1 Introduction

Given a domain $D$ in $\mathbb{R}^2$, it is well known that the area $A$ of $D$ and the length $L$ of $\partial D$ satisfy

$$4\pi A \leq L^2$$

and that equality holds if and only if $D$ is a disk. This isoperimetric inequality is perhaps the most beautiful inequality in geometry. In the hope of generalizing (1) one can ask the following question. Are there any surfaces that satisfy (1)? There are two natural candidates for this question: flat surfaces and minimal surfaces.

A flat surface is, by definition, a two-dimensional surface with flat metric. Therefore a flat surface can be obtained from a generalized domain in $\mathbb{R}^2$ with some identifications along the boundary. Then, does every flat surface satisfy the isoperimetric inequality (1.1)? To this very natural question one can easily find a counterexample: a long cylinder. Let $ABCD$ be a rectangle in $\mathbb{R}^2$. Then identifying the parallel sides $AD$ and $BC$ gives rise to a cylinder. Identifying the sides decreases the circumference of the rectangle, thereby causing the isoperimetric inequality (1.1) to fail if $AD$ is sufficiently longer than $AB$.

Being locally area minimizing and having zero mean curvature, minimal surfaces in $\mathbb{R}^n$ are thought of as generalized planes. Therefore it has long been conjectured that minimal surfaces should satisfy (1.1) as well.

In this note we will prove the isoperimetric inequality for some flat surfaces. And we will see how the isoperimetric inequality for some minimal surfaces in $\mathbb{R}^3$ can be derived from that of the associated flat surfaces.

2 flat surfaces

There are dozens of proofs for the isoperimetric inequality in $\mathbb{R}^2$. To cite a few, see Steiner [Sp, p.439], [Cv, p.283], Bonnesen [Os, p.1199], Hurwitz [Os, p.1184], Brunn-Minkowski [Os, p.1190], Hadwiger [Ha, p.153], Knothe [T, p.22], Schmidt
[dC, p.32], Gromov [Cv, p.276], and Hélein [He]. Among these Knothe’s proof will be used in this section.

**Definition.** A flat surface is an open surface with or without boundary which has a flat metric. Every flat surface can be isometrically immersed as a generalized domain in $\mathbb{R}^2$. This generalized domain may have multiplicity and branch points, and some parts of its boundary may be identified. Therefore a flat surface can be obtained as a layered surface by applying cutting and pasting to separate pieces of paper (=domain) and by identifying some parts of the boundary.

**Theorem 1.** Let $F$ be a flat surface with nonempty boundary and suppose that the rays emanating from each boundary point of $F$ never intersect each other. Then $F$ satisfies the classical isoperimetric inequality $4\pi A \leq L^2$, and equality holds if and only if $F$ is a disk in $\mathbb{R}^2$.

Proof. There are two ways to find the area of a domain by scanning the interior from its boundary points. First, let $D$ be a convex domain in $\mathbb{R}^2$, $p$ a boundary point, and $\rho$ a ray tangent to $\partial D$ at $p$ in the counterclockwise direction along $\partial D$. Let $(r, \theta)$ be the polar coordinates with $r$ the distance from $p$ and $\theta$ the angle measured from $\rho$. Define $r(\theta) = r$ if there exists a boundary point with polar coordinates $(r, \theta)$. Then
\[ A = \frac{1}{2} \int_0^{2\pi} r(\theta)^2 d\theta. \] (2.2)

If $D$ is not convex, define $r(\theta) = \min\{r : (r, \theta) are polar coordinates of a boundary point\}$. Then we just have
\[ A \geq \frac{1}{2} \int_0^{2\pi} r(\theta)^2 d\theta. \] (2.3)

The second way is by Crofton’s formula [Sa]. Parametrize $\Gamma = \partial D$ by arclength $s$, $0 \leq s \leq L = \text{Length}(\Gamma)$. Let $(r(s), \theta(s))$ be the polar coordinates measured with respect to the point $\Gamma(s)$ and the ray tangent to $\Gamma$ at $\Gamma(s)$. Define $r(s, \theta) = \min\{r(s) : (r(s), \theta(s)) are polar coordinates of a point in \Gamma\}$. Then Crofton’s formula states, whether $D$ is convex or not,
\[ 2\pi A = \int_0^\pi \int_0^L r(s, \theta) \sin \theta \, ds \, d\theta. \] (2.4)
In general, (2.2) and (2.3) do not hold on a flat surface. This is because the scanning map from a fixed boundary point may not be one-to-one on a flat surface. But if no two rays emanating from any boundary point intersect each other, which is guaranteed by the hypothesis of this theorem, then (2.3) holds. However, (2.4) holds even without this hypothesis for flat surfaces. Therefore, using (2.3) and (2.4) and integrating on $\partial D \times \partial D$, we have

\[
0 \leq \int_0^\pi \int_0^\pi (r_1 \sin \theta_2 - r_2 \sin \theta_1)^2 \, d\theta_1 \, d\theta_2 \, ds_1 \, ds_2
\]

\[
= 2 \int_0^\pi \int_0^\pi \int_0^\pi r_1^2 \sin^2 \theta_2 \, ds_1 \, d\theta_1 \, d\theta_2 \, ds_2
\]

\[
-2 \int_0^\pi \int_0^\pi \int_0^\pi r_1 \sin \theta_1 r_2 \sin \theta_2 \, ds_1 \, d\theta_1 \, d\theta_2 \, ds_2
\]

\[
\leq 4A \int_0^\pi \int_0^\pi \sin^2 \theta_2 \, ds_1 \, d\theta_2 \, ds_2 - 2 \left( \int_0^\pi r_1 \sin \theta_1 \, d\theta_1 \right)^2
\]

\[
= 2\pi A(L^2 - 4\pi A).
\]

$4\pi A = L^2$ implies that

\[
r_1 \sin \theta_2 - r_2 \sin \theta_1 = 0
\]

for all values of $\theta_1$, $\theta_2$, $s_1$ and $s_2$. Hence for $s_1 = s_2 = 0$ and $\theta_2 = \pi/2$ we have

\[
r_1(0, \theta_1) = r_2(0, \pi/2) \sin \theta_1,
\]

therefore $\partial D$ is a circle with diameter $r_2(0, \pi/2)$. □

We now turn to the second condition which guarantees the isoperimetric inequality for the flat surfaces.

Let $C$ be a smooth immersion of a circle in a flat surface $F$. The rotation number of $C$ in $F$ is defined as follows. Suppose $C$ is parametrized by arclength $0 \leq s \leq \ell$. Since $F$ is locally in $\mathbb{R}^2$, $C'(s)$ is a well-defined map from $[0, \ell]$ into the unit circle $S^1$.

**Definition.** The rotation number of $C$ in $F$ is defined to be the degree of the map $C'(s) : [0, \ell] \to S^1$. The rotation number is not necessarily an integer.

**Theorem 2.** If a flat surface with integer rotation number $F$ contains no straight loop (i.e., a loop with vanishing curvature), then $4\pi A \leq L^2$ holds for $F$, equality holding if and only if $F$ is a disk.

**Proof.** See [CS].
3 Minimal Surfaces

A minimal surface is not flat; its Gaussian curvature is negative except possibly at isolated flat points. However, a minimal surface is locally area minimizing away from isolated singular points. It is in this sense that a minimal surface is called a generalized plane and it is for this reason that one conjectures that a minimal surface should satisfy the same isoperimetric inequality as in $\mathbb{R}^2$: $4\pi A \leq L^2$.

In this note we will show how to obtain from a minimal surface two flat surfaces whose area and boundary length appropriately control those of the minimal surface.

Given a minimal surface $\Sigma$ in $\mathbb{R}^3$, one can construct three flat surfaces from $\Sigma$ by way of the Weierstrass representation formula as we shall see below.

Let $\Sigma$ be a surface in $\mathbb{R}^3$ and let $x_1, x_2, x_3$ be the rectangular coordinates of $\mathbb{R}^3$. If we denote the vector-valued function $\Psi$ on $\Sigma$ by $\Psi = (x_1, x_2, x_3)$, then

$$\Delta \Psi = (\Delta x_1, \Delta x_2, \Delta x_3) = 0,$$

where the Laplacian is taken with respect to the metric $ds^2$ of $\Sigma$ and $\vec{H}$ is the mean curvature vector of $\Sigma$. Hence if $\Sigma$ is a minimal surface, then $x_1, x_2, x_3$ are harmonic on $\Sigma$. Let $z = x + iy$ be a complex coordinate on $\Sigma$ such that

$$ds^2 = 2\lambda |dz|^2.$$

Then

$$\Delta \Psi = \frac{2}{\lambda} \Psi_{zz} = 0.$$

Define

$$(\varphi_1, \varphi_2, \varphi_3) = 2\left(\frac{dx_1}{dz}, \frac{dx_2}{dz}, \frac{dx_3}{dz}\right).$$

From (3.5) it follows that $\varphi_1, \varphi_2, \varphi_3$ are holomorphic in $z$. Furthermore, since $\Psi : (x, y) \mapsto (x_1, x_2, x_3)$ is a conformal map, we have

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = |\Psi_x|^2 - |\Psi_y|^2 - 2i < \Psi_x, \Psi_y > = 0.$$

Now, defining a holomorphic function $f$ and a meromorphic function $g$ by

$$f = \varphi_1 - i\varphi_2, \quad g = \frac{\varphi_3}{\varphi_1 - i\varphi_2},$$

we obtain the Weierstrass representation formula for $\Sigma$:

$$(x_1, x_2, x_3) = \text{Re} \left(\frac{1}{2} \int_1^z f(1 - g^2) \, dz - \frac{i}{2} \int_1^z f(1 + g^2) \, dz, \int_1^z fg \, dz\right).$$

Note here that

$$4\lambda = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 = \frac{|f|^2(1 + |g|^2)^2}{2}.$$
and that the Gaussian curvature $K$ of $\Sigma$ is given by

$$K = -\frac{1}{2} \Delta \log \lambda \leq 0.$$  

However, since $\log |f|^2$, $\log |fg|^2$, and $\log |fg|^2$ are harmonic we can define three flat metrics $g_1$, $g_2$, $g_3$ on $\Sigma$:

$$g_1 = \frac{1}{4} |f|^2 |dz|^2, \quad g_2 = \frac{1}{2} |fg|^2 |dz|^2, \quad g_3 = \frac{1}{4} |fg|^2 |dz|^2.$$  

Let $F_i$ be the flat surface $\Sigma$ with metric $g_i$, $i = 1, 2, 3$. $F_i$ can be constructed as a generalized domain in $\mathbb{R}^2$ by complex integration on $\Sigma$ as follows.

$$(3.7) \quad F_1 : \int z^1 \frac{1}{2} f \, dz, \quad F_2 : \int z^1 \frac{1}{2} fg \, dz, \quad F_3 : \int z^1 \frac{1}{2} fg^2 \, dz.$$  

In fact, $F_1$ and $F_3$ are the limits of the sequences of minimal surfaces $\{\Sigma^1_n\}$ and $\{\Sigma^3_n\}$, respectively, defined as follows.

$$\Sigma^1_n : \text{Re} \left( \frac{1}{2} \int z^1 f(1 - \frac{g^2}{n^2}) \, dz, \frac{i}{2} \int z^1 f(1 + \frac{g^2}{n^2}) \, dz, \int z^1 \frac{fg}{n} \, dz \right),$$

$$\Sigma^3_n : \text{Re} \left( \frac{1}{2} \int z^1 f(1 - n^2 g^2) \, dz, \frac{i}{2} \int z^1 f(1 + n^2 g^2) \, dz, \int z^1 \frac{fg}{n} \, dz \right),$$

which are obtained from $\Sigma$ by changing its Weierstrass data from $f$ and $g$ to $f$ and $g/n$, $f/n$ and $ng$, respectively. Alternatively, $F_1$ and $F_3$ can be introduced as follows. Let $\bar{x}_j$ be the harmonic conjugate of $x_j$, $j = 1, 2$, and define $x_{j}^* = x_j + i\bar{x}_j$. Then we have the two maps $x_{j}^* - ix_j$ and $-x_{j}^* - ix_j$ from $\Sigma$ into $\mathbb{C}$, and we can easily see that the images of these maps are the flat surfaces $F_1$ and $F_3$.

The following theorem states that $F_1$ and $F_3$ play a pivotal role for the isoperimetric inequality of the minimal surface $\Sigma$.

**Theorem 3.** If the flat surfaces $F_1$ and $F_3$ of a minimal surface $\Sigma$ in $\mathbb{R}^3$ both satisfy the isoperimetric inequality $4\pi A \leq L^2$, then so does $\Sigma$.

**Proof.** Let $A = \text{Area}(\Sigma)$, $A_i = \text{Area}(F_i)$, $L = \text{Length}(\partial \Sigma)$, $L_i = \text{Length}(\partial F_i)$. Then

$$A = \int_{\Sigma} \frac{1}{4} |f|^2(1 + |g|^2)^2 \, dx \, dy$$

$$= \int_{\Sigma} \frac{1}{4} |f|^2 \, dx \, dy + \int_{\Sigma} \frac{1}{2} |fg|^2 \, dx \, dy + \int_{\Sigma} \frac{1}{4} |fg|^2 \, dx \, dy$$

$$= A_1 + A_2 + A_3.$$
and
\[ L = \int_{\partial \Sigma} \frac{1}{2} |f|(1 + |g|^2) \, ds = \int_{\partial \Sigma} \frac{1}{2} |f| \, ds + \int_{\partial \Sigma} \frac{1}{2} |fg|^2 \, ds = L_1 + L_3. \]

Moreover, by the Hölder inequality,
\[ (A_2)^2 = \left( \int_{\Sigma} \frac{1}{2} |fg|^2 \, dx \, dy \right)^2 \leq \int_{\Sigma} \frac{1}{2} |f|^2 \, dx \, dy \int_{\Sigma} \frac{1}{2} |fg|^4 \, dx \, dy = 4A_1A_3. \]

Hence
\[ 4\pi A \leq 4\pi A_1 + 8\pi \sqrt{A_1 A_3} + 4\pi A_3 \leq (L_1)^2 + 2L_1 L_3 + (L_3)^2 = (L_1 + L_3)^2 = L^2. \]

Using this theorem we will prove the isoperimetric inequality for some minimal surfaces in \( \mathbb{R}^3 \) in the following theorems. See [CS] for the proofs.

**Theorem 4.** If \( \Sigma \subset \mathbb{R}^3 \) is a minimal surface with fluxes parallel to a line, then \( \Sigma \) satisfies \( 4\pi A \leq L^2 \).

**Theorem 5.** Let \( \Sigma \) be a triply connected minimal surface in \( \mathbb{R}^3 \), that is, \( \Sigma \) has three boundary components and no genus. Then \( \Sigma \) satisfies the isoperimetric inequality \( 4\pi A \leq L^2 \).

**Definition.** Let \( S \subset \mathbb{R}^3 \) be a compact orientable surface whose boundary \( \partial S \) is the union of Jordan curves \( \gamma_1, ..., \gamma_n \). We say that \( \partial S \) is non-twisted if an \( \epsilon \)-tubular neighborhood of \( \gamma_k \) in \( \Sigma \) can be deformed to a trivial strip for all \( k = 1, ..., n \).

**Theorem 6.** If \( \Sigma \) is a compact minimal surface in \( \mathbb{R}^3 \) with non-twisted boundary, then \( \Sigma \) satisfies \( 4\pi A \leq L^2 \).

**References**


J. Choe and R. Schoen, *The sharp isoperimetric inequality for a minimal surface in Euclidean space*.


