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An introduction to the geometry of homogeneous spaces

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Abstract. This article is a survey of the geometry of homogeneous spaces for students. In particular, we are concerned with classical Lie groups and their root systems. Then we describe curvature tensors of the Riemannian connection and the canonical connection of a homogeneous space G/H in terms of the Lie algebra $\mathfrak{g}$.

1 Lie groups

1.1 Definition and examples

Definition 1. *A set G is called a **Lie group** if*

(1) G is an (abstract) group,

(2) G is a C^∞ manifold,

(3) the group operations

$$G \times G \to G, \qquad (x, y) \mapsto xy,$$
$$G \to G, \qquad x \mapsto x^{-1}$$

are C^∞ maps.

Example 1 $\mathbb{R}^n$ is an (abelian) Lie group under vector addition:

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} + \boldsymbol{y}, \qquad \boldsymbol{x} \mapsto -\boldsymbol{x}.$$

Example 2 (Classical Lie groups) Let $M(n, \mathbb{R})$ be the set of all real $n \times n$ matrices. We may identify it as

$$M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}.$$

The subset $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det A \neq 0\}$ is an open submanifold of $M(n, \mathbb{R})$, and becomes a Lie group under the multiplication of matrices. $GL(n, \mathbb{R})$ is called the **general linear group**.

Definition 2. *Let G be a Lie group. A Lie group H is a **Lie subgroup** of G if*

(1) H is a submanifold of the manifold G (H need not have the relative topology),

(2) H is a subgroup of the (abstract) group G.

Example 3

$$
\begin{aligned}
&SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}, \quad \dim = n^2 - 1 \\
&O(n) = \{A \in GL(n,\mathbb{R}) \mid {}^tAA = A\,{}^tA = I_n\}, \quad \dim = \frac{n(n-1)}{2} \\
&SO(n) = \{A \in O(n) \mid \det A = 1\}, \quad \dim = \frac{n(n-1)}{2} \\
&U(n) = \{A \in GL(n,\mathbb{C}) \mid A^*A = A^*A = I_n\}, \quad \dim = n^2 \\
&SU(n) = \{A \in U(n) \mid \det A = 1\}, \quad \dim = n^2 - 1 \\
&Sp(n) = \{A \in GL(n,\mathbb{H}) \mid AA^* = A^*A = I_n\}, \quad \dim = 2n^2 + n
\end{aligned}
$$

where, $\mathbb{H} = \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}\}$ denotes the set of all quaternions $(i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j)$, and $A^* = \overline{{}^tA} = {}^t\overline{A}$ is the conjugate transpose of a matrix A.

Theorem 1.1. *(Cartan) Let H be a closed subgroup of a Lie group G. Then H has a structure of Lie subgroup of G. (The topology of H coincides with the relative topology and the structure of Lie subgroup is unique).*

1.2 Left translation

For any fixed $a \in G$, the map $L_a : G \to G$ defined by

$$L_a(x) = ax \qquad \text{for } x \in G$$

is called the **left translation**. L_a is a C^∞ map on G, and furthermore it is a diffeomorphism of G.

$$L_a{}^{-1} = L_{a^{-1}}.$$

Similarly, the right translation R_a by $a \in G$ is a diffeomorphism defined by

$$R_a(x) = xa \qquad \text{for } x \in G.$$

The **inner automorphism** $i_a : G \to G$

$$i_a := L_a \circ R_{a^{-1}}, \qquad i_a(x) = axa^{-1} \quad (x \in G)$$

is C^∞ map on G.

Definition 3. *Let G be a Lie group. A vector field X on G is* ***left invariant*** *if*

$$(L_a)_*X = X,$$

where

$$((L_a)_*X)f := X(f \circ L_a) \qquad (f \in C^\infty(G)).$$

More precisely,

$$\begin{aligned}(L_a)_{*x}X_x &= X_{ax} \quad \textit{for } x \in G \\ ((L_a)_{*x}X_x)f &= X_x(f \circ L_a) \qquad \textit{for } f \in C^\infty(G)\end{aligned}$$

We denote by $\mathfrak{g}$ *the set of all left invariant vector fields on* G*.*

Proposition 1.2. *(1) For* $X, Y \in \mathfrak{g}$ *and* $\lambda, \mu \in \mathbb{R}$*, we have* $\lambda X + \mu Y \in \mathfrak{g}$*.*
(2) For $X, Y \in \mathfrak{g}$*, we have* $[X, Y] \in \mathfrak{g}$*.*

Remark. For vector fields X, Y on a manifold, bracket $[X, Y]$ is defined by

$$[X, Y]f := X(Yf) - Y(Xf) \qquad (f : C^\infty \text{ function}).$$

1.3 Lie algebras

Definition 4. *An* n*-dimensional vector space* $\mathfrak{g}$ *over a field* $K = \mathbb{R}$ *or* $\mathbb{C}$ *with a* ***bracket*** $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is called a* ***Lie algebra*** *if it satisfies*
(1) $[\ ,\ ]$ *is bilinear,*
(2) $[X, X] = 0$ *for* $\forall X \in \mathfrak{g}$ *and*
(3) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for* $\forall X, Y, Z \in \mathfrak{g}$*.*

Example $M(n, \mathbb{R}), M(n, \mathbb{C})$ and $M(n, \mathbb{H})$ with a commutator $[\ ,\ ]$ defined by

$$[X, Y] = XY - YX$$

are Lie algebras over $\mathbb{R}$.

Definition 5. *A vector subspace* $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$ *is called a* ***Lie subalgebra*** *if* $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$*.*

Example

$$\begin{aligned}\mathfrak{sl}(n, \mathbb{R}) &= \{X \in M(n, \mathbb{R}) \mid \operatorname{trace} X = 0\} \subset M(n, \mathbb{R}) \\ \mathfrak{o}(n) &= \{X \in M(n, \mathbb{R}) \mid X + {}^tX = 0\} \subset M(n, \mathbb{R}) \\ \mathfrak{so}(n) &= \{X \in \mathfrak{o}(n) \mid \operatorname{trace} X = 0\} \subset \mathfrak{o}(n) \\ \mathfrak{u}(n) &= \{X \in M(n, \mathbb{C}) \mid X + X^* = O\} \subset M(n, \mathbb{C}) \\ \mathfrak{su}(n) &= \{X \in \mathfrak{u}(n) \mid \operatorname{trace} X = 0\} \subset \mathfrak{u}(n)\end{aligned}$$

1.4 Lie algebra of a Lie group

Let X, Y be left invariant vector fields on a Lie group G, then

$$(L_a)_*X = X, \quad (L_a)_*Y = Y \qquad \text{for } a \in G,$$

hence we get

$$(L_a)_*[X, Y] = [X, Y].$$

In general, if $\varphi : M \to N$ is a C^∞ map and vector fields X, Y on M and X', Y' on N satisfy

$$\varphi_* X = X', \qquad \varphi_* Y = Y',$$

then we have

$$\varphi_*[X, Y] = [X', Y'].$$

Proposition 1.3. *The set $\mathfrak{g}$ of all left invariant vector fields on a Lie group G is a Lie algebra with respect to the Lie bracket $[X, Y]$ of vector fields X, Y on G.*

$\mathfrak{g}$ is called the Lie algebra of a Lie group G.

Proposition 1.4. *The Lie algebra $\mathfrak{g}$ of a Lie group G may be identified with the tangent space T_eG at the identity $e \in G$.*

$$\mathfrak{g} \ni X \mapsto X_e \in T_eG \qquad \text{(isomorphism between vector spaces)}$$

and conversely

$$T_eG \ni X_e \mapsto X \in \mathfrak{g} \text{ defined by } X_x := (L_x)_{*e}X_e \quad (x \in G).$$

Example An element x in $GL(n, \mathbb{R}) \subset M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ may be expressed as

$$x = (x^i_j),$$

then x^i_j are global coordinates of $GL(n, \mathbb{R})$. A left invariant vector field $X \in \mathfrak{gl}(n, \mathbb{R})$ on $GL(n, \mathbb{R})$ may be written as

$$X = \sum_{i,j=1}^{n} X^i_j \frac{\partial}{\partial x^i_j}.$$

Then $X_e \in T_eGL(n, \mathbb{R})$ may be identified with the $n \times n$ matrix $(X^i_j(e)) \in M(n, \mathbb{R})$.

$$\mathfrak{gl}(n, \mathbb{R}) \ni X = \sum_{i,j=1}^{n} X^i_j \frac{\partial}{\partial x^i_j} \longleftrightarrow (X^i_j(e)) \in M(n, \mathbb{R})$$

Then we may see that the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ of $GL(n, \mathbb{R})$ is isomorphic to $M(n, \mathbb{R})$ as Lie algebras:

$$(\mathfrak{gl}(n, \mathbb{R}), [\ ,\ ]) \cong (M(n, \mathbb{R}), [\ ,\ ])$$

Theorem 1.5. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.*

(1) If H is a Lie subgroup of G, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$

(2) If $\mathfrak{h}$ is a Lie subalgebra, there exists a unique Lie subgroup H of G such that the Lie algebra of H is isomorphic to $\mathfrak{h}$.

1.5 Exponential maps

Definition 6. *A homomorphism $\varphi : \mathbb{R} \to G$ is called a 1-**parameter subgroup** of a Lie group G. $\varphi(t)$ is a curve in G such that*

$$\varphi(0) = e, \qquad \varphi(s+t) = \varphi(s)\varphi(t).$$

For any 1-parameter subgroup $\varphi : \mathbb{R} \to G$, we consider its initial tangent vector

$$\varphi'(0) = \varphi_{*0}\left(\frac{d}{dt}\right)_0 \in T_eG$$

This is a one-to-one correspondense.

For each left invariant vector field $X \in \mathfrak{g}$ on G, we denote by $\varphi_X(t)$ the 1-parameter subgroup with the initial tangent vector ${\varphi_X}'(0) = X_e \in T_eG$.

Definition 7. *The map* $\exp : \mathfrak{g} \to G$ *defined by*

$$\exp X := \varphi_X(1)$$

*is called the **exponential map** of a Lie group G.*

Example For each $X \in M(n, \mathbb{R})$, the series of matrix

$$e^X = I + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + \cdots$$

converges in $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$. The map

$$\exp : M(n, \mathbb{R}) \to M(n, \mathbb{R}), \qquad X \mapsto \exp X = e^X$$

is called the **exponential map** of matrices.

If $[X, Y] = 0$,

$$\exp(X+Y) = \exp X \, \exp Y$$

and we may see that

$$\exp O = I, \quad \exp(-X) = (\exp X)^{-1}.$$

Hence

$$\exp : M(n, \mathbb{R}) \to GL(n, \mathbb{R}).$$

This coincides with the exponential map of the Lie group $GL(n, \mathbb{R})$.

1.6 Adjoint representations

Let G be a Lie group and i_a denotes the inner automorphism by $a \in G$.

$$i_a = L_a \circ R_{a^{-1}} : G \to G, \qquad x \mapsto axa^{-1}$$

We denote its differential by

$$\mathrm{Ad}(a) = \mathrm{Ad}_G(a) := (i_a)_*.$$

For any left invariant vector field $X \in \mathfrak{g}$, $\mathrm{Ad}(a)X = (i_a)_*X$ is also left invariant.

$$\mathrm{Ad}(a) : \mathfrak{g} \to \mathfrak{g} \quad \text{(linear)}$$

satisfies

$$[\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y] = \mathrm{Ad}(a)[X, Y].$$

Thus we have a homomorphism

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \qquad a \mapsto \mathrm{Ad}(a)$$

which called the **adjoint represenation** of G.

Proposition 1.6. *For any $a \in G$ and $X \in \mathfrak{g}$,*

$$exp(Ad(a)X) = i_a(exp\, X).$$

Let $\mathfrak{g}$ be a Lie algebra with a bracket $[\ ,\ ]$. For any X, the linear map

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}, \qquad Y \mapsto \mathrm{ad}(X)Y := [X, Y]$$

satisfies

$$\mathrm{ad}(X)[Y, Z] = [\mathrm{ad}(X)Y, Z] + [Y, \mathrm{ad}(X)Z]$$

(Jacobi identity). The homomorphism

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \qquad X \mapsto \mathrm{ad}(X)$$

is called the **adjoint representation** of $\mathfrak{g}$.

Proposition 1.7. *For any $X, Y \in \mathfrak{g}$,*

$$Ad(exp\, X)Y = e^{ad(X)}Y = \sum_{k=0}^{\infty} \frac{1}{k!} ad(X)^k Y.$$

Definition 8. *Let $\mathfrak{g}$ be a Lie algebra. The bilinear form*

$$B(X, Y) := tr(ad(X) \circ ad(Y)) \qquad (X, Y \in \mathfrak{g})$$

*is called the **Killing form** of $\mathfrak{g}$, where the right-hand-side means the trace of the linear map*

$$ad(X) \circ ad(Y) : \mathfrak{g} \to \mathfrak{g}.$$

Example

$\mathfrak{sl}(n,\mathbb{R})$	$B(X,Y) = 2n\,\mathrm{tr}(XY)$
$\mathfrak{o}(n)$	$B(X,Y) = (n-2)\,\mathrm{tr}(XY)$
$\mathfrak{su}(n)$	$B(X,Y) = 2n\,\mathrm{tr}(XY)$
$\mathfrak{sp}(n)$	$B(X,Y) = (2n+2)\,\mathrm{tr}(XY)$

1.7 Toral subgroups and maximal tori

Definition 9. *Let G be a connected Lie group. A subgroup S of G which is isomorphic to $\underbrace{S^1 \times \cdots \times S^1}_{k}$ is called a **toral subgroup** of G, or simply a **torus**.*

A toral subgroup S is an arcwise-connected compact abelian Lie group.

Definition 10. *A torus T is a **maximal torus** of G if*

$$T' \textit{ is any other torus with } T \subset T' \subset G \Longrightarrow T' = T$$

Theorem 1.8. *Any torus S of G is contained in a maximal torus T.*

Theorem 1.9. *(Cartan–Weil–Hopf) Let G be a compact connected Lie group. Then*

(1) G has a maximal torus T.

(2) $G = \bigcup_{x \in G} xTx^{-1}$.

(3) Any other maximal torus T' of G is conjugate to T, i.e.,

$$\exists a \in G \quad s.t. \quad T' = aTa^{-1}$$

(4) Any maximal torus of G is a maximal abelian subgroup of G.

The dimension $\ell = \dim T$ of T is called the **rank** of G.

Example $G = U(n)$.

$$T := \left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \,\middle|\, \theta_1, \cdots, \theta_n \in \mathbb{R} \right\} \cong \underbrace{S^1 \times \cdots \times S^1}_{n}$$

is a toral subgroup of $U(n)$.

Proposition 1.10. *Any unitary matrix $A \in U(n)$ can be diagonalized by a unitary matrix $P \in U(n)$,*

$$P^{-1}AP = \begin{pmatrix} e^{i\theta_1} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & e^{i\theta_n} \end{pmatrix} =: diag[e^{i\theta_1}, \cdots, e^{i\theta_n}]$$

Example $G = SO(n)$. We denote by R_θ the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and if $n = 2m$ put

$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & R_{\theta_m} \end{pmatrix} \middle| \ \theta_1, \cdots, \theta_m \in \mathbb{R} \right\} \cong \underbrace{S^1 \times \cdots \times S^1}_{m}$$

and if $n = 2m+1$ we put

$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & & \text{\Large 0} \\ & \ddots & & \\ & & R_{\theta_m} & \\ \text{\Large 0} & & & 1 \end{pmatrix} \middle| \ \theta_1, \cdots, \theta_m \in \mathbb{R} \right\} \cong \underbrace{S^1 \times \cdots \times S^1}_{m}$$

then T is a maximal torus of $SO(n)$.

Proposition 1.11. *Any orthogonal matrix $A \in O(n)$ may be put into the following* *form by an orthogonal matrix $Q \in O(n)$,*

$$
{}^tQAQ = \begin{pmatrix}
1 & & & & & & & & \\
& \ddots & & & & & & 0 & \\
& & 1 & & & & & & \\
& & & -1 & & & & & \\
& & & & \ddots & & & & \\
& & & & & -1 & & & \\
& & & & & & \boxed{\begin{matrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{matrix}} & & \\
& & & & & & & \ddots & \\
0 & & & & & & & & \boxed{\begin{matrix} \cos\theta_r & \sin\theta_r \\ -\sin\theta_r & \cos\theta_r \end{matrix}}
\end{pmatrix}
$$

Remark.

$$
R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \qquad R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2,
$$

1.8 Root systems

Let G be a compact connected Lie group.
Let (ρ, V) be any representation of G.

$$\rho : G \to GL(V) \quad \text{(homomorphism)}$$

(V is a vector space over $\mathbb{R}$ or $\mathbb{C}$). We may choose a $\rho(G)$-invariant inner product on V. For any $\rho(G)$-invariant subspace $W \subset V$, we have

$$V = W \oplus W^\perp \quad (\rho(G)\text{-invariant decomposition})$$

Therefore, (ρ, V) is completely reducible.

$$(\rho, V) = (\rho_1, V_1) \oplus \cdots \oplus (\rho_k, V_k)$$

where (ρ_i, V_i) is irreducible.

In particular, the adjoint representation $(\mathrm{Ad}, \mathfrak{g})$ of G is completely reducible, and we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

where

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \mathrm{ad}(Y)X = 0 \text{ for } \forall Y \in \mathfrak{g}\} \quad \text{(center of } \mathfrak{g})$$

and $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ (i.e., $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_i$) and $\mathfrak{g}_i$ is irreducible.

Proposition 1.12. *Let A, B be two diagonalizable matrices. If $[A,B]=0$, then A and B are simultaneously diagonalizable, i.e.,*

$$\exists \boldsymbol{x} \neq \mathbf{0} \quad s.t. \quad A\boldsymbol{x} = \lambda \boldsymbol{x}, \quad B\boldsymbol{x} = \mu \boldsymbol{x} \quad (\lambda, \mu \ : \textit{eigenvalues})$$

Let G be a compact connected Lie group, and S a torus of G. We choose an $\mathrm{Ad}(S)$-invariant inner product on $\mathfrak{g}$. Then the representation $(\mathrm{Ad}, \mathfrak{g})$ of S is completely reducible. If we consider the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and the representation $(\mathrm{Ad}, \mathfrak{g}^{\mathbb{C}})$ of S, the eigenvalues of $\mathrm{Ad}(s)$ are complex numbers with absolute value 1, and corresponding eigenspaces in $\mathfrak{g}^{\mathbb{C}}$ are of $\dim_{\mathbb{C}} = 1$. Then $\mathfrak{g}$ may be decomposed into irreducible subspaces.

$$\mathfrak{g} = V_0 \oplus V_1 \oplus \cdots \oplus V_m$$

where

$$\mathrm{Ad}(s)|_{V_0} = id \qquad \text{on} \quad V_0$$

and for some linear form $\theta_k : \mathfrak{s} \to \mathbb{R}$

$$\mathrm{Ad}(\exp v)|_{V_k} = \begin{pmatrix} \cos\theta_k(v) & -\sin\theta_k(v) \\ \sin\theta_k(v) & \cos\theta_k(v) \end{pmatrix} \qquad \text{on } V_k \quad \text{for } v \in \mathfrak{s}$$

w.r.t. an orthonormal basis on V_k. These θ_k are unique up to sign and order, and do not depent on the inner product and an orthonormal basis.

Definition 11. *Linear forms $\pm\theta_k$ $(k = 1, \cdots, m)$ are called* ***roots*** *of G relative to S. If S is a maximal torus T of G, they are called simply the* ***roots*** *of G. The set Δ of all roots of G is called the* ***root system*** *of G.*

Remark. Sometimes we define roots as

$$\pm \frac{1}{2\pi}\theta_k$$

so as to take $\mathbb{Z}$-values on the unit lattice $\exp^{-1}(e) \subset \mathfrak{t}$.

Example $G = SU(n)$ and T is the maximal torus defined as before.

$$T = \{\mathrm{diag}[e^{ix_1}, \cdots, e^{ix_n}] \mid x_k \in \mathbb{R}, \sum_{k=1}^{n} x_k = 0\}$$

The Lie algebra $\mathfrak{t}$ of T is

$$\mathfrak{t} = \{\mathrm{diag}[ix_1, \cdots, ix_n] \mid x_k \in \mathbb{R}, \sum_{k=1}^{n} x_k = 0\}$$

$x_1, \cdots, x_n \in \mathfrak{t}^*$ are deined in natural way.

$$\text{linear form} \quad \mathrm{diag}[ix_1, \cdots, ix_n] \mapsto x_k \quad \text{is denoted by } x_k$$

We take the inner product (,) on $\mathfrak{g}$ defined by

$$(X, Y) := -\frac{1}{2}\mathrm{tr}\,(XY)$$

and for $1 \leq j < k \leq n, 1 \leq \ell \leq n$, put

$$U_{jk} = \begin{pmatrix} & & 1 & \\ & & & \\ & -1 & & \\ & & & \end{pmatrix}, \quad U'_{jk} = \begin{pmatrix} & & i & \\ & & & \\ & i & & \\ & & & \end{pmatrix}, \quad \sqrt{2}E_\ell = \sqrt{2}\begin{pmatrix} & & \\ & i & \\ & & \end{pmatrix}$$

which form an orthonormal basis of $\mathfrak{g}$. Then we may see that for any $v = \mathrm{diag}[ix_1, \cdots, ix_n] \in \mathfrak{t}$

$$\mathrm{Ad}(\exp v) = \begin{pmatrix} \cos(x_j - x_k) & -\sin(x_j - x_k) \\ \sin(x_j - x_k) & \cos(x_j - x_k) \end{pmatrix}$$

on $\mathrm{span}_{\mathbb{R}}\{U_{jk}, U'_{jk}\}$.

Thus the root system Δ of $SU(n)$ is

$$\Delta = \{\pm(x_j - x_k) \mid 1 \leq j < k \leq n\}$$

Definition 12. *Let $\{\lambda_1, \cdots, \lambda_\ell\}$ be a basis of $\mathfrak{t}^*$. The lexicographic order $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0$ is defined by*

$$v = \sum_{k=1}^{\ell} v^i \lambda_i > 0 \Longleftrightarrow \begin{cases} v^1 = \cdots = v^{k-1} = 0 \\ v^k > 0 \quad \text{for some } k \end{cases}$$

*A root $\alpha \in \Delta$ is **positive** (resp. negative) if $\alpha > 0$ (resp. $\alpha < 0$) relative to the lexicographic order w.r.t. the basis.*

$$\Delta = \Delta^+ \cup \Delta^-$$

*A positive root α is called a **simple root** if it can not be expressed as a sum of two positive roots. The set $\{\alpha_1, \cdots, \alpha_\ell\}$ of all simple roots of G is called the **simple root system** of G (where $\ell = \mathrm{rank}\,G$).*

Example $G = SU(n)$ and T a maximal torus of all diagonal matrices. $\{x_1, \cdots, x_{n-1}\}$ is a basis of $\mathfrak{t}^*$. Consider the lexicographic order $x_1 > x_2 > \cdots > x_{n-1} > 0$. Then

$$\begin{array}{ll} \Delta^+ = \{x_j - x_k \mid 1 \le j < k \le n\} & \text{positive root system} \\ \Delta^- = \{-(x_j - x_k) \mid 1 \le j < k \le n\} & \text{negative root system} \end{array}$$

and the simple root system is

$$\{\alpha_j = x_j - x_{j+1} \mid 1 \le j \le n-1\}$$

1.9 Root systems – again –

Here, we shall calculate the root system in terms of Lie algebra.

Let G be a compact connected semisimple Lie group, T a maximal torus of G, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{t}$ the Lie algebra of T.

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, \qquad \mathfrak{t}^{\mathbb{C}} := \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$$

We extend $[\ ,\ ]$ complex-linearly on $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{g}^{\mathbb{C}}$ becomes a complex Lie algebra.

$\mathfrak{t}^{\mathbb{C}}$ becomes a maximal abelian Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

The Killing form $B^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ is the natural extension of the Killing form B of $\mathfrak{g}$.

If we take an $\mathrm{Ad}(G)$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$, we can make an $\mathrm{Ad}(G)$-invariant Hermitian inner product $(\ ,\ )$ on $\mathfrak{g}^{\mathbb{C}}$ in natural way.

$$(\mathrm{ad}(v)X, Y) + (X, \mathrm{ad}(v)Y) = 0$$

for $X, Y \in \mathfrak{g}^{\mathbb{C}}$, $v \in \mathfrak{t}^{\mathbb{C}}$, that is, $\mathrm{ad}(v)$ is a skew-Hermitian transformation. Thus for any $v \in \mathfrak{t}^{\mathbb{C}}$, $\mathrm{ad}(v)$ is diagonalizable.

Hence, $\mathfrak{t}^{\mathbb{C}}$ is a **Cartan subalgebra** of $\mathfrak{g}^{\mathbb{C}}$.

Definition 13. *A Lie aubalgebra $\mathfrak{h}$ is called a **Cartan subalgebra** of a complex Lie algebra $\mathfrak{g}$ if*

(1) $\mathfrak{h}$ is maximal abelian subalgebra,

(2) ad(H) is diagonalizable for any $H \in \mathfrak{h}$.

1.10 $SU(n)$

$G = SU(n)$, T is the maximal torus defined by

$$T = \{\mathrm{diag}\,[e^{i\theta_1}, \cdots, e^{i\theta_n}] \mid \theta_1 + \cdots + \theta_n = 0\}$$

$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^* = O, \quad \mathrm{tr}\, X = 0\}$

A basis of $\mathfrak{g}$ is given by

$$U_{jk} = \begin{pmatrix} & & \\ & & 1 \\ & & \\ & -1 & \\ & & \end{pmatrix}, \quad U'_{jk} = \begin{pmatrix} & & \\ & & i \\ & & \\ & i & \\ & & \end{pmatrix}, \quad E_\ell = \begin{pmatrix} i & \\ & -i \end{pmatrix}.$$

Then $\mathfrak{g}^{\mathbb{C}}$ is spanned (over $\mathbb{R}$) by

$$\begin{array}{ccc} U_{jk}, & U'_{jk}, & E_\ell, \\ iU_{jk}, & iU'_{jk}, & iE_\ell. \end{array}$$

Hence $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ and

$$\begin{aligned} \mathfrak{t} &= \{\mathrm{diag}[i\theta_1, \cdots, i\theta_n] \mid \theta_k \in \mathbb{R}, \theta_1 + \cdots + \theta_n = 0\}, \\ \mathfrak{t}^{\mathbb{C}} &= \{\mathrm{diag}[\lambda_1, \cdots, \lambda_n] \mid \lambda_k \in \mathbb{C}, \lambda_1 + \cdots + \lambda_n = 0\}. \end{aligned}$$

For $v = \mathrm{diag}[\lambda_1, \cdots, \lambda_n] \in \mathfrak{t}^{\mathbb{C}}$, if we put

$$E_{jk} = \begin{pmatrix} & & 1 & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

then we have

$$\mathrm{ad}(v)E_{jk} = [v, E_{jk}] = (\lambda_j - \lambda_k)E_{jk}.$$

Let $x_j : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ be a linear form defined by

$$x_j(\mathrm{diag}[\lambda_1, \cdots, \lambda_n]) = \lambda_j,$$

then we have

$$\mathrm{ad}(v)E_{jk} = [v, E_{jk}] = (x_j - x_k)(v)E_{jk}.$$

Thus if we put $\alpha = x_j - x_k \in (\mathfrak{t}^{\mathbb{C}})^*$,

$$\mathrm{ad}(v)X = \alpha(v)X \qquad \text{for } X \in \mathfrak{g}^{\mathbb{C}}_\alpha = \mathbb{C}E_{jk}.$$

The root system of $\mathfrak{g} = \mathfrak{su}(n)$ relative to $\mathfrak{t}$ is

$$\Delta = \{\pm(x_j - x_k) \mid 1 \leq j < k \leq n\},$$

and we have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Delta}, \mathfrak{g}^{\mathbb{C}}_\alpha,$$

where

$$\mathfrak{g}^{\mathbb{C}}_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}(v)X = \alpha(v)X \quad \text{for } v \in \mathfrak{t}^{\mathbb{C}}\}$$

is the **root space** corresponding to a root α. ($\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}}_\alpha = 1$).

In usual, we denote by $\mathfrak{h}$ a Cartan subalgebra, so in our case, $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$. and then

$$\mathfrak{h}_{\mathbb{R}} := \{v \in \mathfrak{h} \mid \alpha(v) \in \mathbb{R} \ (\forall \alpha \in \Delta)\}$$

is just $i\mathfrak{t}$.

We take a basis $\{v_1, \cdots, v_\ell\}$ of $\mathfrak{h}_{\mathbb{R}}$ as

$$v_j := \mathrm{diag}[0, \cdots, 1, \cdots, 0] \in i\mathfrak{t}$$

and put $\{\lambda_1, \cdots, \lambda_\ell\}$ its dual basis of ${\mathfrak{h}_{\mathbb{R}}}^*$,

$$\lambda_j = x_j.$$

We consider the lexicographic order in ${\mathfrak{h}_{\mathbb{R}}}^*$. The root system is a union of the positive root system and the negative root system.

$$\Delta = \Delta^+ \cup \Delta^-$$

If we put

$$\alpha_j = x_j - x_{j+1} \quad (1 \leq j \leq n-1 = \ell)$$

then they are the simple roots and $\alpha \in \Delta^+$ may be written as a $\mathbb{Z}_{\geq 0}$-coefficients linear combination of simple roots.

The restriction to $\mathfrak{h}$ of the Killing form B of $\mathfrak{g}^{\mathbb{C}}$ is nondegenerate, and we denote it by $(\ ,\ )$.

For $\lambda \in \mathfrak{h}^*$, we define $v_\lambda \in \mathfrak{h}$ by

$$(v_\lambda, v) = \lambda(v) \quad \text{for } v \in \mathfrak{h},$$

and we define an inner product on $\mathfrak{h}^*$ by

$$(\lambda, \mu) := (v_\lambda, v_\mu).$$

We put

$$(\ ,\ )_{\mathbb{R}} := (\ ,\ )|_{\mathfrak{h}_{\mathbb{R}}}, \qquad (\ ,\ )_{\mathbb{R}} := (\ ,\ )|_{\mathfrak{h}_{\mathbb{R}}^*},$$

which are positive definite. For convenience, we denote these simply by $(\ ,\ )$.

For $\alpha, \beta \in \Delta$ $(\alpha \neq \pm\beta)$, we consider the sequence

$$\beta - p\alpha,\ \cdots, \beta - \alpha, \beta, \beta + \alpha, \cdots, \beta + q\alpha \in \Delta$$

where $\beta - (p+1)\alpha \notin \Delta$ and $\beta + (q+1)\alpha \notin \Delta$.

$$\{\beta + m\alpha \mid -p \leq m \leq q\}$$

is called the α-sequence containing β. We may see that

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = p - q \in \mathbb{Z}.$$

In particular, if $p = 0$, i.e., $\beta - \alpha \notin \Delta$,

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \leq 0.$$

Therefore for simple roots α_j, α_k

$$\frac{2(\alpha_j, \alpha_k)}{(\alpha_j, \alpha_j)} \leq 0.$$

On the other hand
$$\frac{2(\alpha_j,\alpha_k)}{(\alpha_j,\alpha_j)}\frac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)} = 4\cos^2\theta,$$
where θ is the angle of α_j and α_k.
$$\cos^2\theta = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \quad (\text{or } \ 1).$$
Since $\pi/2 \le \theta < \pi$, we have four cases.
$$\begin{aligned}
&(1) \quad \theta = \frac{\pi}{2},\\
&(2) \quad \theta = \frac{2\pi}{3}, \quad \|\alpha_j\| = \|\alpha_k\|\\
&(3) \quad \theta = \frac{3\pi}{4}, \quad \|\alpha_j\| : \|\alpha_k\| = 1 : \sqrt{2} \text{ or } \sqrt{2} : 1\\
&(4) \quad \theta = \frac{5\pi}{6}, \quad \|\alpha_j\| : \|\alpha_k\| = 1 : \sqrt{3} \text{ or } \sqrt{3} : 1
\end{aligned}$$
Then we draw a diagram as follows:
(1) we do not join α_j and α_k $\quad \circ \qquad \circ$
(2) join by single line $\quad \circ\!-\!\circ$
(3) join by double line $\quad \circ\!\Longrightarrow\!\circ$ (arrow from longer root to shorter one)
(4) join by triple line $\quad \circ\!\Rrightarrow\!\circ$ (arrow from longer root to shorter one)

Thus the **Dynkin diagram** has vertices $\{\alpha_1, \cdots, \alpha_\ell\}$ and edges (1) – (4). The **highest root** μ (that is, μ is a positive root such that $\mu \ge \alpha$ for any $\alpha \in \Delta$) can be written as
$$\mu = m_1\alpha_1 + \cdots + m_\ell\alpha_\ell$$
in the Dynkin diagram, the integers m_k are written on each α_k.

The highest root μ of $\mathfrak{su}(n)$ is
$$\mu = x_1 - x_n = \alpha_1 + \cdots + \alpha_\ell, \qquad m_1 = \cdots = m_\ell = 1$$

2 Geometry of homogeneous spaces

2.1 Homogeneous spaces

Definition 14. *A C^∞ manifold M is* ***homogeneous*** *if a Lie group G acts transitively on M, i.e.,*

(1) There exists a C^∞ map : $G \times M \to M$, $\quad (a, x) \mapsto ax \quad$ such that

(i) $a(bx) = (ab)x$

(ii) $ex = x$ for any $x \in M$

(2) For any $x, y \in M$, there exists an element $a \in G$ such that $y = ax$

For an arbitraryly fixed point $x_0 \in M$, the closed subgroup

$$H := \{a \in G \mid ax_0 = x_0\}$$

is called an **isotropy subgroup** of G at x_0.

Then M is diffeomorphic to the coset space G/H.

$$M \cong G/H, \qquad x \mapsto aH,$$

where, $a \in G$ is an element of G s.t. $x = ax_0$.

Conversely, if H is a closed subgroup of a Lie group G, then the coset space G/H becomes a C^∞ manifold, and G acts on M by

$$G \times G/H \to G/H, \qquad (g, aH) \mapsto (ga)H,$$

and the map

$$\tau_g : G/H \to G/H, \qquad aH \mapsto (ga)H$$

is called a **left translation** by g.

$$\tau_g \circ \pi = \pi \circ L_g$$

2.2 G-invariant Riemannian metrics

Let $M = G/H$ be a homogeneous space of a Lie group G and a closed subgroup H. A Riemannian metric $\langle\ ,\ \rangle$ on M is **G-invariant** if

$$\langle (\tau_g)_* X, (\tau_g)_* Y \rangle = \langle X, Y \rangle \qquad \text{for } \forall g \in G$$

for any $X, Y \in TM$. Then τ_g is an isometry of the Riemannian manifold $(M, \langle\ ,\ \rangle)$.

Definition 15. *A homogeneous space $M = G/H$ is **reductive** if there exists a decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad (\textit{direct sum})$$

such that

$$Ad(H)\mathfrak{m} \subset \mathfrak{m}, \qquad \textit{i.e.,} \quad Ad(h)\mathfrak{m} \subset \mathfrak{m} \quad (\forall h \in H).$$

If $\mathfrak{g}$ has an $\mathrm{Ad}(H)$-invariant inner product $(\ ,\ )$, then the orthogonal complement $\mathfrak{m} := \mathfrak{h}^\perp$ satisfies the above property.

$$\pi : G \to M = G/H$$

is the natural projection, $o = eH \in G/H$.

$$\pi_{*e} : T_eG \to T_oM \qquad \text{onto linear}$$

induces an isomorphism

$$T_oM \cong T_eG/\ker \pi_{*e} \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$$

For any $X_{gH}, Y_{gH} \in T_{gH}M$

$$\begin{aligned}\langle X_{gH}, Y_{gH}\rangle_{gH} &= \langle (\tau_{g^{-1}})_{*gH}X_{gH}, (\tau_{g^{-1}})_{*gH}Y_{gH}\rangle_{eH} \\ &= ((\tau_{g^{-1}})_{*gH}X_{gH}, (\tau_{g^{-1}})_{*gH}Y_{gH}).\end{aligned}$$

Let G/H be a reductive homogeneous space,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad (\mathrm{Ad}(H)\text{-invariant decomposition}).$$

For any $h \in H$

$$\pi(i_h(g)) = \tau_h(\pi(g)) \qquad (g \in G).$$

Hence for any $X \in \mathfrak{m} \subset \mathfrak{g}$

$$\pi_{*e}(\mathrm{Ad}(h)X) = (\tau_h)_{*e}X \qquad (\pi_{*e} : \mathfrak{m} \cong T_oM).$$

Then a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on $M = G/H$ induces an inner product $(\ ,\ )$ on $\mathfrak{m}$.

$$(X, Y) = \langle X, Y\rangle_o$$

Then $(\ ,\ )$ is $\mathrm{Ad}(H)$-invariant.

$$(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y) = (X, Y), \quad \text{for} \quad \forall h \in H, \quad X, Y \in \mathfrak{m}.$$

Hence, we have the one-to-one correspondence:

$$\begin{array}{c}\langle\ ,\ \rangle : G\text{-inv. Riem. metric on } G/H \\ \updownarrow \\ (\ ,\ ) : \mathrm{Ad}(H)\text{-inv. inner product on } \mathfrak{m}\end{array}$$

In the sequel, we use the same notation $\langle\ ,\ \rangle$ for an $\mathrm{Ad}(H)$-inv. inner product on $\mathfrak{m}$.

Definition 16. *A reductive homogeneous space $M = G/H$ with a Riemannian metric $\langle\ ,\ \rangle$ is* ***naturally reductive*** *if there exists an $Ad(H)$-invariant decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad (\textit{direct sum}), \qquad \mathit{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$$

such that

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0$$

for any $X, Y, Z \in \mathfrak{m}$, where $[X, Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[X, Y]$.

Remark. Accoring to the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,

$$[X, Y] = [X, Y]_{\mathfrak{h}} + [X, Y]_{\mathfrak{m}}$$

Definition 17. *A homogeneous space $M = G/H$ with a Riemannian metric $\langle\ ,\ \rangle$ is* ***normal homogeneous*** *if there exists a bi-invariant Riemannian metric $(\ ,\ )$ on G such that, if we identify*

$$\mathfrak{m} := \mathfrak{h}^{\perp} \quad (\textit{with respect to } (\ ,\ ))$$

with T_oM, then

$$\langle\ ,\ \rangle = (\ ,\ )|_{\mathfrak{m}\times\mathfrak{m}}.$$

Remark. If $M = G/H$ is normal homogeneous, M is naturally reductive.

Example If the Kiling form B of a compact Lie group G is negative definite (then G is **semisimple**), using the inner product $-B$ on $\mathfrak{g}$, G/H is normal homogeneous.

2.3 Riemannian connection and the canonical connection

Let G/H be a homogeneous space. For $X \in \mathfrak{g}$,

$$\gamma(t) = \tau_{\exp tX}(gH)$$

is a curve in M through a point gH. The vector field X^* defined by

$$X^*_{gH} := \gamma'(0) = \left.\frac{d}{dt}\right|_{t=0} \tau_{\exp tX}(gH)$$

satisfies at the origin $o = eH$

$$X^*_o = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp tX) = \pi_{*e}(X).$$

If G/H is reductive, the isomorphism $\pi_{*e} : \mathfrak{m} \cong T_oM$ may be expressed as

$$\mathfrak{m} \ni X \leftrightarrow X^*_o \in T_oM.$$

Proposition 2.1. *For $X, Y \in \mathfrak{g}$,*

$$[X^*, Y^*] = -[X, Y]^*.$$

Let G/H be a reductive homogeneous space with a G-invariant Riemannian metric $\langle\ ,\ \rangle$. We denote by ∇ the Riemannian connection of $(M/H, \langle\ ,\ \rangle)$.

Proposition 2.2. *For any $X \in \mathfrak{g}$, the vector field X^* is a Killing vector field on G/H, i.e.,*

$$X^*\langle Y, Z\rangle = \langle [X^*, Y], Z\rangle + \langle Y, [X^*, Z]\rangle$$

for any vector fields $Y, Z \in \mathfrak{X}(G/H)$.

Since

$$X^*\langle Y, Z\rangle = \langle \nabla_{X^*} Y, Z\rangle + \langle Y, \nabla_{X^*} Z\rangle,$$

we have

$$\langle \nabla_Y X^*, Z\rangle + \langle Y, \nabla_Z X^*\rangle = 0.$$

Proposition 2.3. *Let $(G/H, \langle\ ,\rangle)$ be a naturally reductive homogeneous space. Then*

$$\nabla_v X^* = \begin{cases} [X, v] & \text{if } X \in \mathfrak{h} \\ \frac{1}{2}[X, v]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m} \end{cases}$$

for $v \in \mathfrak{m} \cong T_o M$.

Definition 18. *Let $(G/H, \langle\ ,\ \rangle)$ be a naturally reductive homogeneous space. Then there exists a metric connection D with torsion tensor T and the curvature tensor B such that*

$$DT = 0, \qquad DB = 0.$$

*D is called the **canonical connection**.*

The canonical connection D is given by

$$D_v X^* = \begin{cases} [X, v] & \text{if } X \in \mathfrak{h} \\ [X, v]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m} \end{cases}$$

and

$$\begin{aligned} D_X Y &= \nabla_X Y + \frac{1}{2} T(X, Y), \\ T(X, Y) &= -[X, Y]_{\mathfrak{m}}, \\ B(X, Y)Z &= -[[X, Y]_{\mathfrak{h}}, Z], \end{aligned}$$

for $X, Y, Z \in \mathfrak{m}$.

2.4 Curvature tensor

Theorem 2.4. *Let $(G/H, \langle\ ,\ \rangle)$ be a naturally reductive homogeneous space. Then the Riemannian curvature tensor R is given by*

$$\begin{aligned}(R(X^*, Y^*)Z^*)_o &= -[[X,Y]_{\mathfrak{h}}, Z] - \frac{1}{2}[[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} \\ &\quad -\frac{1}{4}[[Y,Z]_{\mathfrak{m}}, X]_{\mathfrak{m}} - \frac{1}{4}[[Z,X]_{\mathfrak{m}}, Y]_{\mathfrak{m}}\end{aligned}$$

for $X, Y, Z \in \mathfrak{m}$.

Then we have

$$(R(X^*, Y^*)Y^*)_o = -[[X,Y]_{\mathfrak{h}}, Y] - \frac{1}{4}[[X,Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}}$$

and the sectional curvature is given by

$$\langle R(X^*, Y^*)Y^*, X^*\rangle_o = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle + \langle [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y\rangle.$$

If $(G/H, \langle\ ,\ \rangle)$ is normal homogeneous

$$\langle R(X^*, Y^*)Y^*, X^*\rangle_o = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle + \langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}}\rangle \geq 0.$$

2.5 Symmetric spaces

Definition 19. *A connected Riemannian manifold (M, g) is called a* ***Riemannian (globally) symmetric space*** *if*

$$\forall p \in M, \exists s_p \in I(M,g) \text{ s.t. } {s_p}^2 = id., \quad p \text{ is an isolated fixed point of } s_p$$

s_p is the geodesic symmetry:

$$s_p(\gamma_v(t)) = \gamma_v(-t),$$

where $\gamma_v(t)$ is the geodesic s.t.

$$\gamma_v(0) = p, \quad {\gamma_v}'(0) = v \in T_pM.$$

The connected Lie group $G = I_0(M, g)$ (identity component) acts transitively on M, and M becomes a Riemannian homogeneous space G/H, where a closed subgroup H is an isotropy subgroup at $p_0 \in M$. The map

$$\sigma : G \to G, \quad a \mapsto s_{p_0} \circ a \circ s_{p_0}$$

is an involutive automorphism of G, and

$$G_0^\sigma \subset H \subset G^\sigma,$$

where

$$G^\sigma = \{a \in G \mid \sigma(a) = a\}, \qquad G_0^\sigma \text{ is the identity component of } G^\sigma.$$

Such a pair (G, H) is called a **symmetric pair**.

Examples

$M = G/H$	G	H	
S^n	$SO(n+1)$	$SO(n)$	sphere
$Gr_k(\mathbb{R}^n)$	$O(n)$	$O(k) \times O(n-k)$	real Grassmann mfd.
$Gr_k(\mathbb{C}^n)$	$U(n)$	$U(k) \times U(n-k)$	complex Grassmann mfd.

The Lie algebra $\mathfrak{g}$ may be decomposed into ± 1-engenspaces of the differential $d\sigma : \mathfrak{g} \to \mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad d\sigma|_{\mathfrak{h}} = id, \quad d\sigma|_{\mathfrak{m}} = -id$$

Then we see that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

and hence the canonical connection D coincides with the Riemannian connection ∇, thus

$$\begin{aligned} &\nabla R = 0, \\ &R(X, Y)Z = -[[X, Y], Z] \end{aligned}$$

for $X, Y, Z \in \mathfrak{m}$.

2.6 The root system of a symmetric space

Let $M = G/H$ be a Riemannian symmetric space. Then as before, we have the $\mathrm{Ad}(H)$-invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

We take a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}$. For any $H \in \mathfrak{a}$, we consider the linear maps $\mathrm{ad}(H) : \mathfrak{g} \to \mathfrak{g}$, which are simultaneously diagonalizable. If there exists $X \in \mathfrak{g}$ such that

$$\mathrm{ad}(H)X = \alpha(H)X$$

for all $H \in \mathfrak{a}$ for some nonzero linear form $\alpha \in \mathfrak{a}^*$, α is called a **root** of M relative to $\mathfrak{a}$. The set Σ of all roots of M relative to $\mathfrak{a}$ is called a restricted root system of M.

Σ can be obtained as follows: take a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{h}^{\mathbb{C}} \supset \mathfrak{a}$, let Δ be the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$. Then Σ coinsides with $\{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta\}$ (as well as their multiplicities).

3 Homogeneous structures

When does a Riemmanian manifold (M, g) become a Riemannian homogeneous manifold?

W. Ambrose and I. M. Singer characterized it by the existense of some tensor field T of type $(1, 2)$ on M, which is called a **homogeneous structure**.

Theorem 3.1. *([1]) Let (M, g) be a connected, simply connected, complete Riemannian manifold. (M, g) is a Riemannian homogeneous space if and only if there exists a tensor field T of type $(1, 2)$ on M satisfying*

(1) $g(T(X)Y, Z) + g(Y, T(X)Z) = 0$

(2) $\nabla_X R = T(X) \cdot R$

(3) $\nabla_X T = T(X) \cdot T$

where ∇ and R denotes the Riemannian connection and the Riemannian curvature tensor of (M, g), respectively.

$$\tilde{\nabla} := \nabla - T$$

is called a A–S connection. (1)–(3) are equivalent to $\tilde{\nabla} g = 0$, $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$.

Remark. Here, we consider the tensor field $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ of type $(1, 2)$ as, for $X \in \mathfrak{X}(M)$,

$$T(X) : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad Y \mapsto T(X)Y$$

and

$$\begin{aligned}
(T(X) \cdot R)(Y, Z)W &:= T(X)(R(Y, Z)W) - R(T(X)Y, Z)W \\
&\quad - R(Y, T(X)Z)W - R(Y, Z)T(X)W \\
(T(X) \cdot T)(Y)Z &:= T(X)T(Y)Z - T(T(X)Y)Z - T(Y)T(X)Z
\end{aligned}$$

About the homogeneity of almost Hermitian manifold (M, g, J), (J is an almost complex structure; $J^2 = -I$), the following result was shown by K. Sekigawa.

Theorem 3.2. *([6]) Let (M, g, J) be a connected, simply connected, complete almost Hermitian manifold. M is a homogeneous almost Hermitian manifold if and only if there exists a tensor field T of type $(1, 2)$ on M satisfying*

(1) $g(T(X)Y, Z) + g(Y, T(X)Z) = 0$

(2) $\nabla_X R = T(X) \cdot R$

(3) $\nabla_X T = T(X) \cdot T$

(4) $\nabla_X J = T(X) \cdot J$

An **almost contact metric manifold** (M, ϕ, ξ, η, g) is a C^∞ manifold M with an almost contact metric structure (ϕ, ξ, η, g) such that
(1) ϕ is a tensor field of type $(1, 1)$,
(2) $\xi \in \mathfrak{X}(M)$ is called a characteristic vector field,
(3) η is a 1-form,
(4) g is a Riemannian metric
satisfying

$$
\begin{aligned}
&\phi^2 X = -X + \eta(X)\xi, \\
&\eta(\xi) = 1, \\
&g(X, \xi) = \eta(X), \\
&g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\end{aligned}
$$

About the homogeneity of (M, ϕ, ξ, η, g), the following result was shown.

Theorem 3.3. *([5]) Let (M, ϕ, ξ, η, g) be a connected, simply connected, complete almost contact metric manifold. M is a homogeneous almost contact metric manifold if and only if there exists a skew-symmetric tensor field T of type $(1, 2)$ on M satisfying*

(1) $g(T(X)Y, Z) + g(Y, T(X)Z) = 0$

(2) $\nabla_X R = T(X) \cdot R$

(3) $\nabla_X T = T(X) \cdot T$

(4) $\nabla_X \eta = -\eta \circ T(X)$

(5) $\nabla_X \phi = T(X) \cdot \phi$

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