Abstract. We prove that the Ricci tensor with respect to the generalized Tanaka-Webster connection of a real hypersurface in a complex projective space of complex dimension \( n \geq 3 \) vanishes identically if and only if the real hypersurface is locally congruent to some geodesic hypersphere.

1 Introduction

Tanaka-Webster connection is a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifold which associated with the almost contact structure ([7], [9]). Tanno [8] gave the generalized Tanaka-Webster connection (g-Tanaka-Webster connection) for contact metric manifolds, which coincides with Tanaka-Webster connection if the associated CR-structure is integrable. For a real hypersurface in a Kählerian manifold with an almost contact metric structure \((\eta, \phi, \xi, g)\), in [1] and [2], Cho defined the g-Tanaka-Webster connection \( \hat{\nabla}^{(k)} \) for a non-zero real number \( k \). Then we can see that \( \hat{\nabla}^{(k)} \eta = 0 \), \( \hat{\nabla}^{(k)} \xi = 0 \), \( \hat{\nabla}^{(k)} g = 0 \), \( \hat{\nabla}^{(k)} \phi = 0 \). Moreover, if the shape operator \( A \) of a real hypersurface satisfies \( \phi A + A \phi = 2 k \phi \), then the g-Tanaka-Webster connection \( \hat{\nabla}^{(k)} \) coincides with the Tanaka-Webster connection.

For real hypersurfaces in a complex space form \( \mathbb{M}^n(c) \) of constant holomorphic sectional curvature \( 4c \neq 0 \), one of the major problem is to determine real hypersurfaces satisfying certain geometrical assumptions. Cho [3] determined flat Hopf hypersurfaces with respect to the g-Tanaka-Webster connection of a non-flat complex space form. Besides, he classified Hopf hypersurfaces of a non-flat complex space form which admits a pseudo-Einstein CR-structure for the g-Tanaka-Webster connection.

The purpose of this paper is to study real hypersurfaces in a complex projective space whose Ricci tensor with respect to the g-Tanaka-Webster connection \( \hat{\nabla}^{(k)} \) vanishes identically. We show the following

**Theorem.** Let \( M \) be a real hypersurface of a complex projective space \( \mathbb{C}P^n \),
If the Ricci tensor \( \hat{S} \) of the generalized Tanaka-Webster connection \( \hat{\nabla}^{(k)} \) vanishes identically, then \( M \) is locally congruent to a geodesic hypersphere with \( k^2 \geq 4n(n-1) \).

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2 Preliminaries

Let \( \mathbb{C}P^n \) denote the complex projective space of complex dimension \( n \) (real dimension \( 2n \)) of constant holomorphic sectional curvature 4. We denote by \( J \) the almost complex structure of \( \mathbb{C}P^n \). The Hermitian metric of \( \mathbb{C}P^n \) will be denoted by \( G \).

Let \( M \) be a real \((2n-1)\)-dimensional hypersurface immersed in \( \mathbb{C}P^n \). We denote by \( g \) the Riemannian metric induced on \( M \) from \( G \). We take the unit normal vector field \( V \) of \( M \) in \( \mathbb{C}P^n \). For any vector field \( X \) tangent to \( M \), we define \( \phi, \eta \) and \( \xi \) by

\[
JX = \phi X + \eta(X)V, \quad JV = -\xi,
\]

where \( \phi X \) is the tangential part of \( JX \), \( \phi \) is a tensor field of type (1,1), \( \eta \) is a 1-form, and \( \xi \) is the unit vector field on \( M \). Then they satisfy

\[
\phi^2X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

Thus \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \).

We denote by \( \nabla \) the operator of covariant differentiation in \( \mathbb{C}P^n \), and by \( \nabla \) the one in \( M \) determined by the induced metric. Then the Gauss and Weingarten formulas are given respectively by

\[
\nabla_X Y = \nabla_X Y + g(AX, Y)V, \quad \nabla_X V = -AX
\]

for any vector fields \( X \) and \( Y \) tangent to \( M \). We call \( A \) the shape operator of \( M \).

From the Gauss and Weingarten formulas, we have

\[
\nabla_X \xi = \phi AX, \quad (\nabla_X \phi) Y = \eta(Y)AX - g(AX, Y)\xi.
\]

We denote by \( R \) the Riemannian curvature tensor field of \( M \). Then the equation of Gauss is given by

\[
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY,
\]

and the equation of Codazzi by

\[
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi.
\]

If \( A\xi = \lambda \xi \), \( \lambda \) being a function, then \( M \) is called a Hopf hypersurface. For a Hopf hypersurface, we see
Proposition A ([6]). Let $M$ be a Hopf hypersurface of $\mathbb{CP}^n$, $n \geq 2$. If $X \perp \xi$ and $AX = \lambda X$, then $\alpha = g(A\xi, \xi)$ is constant and

$$A\phi X = \frac{\lambda \alpha + 2}{2\lambda - \alpha} \phi X.$$ 

Theorem B ([5]). Let $M$ be a Hopf hypersurface of $\mathbb{CP}^n$. Then $M$ has constant principal curvatures if and only if $M$ is locally congruent to one of the following:

(A1) a geodesic hypersphere (that is, a tube over a hyperplane $\mathbb{CP}^{n-1}$),

(A2) a tube over a totally geodesic $\mathbb{CP}^k$ ($1 \leq k \leq n - 2$),

(B) a tube over a complex quadric $Q_{n-1}$,

(C) a tube over $\mathbb{CP}^1 \times \mathbb{CP}^{(n-1)/2}$ and $n(\geq 5)$ is odd,

(D) a tube over a complex Grassmann $G_{2,5}(\mathbb{C})$ and $n = 9$,

(E) a tube over a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$.

Next we introduce the notion of Tanaka-Webster connection and its generalization. Tanaka [7] defined the canonical affine connection on a non-degenerate, pseudo-Hermitian $\mathcal{CR}$ manifold. As a generalization of Tanaka-Webster connection, Tanno [8] defined the $g$-Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}X Y = \nabla X Y + \phi AX Y \xi - \eta(Y) \phi AX - k\eta(X) \phi Y,$$

where $(\phi, \xi, \eta, g)$ is a contact metric structure. Using the naturally extended affine connection of Tanno’s $g$-Tanaka-Webster connection, the $g$-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifold is given by

$$\hat{\nabla}^{(k)}X Y = \nabla X Y + \phi AX Y \xi - \eta(Y) \phi AX - k\eta(X) \phi Y$$

for a non-zero real number $k$ (see Cho [1], [2]). Then we see that

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0.$$ 

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the $g$-Tanaka-Webster connection coincides with the Tanaka-Webster connection. Next we define the $g$-Tanaka-Webster curvature tensor $\hat{R}$ with respect to $\hat{\nabla}^{(k)}$ by

$$\hat{R}(X, Y)Z = \hat{\nabla}X \hat{\nabla}Y Z - \hat{\nabla}Y \hat{\nabla}X Z - \hat{\nabla}[X,Y]Z$$

for all vector fields $X$, $Y$ and $Z$ in $M$. We denote by $\hat{S}$ the $g$-Tanaka Webster Ricci tensor, which is defined by

$$\hat{S}(Y, Z) = \text{trace of } \{X \mapsto \hat{R}(X, Y)Z\}.$$
3 The Ricci tensor of real hypersurfaces in a complex space form

To prove the theorem, we prepare the following lemma.

**Lemma 3.1.** Let $M$ be a real hypersurface of a complex projective space $CP^n$, $n \geq 3$. If there exists an orthonormal frame $\{e_1, \cdots, e_{2n-2}, \xi\}$ on a neighborhood $N$ of $x \in M$ such that the shape operator $A$ can be represented as

$$
A = \begin{pmatrix}
  a_1 & 0 & h_1 \\
  \cdots & \ddots & \vdots \\
  0 & \cdots & a_{2n-2} \\
  h_1 & 0 & \cdots & 0 & \alpha
\end{pmatrix},
$$

where $Ae_1 = a_1e_1 + h_1 \xi$, $Ae_i = a_ie_i$ ($i = 1, \cdots, 2n - 2$) and $A\xi = h_1e_1 + \alpha \xi$, then we have

\begin{align*}
(a_1 - a_j)g(\nabla e_i e_1, e_j) &+ (a_j - a_i)g(\nabla e_i e_j, e_1) + a_i h_1 g(\phi e_i, e_j) \\
= 0, \\
(a_j - a_1)g(\nabla e_i e_j, e_1) - (a_1 - a_j)g(\nabla e_j e_i, e_1) &+ h_1(a_i + a_j)g(\phi e_i, e_j) \\
= 0, \\
\{2 - 2a_1a_j + \alpha(a_i + a_j)\}g(\phi e_i, e_j) - h_1g(\nabla e_i e_j, e_1) + h_1g(\nabla e_j e_i, e_1) \\
= 0, \\
(a_1 - a_i)g(\nabla e_i e_1, e_i) &- (e_1a_i) = 0, \\
h_1(2a_i + a_1)g(\phi e_i, e_1) + (a_1 - a_j)g(\nabla e_i e_i, e_1) + (e_1a_i) &+ 0, \\
1 + a_1\alpha - a_1a_i - h_1^2 g(\phi e_1, e_i) - (a_1 - a_i)g(\nabla \xi e_1, e_i) \\
&+ h_1g(\nabla e_1 e_1, e_i) = 0.
\end{align*}

**Proof.** By the equation of Codazzi, we have

$$
g((\nabla e_i A)e_1 - (\nabla e_i A)e_i, e_j) = 0,
$$
where \(i, j = 2, \cdots, 2n - 2\). On the other hand, we have

\[
    g((\nabla_{e_i} A)e_1 - (\nabla_{e_1} A)e_i, e_j) \\
    = g(\nabla_{e_i}(Ae_1) - A\nabla_{e_i} e_1 - \nabla_{e_1}(Ae_i) + A\nabla_{e_1} e_i, e_j) \\
    = (a_1 - a_j)g(\nabla_{e_i} e_1, e_j) + (a_j - a_i)g(\nabla_{e_1} e_i, e_j) + a_i b_1 g(\phi e_i, e_j).
\]

Thus we obtain (3.1). By the similar computation, we have our results.

\[\square\]

**Lemma 3.2.** Let \(M\) be a real hypersurface of a complex projective space \(\mathbb{C}P^n\), \(n \geq 3\). If the Ricci tensor \(\hat{S}\) of the generalized Tanaka-Webster connection \(\hat{\nabla}^{(k)}\) vanishes identically, then \(M\) is a Hopf hypersurface.

**Proof.** Suppose \(M\) is not a Hopf hypersurface. By the definition of the g-Tanaka-Webster connection, we have (see [3])

\[
\hat{R}(X, Y)Z = R(X, Y)Z + g(\phi((\nabla_X A)Y - (\nabla_Y A)X), Z)\xi \\
+ 2g(\phi AX, Z)\phi AX - 2g(\phi AX, Z)\phi AY \\
+ g((\nabla_X \phi)AY - (\nabla_Y \phi)AX, Z)\xi \\
- \eta(Z)\left(g((\nabla_X A)Y - (\nabla_Y A)X) + (\nabla_X \phi)AY - (\nabla_Y \phi)AX\right) \\
- k\left(g((\phi A + A\phi)X, Y)\phi Z + \eta(Y)(\nabla_X \phi)Z - \eta(X)(\nabla_Y \phi)Z\right) \\
+ (\phi AX, F_Y Z)\xi - \eta(F_Y Z)\phi AX - k\eta(X)\phi F_Y Z \\
- g(\phi AY, F_X Z)\xi + \eta(F_X Z)\phi AY + k\eta(Y)\phi F_X Z.
\]

(3.7)

By the definition of g-Tanaka-Webster Ricci tensor, equations of Gauss and Codazzi, direct calculation shows that

\[
\hat{S}(Y, Z) = 2ng(Y, Z) + (\text{tr}A - \eta(AX) + k)g(AY, Z) \\
- g(A^2 Y, Z) - g(\phi A \phi AY, Z) - kg(\phi A \phi Y, Z) + \eta(AY)g(AX, Z) \\
+ \eta(Z)\left(-2n\eta(Y) - \eta(AY)\text{tr}A + \eta(A^2 Y) - k\eta(AY)\right).
\]

We can choose an orthonormal basis \(\{e_1, \cdots, e_{2n-2}, \xi\}\) of \(T^*_x M\) such that the shape
operator $A$ is represented by a matrix form

$$A = \begin{pmatrix} a_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{2n-2} & h_{2n-2} \\ h_1 & \cdots & h_{2n-2} & \alpha \end{pmatrix},$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \cdots, 2n-2$ and $\alpha = g(A\xi, \xi)$. By the direct computation using the previous equation, we have

$$\hat{S}(\xi, \xi) = 0, \quad \hat{S}(e_i, \xi) = 0,$$

(3.8)

$$\hat{S}(\xi, e_i) = (\text{tr}A - \alpha + k - a_i)h_i - g(\phi A\phi A\xi, e_i) = 0,$$

(3.9)

$$\hat{S}(e_i, e_i) = 2n + (\text{tr}A)a_i - a_i^2 - \alpha a_i + ka_i + (a_i + k)g(A\phi e_i, \phi e_i) = 0,$$

(3.10)

$$\hat{S}(e_i, e_j) = (a_i + k)g(A\phi e_i, \phi e_j) = 0 \quad (i \neq j).$$

From (3.10), if $a_i \neq -k$, then $g(A\phi e_i, \phi e_j) = 0$ for all $j \neq i$. Thus we set

$$A\phi e_i = \bar{a}_i \phi e_i + \bar{h}_i \xi,$$

where we have put $\bar{a}_i = g(A\phi e_i, \phi e_i)$ and $\bar{h}_i = g(A\phi e_i, \xi)$. If $a_i = -k$ for any $i$, then (3.9) and $\text{tr}A = (2n-2)a + \alpha$ imply that

$$2n + (2n - 4)a^2 = 0.$$

This is a contradiction. Thus $a_i \neq -k$ for some $i$. We also have

(3.11) $$\hat{S}(\phi e_i, \phi e_i) = 2n + (\text{tr}A)\bar{a}_i - \bar{a}_i^2 - \alpha \bar{a}_i + k\bar{a}_i + (\bar{a}_i + k)a_i = 0.$$ Using (3.9) and (3.11), we obtain

$$(a_i - \bar{a}_i)(\text{tr}A - \alpha - a_i - \bar{a}_i) = 0.$$ When $a_i = \bar{a}_i$, (3.9) implies

(3.12) $$2n = a_i(\alpha - 2k - \text{tr}A).$$
In this case, we put $a_i = \bar{a}_i = a$. Otherwise, if $a_i \neq \bar{a}_i$, then $\text{tr} A = a_i + \bar{a}_i + \alpha$. Using (3.9) and (3.11), we obtain

$$2a_i^2 - 2(\text{tr} A - \alpha)a_i - k(\text{tr} A - \alpha) - 2n = 0,$$
$$2\bar{a}_i^2 - 2(\text{tr} A - \alpha)\bar{a}_i - k(\text{tr} A - \alpha) - 2n = 0.$$

Therefore, $b = a_i$ and $\bar{b} = \bar{a}_i$ are solutions of the following

$$X^2 - (\text{tr} A - \alpha)X - \frac{k}{2}(\text{tr} A - \alpha) - n = 0.$$

Thus $b$ and $\bar{b}$ satisfies

(3.13) $b + \bar{b} = \text{tr} A - \alpha$,

(3.14) $\bar{b}b = -\frac{k}{2}(\text{tr} A - \alpha) - n$.

We suppose that there exists $i$ such that $a_i = \bar{a}_i \neq -k$. Let $Ae_x = -ke_x + h_x \xi$.

Then (3.9) implies

$$\hat{S}(e_x, e_x) = 2n - k(\text{tr} A) - 2k^2 + \alpha k = 0.$$

From this and (3.12), we have

$$(a_i + k)(\alpha - 2k - \text{tr} A) = 0.$$

Since $a_i + k \neq 0$, we see $\alpha - 2k - \text{tr} A = 0$. This is a contradiction by (3.12). Hence, taking a suitable orthonormal basis if necessary, we have

$$A = \begin{pmatrix}
  b & \ddots & \vdots \\
  \ddots & b & \ddots \\
  \vdots & \ddots & b \\
  h_1 & \cdots & h_{2n-2}
\end{pmatrix}
\begin{pmatrix}
  h_1 \\
  \vdots \\
  h_{2n-2}
\end{pmatrix}.$$
where \( d = a \) or \( d = -k \). In the following, we use integers \( x, y, z, \cdots \) for \( Ae_x = be_x + \xi_x \) and \( s, t, u, \cdots \) for \( Ae_s = de_s + \xi_s \). We denote by \( H_1, H_2 \) and \( H_3 \) the subspaces of a tangential space spanned by \( \{ e_x \}, \{ \phi e_x \} \) and \( \{ e_s \} \), respectively. We notice that \( d \) satisfies

\[
d = g(Ae_s, e_s) = g(A\phi e_s, \phi e_s),
\]

(3.15)

\[
2n = d(\alpha - 2k - \text{tr}A).
\]

By a direct computation using (3.8), we have

\[
(\text{tr}A - \alpha + k - b + \bar{b})h_x = 0,
\]

(3.16)

\[
(\text{tr}A - \alpha + k + b - \bar{b})\bar{h}_x = 0,
\]

(3.17)

\[
(\text{tr}A - \alpha + k)h_s = 0.
\]

(3.18)

If there exists \( e_s \in H_3 \) that satisfies \( h_s \neq 0 \), then \( \text{tr}A - \alpha + k = 0 \). Using (3.13) and (3.15), we have

\[
b + \bar{b} = -k, \quad 2n = -dk.
\]

On the other hand, we can represent \( \text{tr}A \) as

\[
\text{tr}A = p(b + \bar{b}) + qd + \alpha,
\]

where \( p \) and \( q \) denote numbers of \( b \) and \( d \), respectively. From these equations and (3.13), we obtain

\[
-(p - 1)dk + qd^2 = 2(p - 1)n + qd^2 = 0.
\]

Since \( p, q \) and \( n \) are natural numbers, this is a contradiction. Hence we see that \( h_s = 0 \) for all \( e_s \in H_2 \).

If there exist \( e_x \in H_1 \) and \( \phi e_y \in H_2 \) that satisfy \( h_x \neq 0 \) and \( \bar{h}_y \neq 0 \), (3.16) and (3.17) implies \( b = \bar{b} \). This is a contradiction.

So it is sufficient to consider the case that \( \bar{h}_x = 0 \) for any \( \phi e_x \in H_2 \) and \( h_y \neq 0 \) for some \( e_y \in H_1 \). Using (3.13) and (3.16), we have

\[
b = \text{tr}A - \alpha + \frac{k}{2}, \quad \bar{b} = -\frac{k}{2}.
\]

(3.19)
On the other hand, the direct computation shows that
\[ |xE - A| = (x - d)^q(x - b)^p(x - b)^{p-1}((x - b)(x - \alpha) - \sum_{x=1}^p h_x^2). \]
Thus the eigenvalue \( b \) of the symmetric matrix \( A \) has multiplicity at least \( p - 1 \).
Suppose \( e' \) satisfies \( Ae' = be' \). We can represent \( e' = X + \beta \xi \), where \( X \in H_1 \). Since \( AX = bX + h\xi \) for some \( h \), we obtain
\[ Ae' = bX + h\xi + \beta(\sum h_xe_x + \alpha\xi). \]
On the other hand, we have
\[ Ae' = b(X + \beta\xi) = bX + b\beta\xi. \]
From these equations, we obtain
\[ \beta \sum h_xe_x + (h + \alpha\beta - b\beta)\xi = 0. \]
Since \( h_x \neq 0 \) for some \( e_x \), we have \( \beta = 0 \), that is, \( g(e', \xi) = 0 \). Thus we can represent the shape operator \( A \) by a following matrix with respect to an orthonormal basis \( \{e_1, \cdots, e_p, \phi e_1, \cdots, \phi e_p, e_{2p+1}, \cdots, e_{2n-2}, \xi\} \):
\[
A = \begin{pmatrix}
  b & \cdots & b \\
  \ddots & \ddots & \ddots \\
  b & \bar{b} & \ddots \\
  & \bar{b} & d & \ddots \\
  & & d & 0 \\
  h_1 & 0 & \cdots & 0 & \alpha
\end{pmatrix}.
\]
Since \( \text{tr}A = p(b + \bar{b}) +qd + \alpha \), using (3.13), we obtain
\[ (p - 1)(b + \bar{b}) +qd = 0. \]
Case (i): First, we suppose $q \neq 0$. By (3.15) and $n \geq 3$, we see that $d \neq 0$. Thus, from (3.20), we have $p \neq 1$. If $p = 0$, then $M$ is a Hopf hypersurface. This is a contradiction.

In the following we suppose $p \geq 2$. Using (3.15), (3.16) and $\bar{b} = -k/2$, we have

$$2n = d(-b + \bar{b} - k) = d\left(-b - \frac{3}{2}k\right).$$

From this equation and (3.20), we have the following

$$b^2 + kb - \frac{3}{4}k^2 - \frac{2nq}{p-1} = 0.$$ 

Since $b$ is a continuous function on a sufficiently small neighborhood $\mathcal{N}$ of $x$, $p$ and $q$ are constant on $\mathcal{N}$. Hence we see that $b$ and $d$ are also constant on $\mathcal{N}$. We suppose $AU = bU + h_1\xi$ and $AZ = dZ$. By the equation of Codazzi, computing $(\nabla_Z A)U - (\nabla_U A)Z$, we have

$$(b - d)g(\nabla_Z U, \phi Z) + dh_1 = 0$$

on $\mathcal{N}$. Since $d$ is constant on $\mathcal{N}$, using the equation of Codazzi,

$$(\bar{b} - d)g(\nabla_Z \phi U, Z) = 0.$$ 

If $\bar{b} = d$, then (3.15) and (3.16) imply

$$2n = \bar{b}(-b + \bar{b} - k).$$

On the other hand, by (3.13), (3.14) and $\bar{b} = -k/2$, we obtain $\bar{b}^2 = n$. Using these equations, we obtain $b = \bar{b}$. This is a contradiction. Thus we have $b \neq d$, and hence $g(\nabla_Z \phi U, Z) = 0$. Hence we obtain

$$g(\nabla_Z U, \phi Z) = -g(U, (\nabla_Z \phi)Z) - g(U, \phi \nabla_Z Z) = g(\phi U, \nabla_Z Z) = -g(\nabla_Z \phi U, Z) = 0.$$ 

Therefore, we have $dh_1 = 0$. This is a contradiction.

Case (ii): Next, we consider the case that $q = 0$. From (3.14), (3.19) and (3.20),

$$b = -\bar{b} = \frac{k}{2}, \quad bb = -n.$$
We notice that the principal curvatures $b$ and $\tilde{b}$ have multiplicities $n - 2$ and $n - 1$, respectively. Thus we can choose an orthonormal frame \{\(e_1, e_2, \ldots, e_{n-1}, e_n, \ldots, e_{2n-2}, \xi\)\} on a neighborhood $N$ which satisfies $Ae_1 = be_1 + h_1 \xi$, $Ae_x = be_x$ for $x = 2, \ldots, n - 1$ and $A\phi e_x = \tilde{b}\phi e_x = -b\phi e_x$ for $x = 1, \ldots, n - 1$.

Using Lemma 3.1, we have

**Lemma 3.3.** We suppose $q = 0$. Let $\phi e_x$ be perpendicular to $\phi e_1$. Then,

\begin{align*}
\nabla_{e_1} e_1 &= \frac{h_1}{2} \phi e_1, \\
\nabla_{\phi e_x} e_1 &= \frac{2(1 + n)}{h_1} e_x.
\end{align*}

**Proof.** Using (3.5) and $g(\phi e_1, \phi e_x) = 0$, we have $g(\nabla_{e_1} \phi e_x, e_1) = 0$. On the other hand, putting $e_i = \phi e_1$ in (3.5),

\[h_1(2\tilde{b} + b)g(\phi^2 e_1, e_1) + (b - \tilde{b})g(\nabla_{e_1} \phi e_1, e_1) = 0,
\]

from which we obtain

\[g(\nabla_{e_1} \phi e_1) = \frac{h_1}{2}.
\]

By (3.6), we see that $g(\nabla_{e_1} e_1, e_y) = 0$ for any $e_y \in H_1$. Since $g(\nabla_{e_1} e_1, \xi) = -g(e_1, \phi Ae_1) = 0$, we have (3.21).

Next, putting $e_i = \phi e_x$ and $e_j = \phi e_y$ in (3.1), we have $g(\nabla_{\phi e_x} e_1, \phi e_y) = 0$ for any $\phi e_x, \phi e_y \in H_2$. On the other hand, using (3.2), we see that

\[g(\nabla_{\phi e_x} \phi e_y, e_1) = 0
\]

for any $e_y \in H_1$. Thus, putting $e_i = e_y$ and $e_j = \phi e_x$ in (3.3), direct calculation shows that

\[g(\nabla_{\phi e_x} e_1, e_y) = \frac{2 + 2n}{h_1} g(\phi e_y, \phi e_x).
\]

Since $g(\nabla_{\phi e_x} e_1, \xi) = 0$ and $g(\nabla_{\phi e_x} e_1, e_1) = 0$, we have (3.22).

\[\square\]

Using this lemma, we compute the sectional curvature spanned by $e_1$ and $\phi e_x \perp \phi e_1$. From (3.21), we have

\[g(\nabla_{\phi e_x} \nabla_{e_1} e_1, \phi e_x) = -\frac{h_1}{2} g(\phi e_1, \nabla_{\phi e_x} \phi e_x).
\]
Since $g(\phi e_x, \phi e_1) = 0$, we have
\[
g(\nabla_{\phi e_x} \phi e_x, \phi e_1) = -g(\phi e_x, \nabla_{\phi e_x} \phi e_1) = -g(\phi e_x, \phi \nabla_{\phi e_x} e_1)
\]
\[
= -g(\nabla_{\phi e_x} e_1, e_x) = -\frac{2(1+n)}{h_1}.
\]
Thus we obtain
\[
g(\nabla_{\phi e_x} \nabla e_1, \phi e_x) = 1 + n.
\]
On the other hand, by (3.22),
\[
g(\nabla e_1 \nabla e_1, \phi e_x) = g(\nabla_{\phi e_x} e_1, \phi e_x) - g(\nabla_{\phi e_x} e_1, \nabla e_1 \phi e_x)
\]
\[
= -\frac{2(1+n)}{h_1} g(e_x, \nabla e_1 \phi e_x) = 1 + n.
\]
Next, we see that
\[
g(\nabla [\phi_{e_x}, e_1] e_1, \phi e_x)
\]
\[
= g(\nabla \xi e_1, \phi e_x) g(\xi, [\phi_{e_x}, e_1]) + g(\nabla e_1, \phi e_x) g(e_1, [\phi_{e_x}, e_1])
\]
\[
+ \sum_y g(\nabla e_1 e_1, \phi e_x) g(y, [\phi_{e_x}, e_1]) + \sum_z g(\nabla e_1 e_1, \phi e_x) g(\phi e_1, [\phi_{e_x}, e_1])
\]
\[
= g(\nabla e_1, \phi e_x) g(\phi e_1, [\phi_{e_x}, e_1])
\]
\[
= 0.
\]
Here we note that $g(\nabla_{\phi e_x} \phi e_x, e_1) = 0$ from (3.4). Together with (3.23), we have the second equality. Also we remark that (3.2) and (3.22) implies $g(\nabla_{\phi e_x} e_1, \phi e_1) = g(\nabla e_1, \phi e_1) = 0$, which induces the last equality.

From these equations, we see that
\[
g(R(\phi e_x, e_1) e_1, \phi e_x)
\]
\[
= g(\nabla_{\phi e_x} \nabla e_1, \phi e_x) - g(\nabla e_1 \nabla_{\phi e_x} e_1, \phi e_x) - g(\nabla e_1, [\phi_{e_x}, e_1] e_1, \phi e_x)
\]
\[
= 0.
\]
On the other hand, the equation of Gauss implies
\[
g(R(\phi e_x, e_1) e_1, \phi e_x) = 1 + b - 1 - n.
\]
Hence we have $n = 1$. This is a contradiction. So we see that $q \neq 0$. This proves Lemma 3.2.
Real hypersurfaces in complex space forms and the generalized Tanaka-Webster connection

Using Lemma 3.2, we have our main result.

**Theorem 3.4.** Let $M$ be a real hypersurface of a complex projective space $\mathbb{C}P^n$, $n \geq 3$. If the Ricci tensor $\hat{S}$ of the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ vanishes identically, then $M$ is locally congruent to a geodesic hypersphere with $k^2 \geq 4n(n-1)$.

**Proof.** From the proof of Lemma 3.2, $M$ is a Hopf hypersurface with at most 4 distinct constant principal curvatures. From Theorem B, $M$ is locally congruent to one of type $(A_1)$, $(A_2)$ or $(B)$, and $M$ has at most three constant principal curvatures. Suppose $M$ has three constant distinct principal curvatures $b, \bar{b}$ and $\alpha$. Then we have

$$ \text{tr}A = p(b + \bar{b}) + \alpha. $$

Using (3.13), we have

$$ (p - 1)(b + \bar{b}) = 0. $$

Since $n \geq 3$, we see $b = -\bar{b}$. By Proposition A,

$$ \bar{b} = \frac{b\alpha + 2}{2b - \alpha} = -b, $$

from which we obtain $b^2 = -1$. This is a contradiction.

Let $M$ has two distinct constant principal curvatures $d$ and $\alpha$ with multiplicities $2n - 2$ and 1, respectively. Then, from (3.12),

$$ (n - 1)d^2 + kd + n = 0. $$

When $k^2 \geq 4n(n-1)$, the above equality has solutions. Conversely, if $M$ is a Hopf hypersurface with two distinct constant principal curvatures $d$ and $\alpha$ that satisfy the above equation, then its Ricci tensor $\hat{S}$ with respect to the g-Tanaka-Webster connection satisfies (3.8)-(3.10). Therefore we have our result.

**Theorem 3.5.** There are no flat real hypersurface of a complex projective space $\mathbb{C}P^n$, $n \geq 3$ with respect to the generalized Tanaka-Webster connection.
Proof. From Lemma 3.2, $M$ is a Hopf hypersurface. If $\hat{R} = 0$, then (3.7) and the equation of Gauss show

$$-k\left(g(AX, X) + g(A\phi X, \phi X)\right)$$

$$-2g(AX, X)g(A\phi X, \phi X) + 2g(AX, \phi X)^2 = 4$$

for any vector field $X \perp \xi$, $g(X, X) = 1$ (see [3]). We take $X$ such that $AX = dX$. Then we have

$$d^2 + kd + 2 = 0.$$ 

From this and (3.24), we obtain

$$(n - 2)(d^2 + 1) = 0.$$ 

This is a contradiction.

Remark 3.6. In [3], Cho proved that a Hopf hypersurface $M$ of a non-flat complex space form satisfies $\hat{R} = 0$ if and only if $M$ is locally congruent to a horosphere in $\mathbb{C}H^n$, or $\dim M = 3$ and a homogeneous tube over a complex quadric $Q^{n-1}$ and $\mathbb{R}P^n$ (resp. $\mathbb{R}H^n$) in $\mathbb{C}P^n$ (resp. $\mathbb{C}H^n$).

References


