_{\lambda_1^{\Delta_1}} = 1, \quad m_{\lambda_1^{\Delta_2}} = m + 1,$$

and

$$m_{\lambda_1^{\Delta_1}} = \frac{c^{i-1}}{c_{m+i-1}} - \frac{c^{i-3}}{c_{m+i-3}}, \quad i = 3, 4, \ldots,$$

and eigenvalues of the Laplacian $\Delta_2$ on $S^{n-m}(\sqrt{1-c^2})$ are given by

$$\lambda_1^{\Delta_2} = \frac{(j - 1)(n - m + j - 2)}{1 - c^2}, \quad j = 1, 2, 3, \ldots,$$

with multiplicities

$$m_{\lambda_1^{\Delta_2}} = 1, \quad m_{\lambda_1^{\Delta_2}} = n - m + 1,$$

and

$$m_{\lambda_1^{\Delta_2}} = \frac{c^{j-1}}{c_{n-m+j-1}} - \frac{c^{j-3}}{c_{n-m+j-3}}, \quad j = 3, 4, \ldots.$$

Therefore, we infer

$$\lambda_1^J = \lambda_1^\Delta + \frac{(n - 2m)(1 - n)c^4 + 2m(1 - m)c^2 + m(m - 1)}{c^3(1 - c^2)^{3/2}}$$

$$= (nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_1^{\Delta_2}$$

$$+ \frac{(n - 2m)(1 - n)c^4 + 2m(1 - m)c^2 + m(m - 1)}{c^3(1 - c^2)^{3/2}}$$

$$= \left(\frac{nc^2 - m}{c\sqrt{1 - c^2}} + \frac{\sqrt{1 - c^2}}{c} (i - 1)(m + i - 2)\right)$$

$$+ \left(\frac{nc^2 - m}{c\sqrt{1 - c^2}} - \frac{c}{\sqrt{1 - c^2}} \right) \frac{(j - 1)(n - m + j - 2)}{1 - c^2}$$

$$+ \frac{(n - 2m)(1 - n)c^4 + 2m(1 - m)c^2 + m(m - 1)}{c^3(1 - c^2)^{3/2}}.$$ 

It is not difficult to prove

$$(nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_1^{\Delta_2} + \frac{(n - 2m)(1 - n)c^4 + 2m(1 - m)c^2 + m(m - 1)}{c^3(1 - c^2)^{3/2}} = 0.$$
Thus, in order to calculate the weak stability index, it suffices to estimate when 
\[(nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} < (nH - k_1)\lambda_2^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2},\]
for \(i = 1, j > 1\) and \(i > 1, j = 1\). By a direct calculation, we obtain,
\[
(nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} + \frac{(n-2m)(1-n)c_4^2 + 2m(1-m)c_2 + m(m-1)}{c_1^3(1-c_2)^{3/2}}
\]
\[
= \frac{m(c_2 - 1)(n-1)c_2^2 - (m-1)}{c_1^3(1-c_2)^{3/2}} < 0
\]
with multiplicity \(n - m + 1\), and
\[
(nH - k_1)\lambda_2^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} + \frac{(n-m)[(1-n)c_4^2 + mc_2]}{c_1^3(1-c_2)^{3/2}} < 0
\]
with multiplicity \(m + 1\). Therefore the weak stability index \(\text{Ind}_T(M) \geq n + 2\) for 
\(M = S^m(c) \times S^{n-m}(\sqrt{1-c^2})\) with constant scalar curvature \(n(n-1)r (r > 1)\).
Moreover, \(\text{Ind}_T(M) = n + 2\) if and only if
\[
(nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} \geq (nH - k_1)\lambda_2^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2}
\]
and
\[
(nH - k_1)\lambda_3^{\Delta_1} + (nH - k_n)\lambda_3^{\Delta_2} \geq (nH - k_1)\lambda_3^{\Delta_1} + (nH - k_n)\lambda_3^{\Delta_2}.
\]
Since
\[
(nH - k_1)\lambda_1^{\Delta_1} + (nH - k_n)\lambda_2^{\Delta_2} + \frac{(n-2m)(1-n)c_4^2 + 2m(1-m)c_2 + m(m-1)}{c_1^3(1-c_2)^{3/2}}
\]
\[
= \frac{(n-1)(n+2)c_4^2 - 2nm c_2^2 + m(m-1)}{c_1^3(1-c_2)^{3/2}},
\]
\[
(nH - k_1)\lambda_3^{\Delta_1} + (nH - k_n)\lambda_3^{\Delta_2} + \frac{(n-2m)(1-n)c_4^2 + 2m(1-m)c_2 + m(m-1)}{c_1^3(1-c_2)^{3/2}}
\]
\[
= \frac{(n+2)(1-n)c_4^2 + 2nm + 2n - 4)c_2^2 + (m+2)(1-m)}{c_1^3(1-c_2)^{3/2}}
\]
and \(c_2^2 > \frac{m(n-1) + \sqrt{mn(n-1)(n-m)}}{n(n-1)}\), we obtain that \(\text{Ind}_T(M) = n + 2\) if and only if
\[
\frac{nm + \sqrt{m[(2-n)m + (n-1)(n+2)]}}{(n-1)(n+2)} \leq c_2^2 \leq \frac{(nm + n - 2) + \sqrt{(n-m)(3n - 2m + nm - 2)}}{(n-1)(n+2)}.
\]
In addition, it is interesting to point out that the weak stability index of Clifford hypersurfaces \( S^m(c) \times S^{n-m}(\sqrt{1-c^2}) \) converge to infinity as \( c^2 \) converges to 1. In fact, we can obtain that for every \( j \geq 3 \),

\[
(nH - k_1)\lambda_j^{\Delta_1} + (nH - k_n)\lambda_j^{\Delta_2} + \frac{(n - 2m)(1 - n)e^4 + 2m(1 - m)e^2 + m(m - 1)}{c^4(1 - c^2)^{3/2}} < 0
\]

if and only if

\[
\frac{m(n - 1) + \sqrt{m(n - 1)(n - m)}}{n(n - 1)} < c^2 < p_j,
\]

where \( p_j = \frac{m((j - 1)(n - m + j - 2) + 2(m - 1)) + \sqrt{D}}{2((j - 1)(n - m + j - 2) - (n - 2m))}, D = m^2((j - 1)(n - m + j - 2) + 2(m - 1))^2 - 4m(m - 1)(n - 1)((j - 1)(n - m + j - 2) - (n - 2m)). \]

And for every \( i \geq 3 \), we have

\[
(nH - k_1)i^{\Delta_1} + (nH - k_n)i^{\Delta_2} + \frac{(n - 2m)(1 - n)e^4 + 2m(1 - m)e^2 + m(m - 1)}{c^4(1 - c^2)^{3/2}} < 0
\]

if and only if

\[
q_i < c^2 < 1,
\]

where \( q_i = \frac{-(n + m - 2)(i - 1)(m + i - 2) + 2m(1 - m)) - \sqrt{E}}{2((i - 1)(m + i - 2) + (n - 2m))}, E = ((n + m - 2)(i - 1)(m + i - 2) + 2m(1 - m))^2 - 4(1 - n)(m - 1)((i - 1)(m + i - 2) + (n - 2m))(1 - i)(m + i - 2) + m). \]

Hence we know that if \( \frac{m(n - 1) + \sqrt{m(n - 1)(n - m)}}{n(n - 1)} < p_{j+1} \leq c^2 < p_j \), then

\[
\text{Ind}_T(M) = n + 2 + \sum_{j=3}^{i} m\lambda_j^{\Delta_1} = m + C_{n-m+j-2}^{j-2} + C_{n-m+j-1}^{j-1}.
\]

If \( q_i < c^2 \leq q_{i+1} < 1 \), then

\[
\text{Ind}_T(M) = n + 2 + \sum_{j=3}^{i} m\lambda_j^{\Delta_1} = n - m + 1 + C_{m+i-1}^{i-1} + C_{m+i-2}^{i-2}.
\]

Moreover, \( \{q_i\} \not\to 1 \) and \( \text{Ind}_T(M) \not\to \infty \), as \( i \not\to \infty \).

**Example 2.** Some nontrivial example \( M \) which satisfies \( l_e = f_v \).

Let \( v \in R^{n+2} \) be a fixed unit vector and \( c \) a real number with \( |c| < 1 \). Let us define \( S^n(v,c) = \{ x \in S^{n+1} : <x,v>=c \} \). Let \( c_3 = (1,0,\ldots,0) \in R^{n+1} \) and
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c = \frac{4}{7}, then we know that the principal curvatures of \( S^{n-1}(e_1, c) \) are all equal to \( \frac{4}{7} \). By perturbing \( S^{n-1}(e_1, c) \), we can find a hypersurface \( N^{n-1} \subset S^n \) whose mean curvature is not constant and such that all its principal curvatures \( \lambda_i \) satisfy that:

\[ 1 < \lambda_i < 2. \]

Let \( M = S^1 \times N \) and \( \psi : M \to S^{n+1}(1) \) given by

\[ \psi((\cos s, \sin s), x) = \left( \frac{1}{\sqrt{2}} \sin(\sqrt{2}s), \frac{1}{2}(x + \nu(x)) \cos(\sqrt{2}s) + \frac{1}{2}(x - \nu(x)) \right) \]

where \( x \in N \subset S^n \subset R^{n+1} \) denotes the points in \( N \) and \( \nu : N \to S^n \subset R^{n+1} \) is a Gauss map of \( N \). In particular, \( <x, \nu(x)> = 0 \).

Let \( v_1, \cdots, v_{n-1} \) be a basis of \( T_x N \) such that \( -d\nu_x(v_i) = \lambda_i(x)v_i \). Note that \( \frac{\partial}{\partial s} = ((-\sin s, \cos s), 0) \in R^{n+3} \) and \( \tilde{v}_1 = (0, 0, v_1), \cdots, \tilde{v}_{n-2} = (0, 0, v_{n-2}) \) form a basis for the tangent space of \( M \) at \( p = ((\cos s, \sin s), x) \). A direct calculation shows that

\[ d\psi_p(\frac{\partial}{\partial s}) = (\cos(\sqrt{2}s), -\frac{1}{\sqrt{2}}(x + \nu(x)) \sin(\sqrt{2}s)) \]

and

\[ d\psi_p(\tilde{v}_i) = \frac{1}{2} \left( 0, (1 - \lambda_i(x)) \cos(\sqrt{2}s) + 1 + \lambda_i(x))v_i \right). \]

The expression \( (1 - \lambda_i(x)) \cos(\sqrt{2}s) + (1 + \lambda_i(x)) \neq 0 \), so \( \psi \) is an immersion. Moreover, \( \tilde{\nu} : M \to S^{n+1} \) given by

\[ \tilde{\nu}(p) = \left( \frac{1}{\sqrt{2}} \sin(\sqrt{2}s), \frac{1}{2}(x + \nu(x)) \cos(\sqrt{2}s) - \frac{1}{2}(x - \nu(x)) \right) \]

is a Gauss map on \( M \). Using the expression for \( \psi \) and for \( \tilde{\nu} \), we get that \( l_v = f_v \) for \( v = (1, 0, \cdots, 0) \in R^{n+2} \).

**Remark 1.** From Example 2, one knows that neither the scalar curvature nor the Gauss-Kronecker curvature of \( M \) is a constant. Moreover, \( M \) is not a hypersurface with constant ratio of the Gauss-Kronecker curvature and the \((n-1)\)-th mean curvature.

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**References**


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