Lagrangian submanifolds with totally geodesic foliation in complex projective space

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Abstract. As a generalization of ruled surfaces, we study Lagrangian submanifolds with 1-parameter family of totally geodesic (n − 1)-dimensional totally real, totally geodesic real projective spaces in $\mathbb{CP}^n$.

1 Introduction

Lagrangian submanifolds have been important geometric objects of study in symplectic geometry. The problem of minimizing the volume of Lagrangian submanifolds under Hamiltonian deformations was proposed by Oh [5], and critical points of the problem, Hamiltonian minimal submanifolds are interesting and important objects among Lagrangian submanifolds (cf. [1]). Using Oh’s result, we can see that compact Lagrangian submanifold $M$ in a Kähler manifold $\tilde{M}$ is Hamiltonian minimal if and only if tangent vector field $JH$ is divergence free, where $J$ and $H$ denote complex structure of $\tilde{M}$ and mean curvature vector field of $M$, respectively. In this note, we study Lagrangian submanifolds with some symmetry, which is obtained by $p$-parameter family of totally geodesic, totally real projective space $\mathbb{RP}^{n-p}$ in complex projective space $\mathbb{CP}^n$, by using some fiber bundles. We consider the case $p = 1$ precisely and for the cases $p \geq 2$, we will discuss elsewhere.

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2 Hamiltonian minimal Lagrangian submanifolds

First we recall about Hamiltonian deformation of Lagrangian submanifolds in Kähler manifolds, defined by Oh [5]. Let $\tilde{M}$ be a complex $n$-dimensional Kähler manifold with Kähler form $\omega$, Riemann metric $\langle \cdot, \cdot \rangle$, and complex structure $J$. Let $x : M \to \tilde{M}$ be a Lagrangian immersion from a real $n$-dimensional manifold $M$ to $\tilde{M}$, i.e., $\omega|_{TM} = 0$. For a vector field $V$ along $x$, we define a 1-form $\alpha_V$ on $M$ as $\alpha_V = \langle J V, \cdot \rangle|_{TM}$. Smooth family of embeddings $\iota_t : M \to P$ is called
**Hamiltonian deformation** if for the variational vector field $V$, the 1-form $\alpha_V$ is exact. A Lagrangian submanifold $M$ is **Hamiltonian minimal** (or $H$-minimal) if $M$ is stationary for any Hamiltonian deformation. Oh [5] showed that when $M$ is compact, $M$ is H-minimal if and only if $\alpha_H$ is co-closed, i.e., $\delta \alpha_H = 0$ where $H$ is the mean curvature vector field of $M$. We have

\[(2.1) \quad M \text{ is Hamiltonian minimal } \Leftrightarrow \ \text{div } JH = 0.\]

With respect to Lagrangian submanifold $M$ in a Kähler manifold $\tilde{M}$ and the induced metric $g$ on $M$, the following relations hold:

\[(2.2) \quad \nabla \sigma = 0 \Rightarrow \nabla \perp H = 0 \Rightarrow \text{div } JH = 0,\]

where $\sigma$ and $\nabla \perp$ denote second fundamental form and normal connection of $M$ in $\tilde{M}$ respectively, and $\nabla \sigma$ is defined by $(\nabla_X \sigma)(Y, Z) = \nabla_X \perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ for tangent vector fields $X, Y, Z$ on $M$. We note that $\nabla_X (JH) = J \nabla \perp X H$.

Let $\tilde{M}^n(4c)$ be an $n$-dimensional complex space form with constant holomorphic sectional curvature $4c$ and let $M = M^n$ be a Lagrangian submanifold in $\tilde{M}^n(4c)$. We note that Lagrangian submanifolds $M^n$ with parallel second fundamental form in complex space forms are classified by Naitoh [2], [3]. By (2.1) and (2.2), they are Hamiltonian minimal.

### 3 A generalization of ruled surfaces to Lagrangian submanifolds in $\mathbb{C}P^n$

Ruled surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$ are obtained as a 1-parameter family of straight lines in $\mathbb{R}^3$. As a generalization, we consider $n$-dimensional Lagrangian submanifolds with $p$-parameter family of totally geodesic $\mathbb{R}P^{n-p}$ in $\mathbb{C}P^n$.

Let

\[(3.1) \quad \tilde{M}_{p,n} := \{ V \mid V \text{ is a real linear subspace of } \mathbb{C}^{n+1}, \dim_{\mathbb{R}} V = n - p + 1, \sqrt{-1} V \perp V \},\]

where $\sqrt{-1} V \perp V$ means that $V$ and $\sqrt{-1} V$ are orthogonal with respect to standard real Euclidean inner product on $\mathbb{C}^{n+1}$. $S^1$ acts on $\tilde{M}_{p,n}$ by the multiplication of unit complex numbers. Then we can see that the quotient space

\[(3.2) \quad M_{p,n} := \tilde{M}_{p,n} / S^1,\]

is naturally identified with the set of totally geodesic, totally real $\mathbb{R}P^{n-p}$ in $\mathbb{C}P^n$.

Let $\Sigma^p$ be a real $p$-dimensional submanifold in $M_{p,n}$. Then we can construct real $n$-dimensional submanifold $M^n$ (which may have some singularities), with $p$-parameter family of totally geodesic, totally real $\mathbb{R}P^{n-p}$, in $\mathbb{C}P^n$.

Let

\[\tilde{E}_{p,n} := \{ (\ell, W) \mid \ell \text{ is a real 1-dimensional subspace in } \mathbb{C}^{n+1}, W \text{ is a real } (n - p)\text{-dimensional subspace in } \mathbb{C}^{n+1}, \ell \perp CW \},\]
Lagrangian geodesic foliation

and let

\[ E_{p,n} := \tilde{E}_{p,n}/S^1 \]

be the quotient by the action of \( S^1 \) on \( \tilde{E}_{p,n} \) as (3.1) and (3.2). Then \( E_{p,n} \) is a total space of \( \mathbb{R}P^{n-p} \)-bundle over \( \mathcal{M}_{p,n} \) with respect to the projection

\[ \pi_E : E_{p,n} \to \mathcal{M}_{p,n}, \quad [\ell, W] \mapsto [\ell + W]. \]

We consider another projection:

\[ pr_1 : E_{p,n} \to \mathbb{C}P^n, \quad [\ell, W] \mapsto C\ell. \]

With respect to the diagram

\[ E_{p,n} @ >> pr_1 > \mathbb{C}P^n \]

\[ V \pi_E VV, \]

\[ \mathcal{M}_{p,n} \]

for each \( x \in \mathcal{M}_{p,n}, pr_1(\pi_E^{-1}(x)) \) is identified with totally real, totally geodesic \( \mathbb{R}P^{n-p} \) in \( \mathbb{C}P^n \).

Let \( \varphi : \Sigma \to \mathcal{M}_{p,n} \) be an immersion from a real \( p \)-dimensional \( \Sigma = \Sigma^p \) to \( \mathcal{M}_{p,n} \), and let \( M = \varphi^*E_{p,n} \) be the pull-back bundle of \( \pi_E : E_{p,n} \to \mathcal{M}_{p,n} \) by \( \varphi \). We have the following diagram:

\[ M = \varphi^*E_{p,n} @ >> \eta > E_{p,n} @ >> pr_1 > \mathbb{C}P^n \]

\[ V \pi_E VV, \]

\[ \mathcal{M}_{p,n} \]

where \( \eta \) denotes the bundle map. Then with respect to \( \Phi = pr_1 \circ \eta : M \to \mathbb{C}P^n \), the image \( \Phi(M) \) is union of \( p \)-parameter family of \( \mathbb{R}P^{n-p} = pr_1(\pi_E^{-1}(\varphi(x))) \) for each \( x \in \Sigma^p \).

We consider the case of \( p = 1 \). Let \( p_M : \mathcal{M}_{1,n} \to \mathbb{C}P^n \) be a projection, defined by \( \mathbb{R}P^{n-1} \to \mathbb{C}P^{n-1} \), where we identify \( \mathbb{C}P^n \) with the set of complex projective hypersplanes \( \mathbb{C}P^{n-1} \subset \mathbb{C}P^n \). Then for \( \mathbb{R}P^{n-1} \in \mathcal{M}_{1,n} \), corresponding \( \mathbb{C}P^{n-1} \) is uniquely determined by which \( \mathbb{R}P^{n-1} \subset \mathbb{C}P^{n-1} \). We note that both \( \mathcal{M}_{1,n} \) and \( \mathbb{C}P^n \) are homogeneous spaces with respect to \( U(n + 1) \), and so Riemannian metrics are defined on both spaces such that \( p_M \) is a Riemannian submersion.

First result is (cf. [1], Proposition 3.1):

**Proposition 3.1.** Let \( \varphi : I \to \mathcal{M}_{1,n} \) be a real 1-dimensional curve (\( I \) is an interval). Then the corresponding map \( \Phi : M^n = \varphi^*E_{1,n} \to \mathbb{C}P^n \) is a Lagrangian immersion (on the open subset of regular points of \( \Phi \)) if and only if the image \( \varphi(I) \subset \mathcal{M}_{1,n} \) is ‘horizontal’ with respect to Riemannian submersion \( p_M : \mathcal{M}_{1,n} \to \mathbb{C}P^n \).
Hence from a curve in $\mathbb{CP}^n$, by taking its horizontal lift to $\mathcal{M}_{1,n}$, we obtain Lagrangian submanifold with 1-parameter family of totally geodesic, totally real $\mathbb{RP}^{n-1}$.

Using this construction, we can see that minimal Lagrangian submanifolds with 1-parameter family of totally geodesic, totally real $\mathbb{RP}^{n-1}$ in $\mathbb{CP}^n$ is totally geodesic. More precisely (cf. [1], Theorem 4.1):

**Theorem 3.2.** Let $\varphi : I \to \mathcal{M}_{1,n}$ be a curve, which is horizontal with respect to Riemannian submersion $p_M : \mathcal{M}_{1,n} \to \mathbb{CP}^n$, and let $\Phi : M^n = \varphi^* E_{1,n} \to CP^n$ be the corresponding Lagrangian immersion. If $\Phi$ is minimal, then $\Phi$ is totally geodesic.

Finally with respect to Hamiltonian minimal Lagrangian submanifolds, we have (cf. cf. [1], Theorem 4.2):

**Theorem 3.3.** Let $\varphi : I \to \mathcal{M}_{1,n}$ be a curve, which is horizontal with respect to Riemannian submersion $p_M : \mathcal{M}_{1,n} \to \mathbb{CP}^n$, and suppose $\varphi(I)$ is a part of orbit of 1-parameter subgroup of $U(n+1)$. If the corresponding Lagrangian immersion $\Phi : M^n = \varphi^* E_{1,n} \to CP^n$ is Hamiltonian minimal, then $\Phi(M)$ is either totally geodesic, or $\varphi(I)$ is a part of orbit of ‘isotropic vector’ in the Lie algebra $\mathfrak{u}(n+1)$.

When $\dim \mathbb{R}M \geq 3$, $M$ must have some singularities, but when $\dim \mathbb{R}M = 2$, the Lagrangian surface $M^2$ in $\mathbb{CP}^2$ is everywhere regular and flat, i.e., the Gaussian curvature $K$ is identically zero and has parallel mean curvature vector $H \neq 0$ with respect to the normal connection. Ogata [4] showed that such immersions are obtained as orbits of Abelian Lie subgroup of $U(3)$.

**References**