Real hypersurfaces in complex space forms with commuting structure Jacobi operator

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Abstract. We introduce recent results of Nikolayevsky on the Osserman conjecture and prove the non existence of real hypersurfaces in complex space forms whose structure Jacobi operator commutes with any other Jacobi operator.

1 On the Osserman conjecture.

A two point homogeneous manifold is a Riemannian manifold \((M, g)\) such that for every two pairs of points \(p_1, q_1\) and \(p_2, q_2\) such that the distances \(d(p_1, q_1), d(p_2, q_2)\) are equal, there is an isometry of the manifold taking \(p_1\) to \(p_2\) and \(q_1\) to \(q_2\). In particular, if \(p_1 = p_2\), then for any two points \(q_1, q_2\) of a geodesic sphere centered at \(p_1\), there is an isometry of the manifold fixing the point \(p_1\), hence necessarily taking the sphere to itself, and mapping \(q_1\) to \(q_2\).

Compact two point homogeneous manifolds are standard spheres and real projective spaces, the complex and quaternionic projective spaces and the projectivized Cayley plane, all with their standard Riemannian metrics.

Let \(M\) be a manifold embedded in Euclidean space. Let \(C\) be a curve in \(M\), and let \(v\) be a vector field along \(C\), that is, at each point \(x(t)\) of \(C\), \(v(t)\) is a tangent vector to \(M\) at \(x(t)\). Then \(v\) is said to be parallel along \(C\) if \(v'(t)\) is normal to \(M\) at \(x(t)\), for all \(t\). It is known that given \(C\) and a tangent vector \(v_0\) at a point \(x(t_0)\) of \(C\), there is a unique parallel vector field \(v(t)\) along \(C\) with \(v(t_0) = v_0\). If \(v(t), w(t)\) are both parallel along \(C\), the \(v(t), w(t)\) is constant. It follows that if \(\pi_0\) is a two-dimensional tangent plane at a point \(x(t_0)\) of \(C\), then we can define parallel two-planes along \(C\) by taking a pair of orthonormal tangent vectors \(v_0, w_0\) spanning \(\pi_0\), letting \(v(t), w(t)\) be the parallel fields along \(C\) extending \(v_0, w_0\) and defining \(\pi(t)\) to be the plane spanned by \(v(t), w(t)\), which are necessarily orthonormal for all \(t\).

A Riemannian manifold is locally symmetric if the sectional curvature is con-
stant along every family of parallel planes $\pi(t)$.

A locally symmetric space has rank one if none of its sectional curvatures is zero.

The two point homogeneous spaces are precisely the Euclidean spaces $\mathbb{R}^n$, the real projective spaces $\mathbb{R}P^n$, and the complete simply-connected locally symmetric spaces of rank one.

Now if $v$ is a unit tangent vector to $M$ at $p$, let $\pi_v$ be the hyperplane in the tangent space orthogonal to $v$. If $R$ is the curvature tensor of $M$ and the dimension of $M$ is $n$, we define the Jacobi operator $R_v$ at $p$ as the endomorphism of $T_p M$ given by $R_v(w) = R(w, v)v$, for any $w \in T_p M$. Thus $R_v(v) = 0$. If we restrict $R_v$ to $\pi_v$, let $\lambda_1(v), \ldots, \lambda_{n-1}(v)$ be the corresponding eigenvalues of the Jacobi operator $R_v$.

Osserman, [14], posed the following question:

(*) For which manifolds the set of eigenvalues $\lambda_1(v), \ldots, \lambda_{n-1}(v)$ is the same for all $p$ and $v$?

Osserman pointed out that for rank one locally symmetric spaces the values of $\lambda_i(v)$ are known explicitly: There are just two distinct values with ratio $1/4$, and they are independent of $p$ and $v$. Thus all such spaces satisfy Condition (*). So he led to

OSSERMAN CONJECTURE: The only nonflat manifolds satisfying Condition (*) are the rank one locally symmetric spaces.

Chi, [3], notes that for a rank one symmetric space

1. $R_v$ have two distinct constant eigenvalues $(1, 1/4)$ for all $v$ if the space is of not constant curvature, and

2. $E_1(v)$, the linear space spanned by $v$ and the eigenspace of $R_v$ with eigenvalue $1$ is the tangent space of a totally geodesic sphere of curvature $1$ through the base point of $v$, and consequently, $E_1(w) = E_1(v)$ whenever $w$ is in $E_1(v)$.

He adapted these properties to the following axioms:

1. Let $R_v$ be the Jacobi operator for $v$ in $SM$, the unit sphere bundle of $M$. Then $R_v$ has precisely two different constant eigenvalues independent of $v$ (counting multiplicities).

2. Let $b, c$ be the two eigenvalues. For $v \in SM$, denote by $E_c(v)$ the subspace spanned by $v$ and the eigenspace of $R_v$ with eigenvalue $c$. Then $E_c(w) = E_c(v)$ whenever $w \in E_c(v)$.

Then he proves the following

**Theorem 1.1.** Locally rank-one symmetric spaces not of constant curvature are characterized by the two axioms.

So he concludes that the conjecture should be true if it implied his two axioms.

In [2] he proved the conjecture for manifolds of dimension 4 or of dimension distinct of $4k$. 
In 2003, Nikolayevsky considered curvature-like tensors in a Euclidean space $\mathbb{R}^n$. They are $(3, 1)$-type tensors having the same symmetries as the curvature tensor of a Riemannian manifold. He introduces similarly the Jacobi operator in the direction of an $X \in \mathbb{R}^n$ for such a curvature-like tensor $R$ and defines:

**DEFINITION 1:** A curvature-like tensor $R$ is Osserman if the eigenvalues of the Jacobi operator $R_X$ do not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

**DEFINITION 2:** A Riemannian manifold $M^n$ is called pointwise Osserman if its curvature tensor is Osserman. If, in addition, the eigenvalues of the Jacobi operator are constant on $M^n$, the manifold $M^n$ is called globally Osserman.

Gilkey, [5], introduced curvature-like tensors with Clifford structure: A curvature-like tensor $R$ in $\mathbb{R}^n$ has a Cliff($\nu$)-structure if

$$R(X, Y)Z = \lambda_0(g(X, Z)Y - g(Y, Z)X) + \sum_{i=1}^{\nu} (\lambda_i/3)(2g(J_iX, Y)J_iZ + g(J_iZ, Y)J_iX - g(J_iZ, X)J_iY)$$

for any $X, Y, Z \in \mathbb{R}^n$, where $g$ denotes the usual scalar product of $\mathbb{R}^n$, $J_1, \ldots, J_\nu$ are skew-symmetric orthogonal operators satisfying the Hurwitz relations $J_iJ_j + J_jJ_i = 0$, $1 \leq i \neq j \leq \nu$, and $\lambda_1, \ldots, \lambda_\nu$ are nonzero numbers.

A Riemannian manifold $M^n$ has a Cliff($\nu$)-structure if its curvature tensor has a Cliff($\nu$)-structure at every point, for some functions $\lambda_0, \lambda_1, \ldots, \lambda_\nu$ on $M^n$.

For any unit vector $X$, the Jacobi operator $R_X$ defined from (1.1) has eigenvalues $\lambda_0 + \lambda_1, \ldots, \lambda_0 + \lambda_\nu$ with corresponding eigenvectors $J_1X, \ldots, J_\nu X$, respectively, and the eigenvalue $\lambda_0$ with the eigenspace $\text{Span}\{X, J_1X, \ldots, J_\nu X\}^\perp$, provided $\nu < n - 1$. Hence a curvature-like tensor (a manifold) with a Cliff($\nu$)-structure is Osserman (pointwise Osserman, respectively).

Therefore, in [6], the following two step approach to the Osserman conjecture was suggested:

1. Show that Osserman curvature-like tensors have a Clifford structure.
2. Classify Riemannian manifolds having Clifford structure.

Nikolayevsky began to study this new approach in [9] given partial answers to questions 1 and 2 above by the following theorems

**Theorem 1.2.** Let $R$ be an Osserman curvature-like tensor in $\mathbb{R}^n$ and suppose that $n$ is not divisible by 8. If $\lambda$ is a simple eigenvalue of the Jacobi operator, then there exists an orthogonal skew symmetric operator $J$ on $\mathbb{R}^n$ such that $JX$ is an eigenvector of $R_X$ corresponding to the eigenvalue $\lambda \|X\|^2$ for all $X \in \mathbb{R}^n$.

**Theorem 1.3.** A Riemannian manifold $M^n$ with a Cliff($\nu$)-structure is two point homogeneous provided that

1. $n \neq 2, 4, 8, 16$, or
2. \( n = 8, \nu < 3, \) or

3. \( n = 16, \nu \neq 8. \)

After this, in [12], Nikolayevsky followed the study of the Osserman conjecture and obtained the following

**Theorem 1.4.** A globally Osserman manifold of dimension \( n \neq 8, 16 \) is two point homogeneous. A pointwise Osserman manifold of dimension \( n \neq 2, 4, 8, 16 \) is two point homogeneous.

And in the case \( n = 16 \) he proved

**Theorem 1.5.** A (pointwise or globally) Osserman manifold with dimension 16 is two point homogeneous if the Jacobi operator has no eigenvalues of multiplicity 7, 8 and 9.

His last result in this topic, [11], states

**Theorem 1.6.** A pointwise Osserman manifold of dimension 8 is flat or locally rank one symmetric.

Combining Theorems 1.4, 1.5 and 1.6, Osserman conjecture is solved in the cases given by the

**Theorem 1.7.** In each of the following cases a Riemannian manifold \( M^n \) is flat or locally rank one symmetric:

1. \( M^n \) is globally Osserman and \( n \neq 16. \)

2. \( M^n \) is pointwise Osserman and \( n \neq 2, 4, 16. \)

3. \( n = 16, M^n \) is (pointwise or globally) Osserman, and its Jacobi operator has no eigenvalues of multiplicity 7, 8 and 9.

Related to the Osserman conjecture, Tsankov, [16], began a similar characterization of manifolds of constant sectional curvature in terms of the Jacobi operators by the following

**Theorem 1.8.** Let \( (M, g) \) be a hypersurface in \( \mathbb{R}^{m+1}, m \geq 3. \) If \( R_X \circ R_Y = R_Y \circ R_X \) for any orthogonal \( X \) and \( Y, \) then \( (M, g) \) has constant sectional curvature.
Recently, Brozos-Vázquez and Gilkey, [1], removed the hypothesis that \((M, g)\) is a hypersurface and proved

**Theorem 1.9.** Let \((M, g)\) be a Riemannian manifold of dimension \(m \geq 3\). Then

1. If \(R_X \circ R_Y = R_Y \circ R_X\), for any \(X, Y\), the manifold \((M, g)\) is flat.

2. If \(R_X \circ R_Y = R_Y \circ R_X\) for any orthogonal \(X\) and \(Y\), then the manifold \((M, g)\) has constant sectional curvature.

Let now \(\mathbb{C}M^m(4\epsilon)\), \(m \geq 2\), \(\epsilon = 1\) (respectively, \(\epsilon = -1\)), be a complex projective space \(\mathbb{C}P^m\) endowed with the metric \(g\) of constant holomorphic sectional curvature 4 (respectively, a complex hyperbolic space \(\mathbb{C}H^m\) endowed with the metric \(g\) of constant holomorphic sectional curvature \(-4\)). Let \(M\) be a connected real hypersurface of \(\mathbb{C}M^m(4\epsilon)\) without boundary. Let \(J\) denote the complex structure of \(\mathbb{C}M^m(4\epsilon)\) and \(N\) a locally defined unit normal vector field on \(M\). Then \(-JN = \xi\) is a tangent vector field to \(M\) called the structure vector field on \(M\). We also call \(D\) the maximal holomorphic distribution on \(M\), that is, the distribution on \(M\) given by all vectors orthogonal to \(\xi\) at any point of \(M\).

If \(R\) is the curvature operator of such a real hypersurface, the Jacobi operator on \(M\) with respect to \(\xi\) is called the structure Jacobi operator on \(M\), \(R_\xi\). Then the structure Jacobi operator \(R_\xi \in \text{End}(T_pM)\) is given by \((R_\xi(Y))(p) = (R(Y, \xi))(p)\) for any \(Y \in T_pM\), \(p \in M\).

Due to the properties of the curvature tensor of such a real hypersurface and Brozos-Vázquez-Gilkey’s result we can assure that there exist no real hypersurfaces in \(\mathbb{C}M^m(4\epsilon)\) such that any two Jacobi operators commute to each other, since no one is flat.

We will say that such a real hypersurface has commuting structure Jacobi operator if this operator commutes with any other Jacobi operator defined on \(M\). That is, \(R_\xi \circ R_X = R_X \circ R_\xi\) for any \(X\) tangent to \(M\). In the following paragraph we will study such real hypersurfaces.

### 2 Real hypersurfaces in \(\mathbb{C}M^m(4\epsilon)\) with commuting structure Jacobi operator

Let \(M\) be a connected real hypersurface in \(\mathbb{C}M^m(4\epsilon)\), \(m \geq 2\), without boundary. Let \(N\) be a locally defined unit normal vector field on \(M\). Let \(\nabla\) be the Levi-Civita connection on \(M\) and \((J, g)\) the Kaehlerian structure of \(\mathbb{C}M^m(4\epsilon)\).

For any vector field \(X\) tangent to \(M\) we write \(JX = \phi X + \eta(X)N\), and \(-JN = \xi\). Then \((\phi, \xi, \eta, g)\) is an almost contact metric structure on \(M\). That is, we have

\[
\phi^2 X = -X + \eta(X)\xi, \tag{2.1}
\]

\[
\eta(\xi) = 1, \tag{2.2}
\]
\[(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\]

for any tangent vectors \(X, Y\) to \(M\). From these facts we obtain

\[(2.4) \quad \phi \xi = 0\]

and

\[(2.5) \quad \eta(X) = g(X, \xi)\]

From the parallelism of \(J\) we get

\[(2.6) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi\]

and

\[(2.7) \quad \nabla_X \xi = \phi AX\]

for any \(X, Y\) tangent to \(M\), where \(A\) denotes the shape operator of the immersion.

As the ambient space has constant holomorphic sectional curvature \(4\epsilon\), the equations of Gauss and Codazzi are given, respectively, by

\[(2.8) \quad R(X, Y)Z = \epsilon\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,\]

and

\[(2.9) \quad (\nabla_X A)Y - (\nabla_Y A)X = \epsilon\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}\]

for any tangent vectors \(X, Y, Z\) to \(M\).

From (2.8) and bearing in mind (2.4) the structure Jacobi operator is given by

\[(2.10) \quad R_\xi(X) = \epsilon\{X - \eta(X)\xi\} + g(A\xi, \xi)AX - g(A\xi, X)A\xi\]

for any \(X\) tangent to \(M\).

In the sequel we need the following results:

**Theorem 2.1.** ([7], [8]) If \(\xi\) is a principal curvature vector with corresponding principal curvature \(\alpha\) and \(X \in \mathcal{D}\) is principal with principal curvature \(\lambda\), then \(\phi X\) is principal with principal curvature \((\alpha \lambda + 2\epsilon)/(2\lambda - \alpha)\).
Theorem 2.2. ([13]) There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, whose shape operator is given by $A\xi = \xi + \beta U$, $AU = \beta \xi + (\beta^2 - 1)U$, $AX = -X$ for any tangent vector $X$ orthogonal to $Span\{\xi, U\}$, where $\beta$ is a nonvanishing smooth function defined on $M$ and $U \in D$ is a unit vector field.

Now suppose that for any $X \in TM$, $R_\xi \circ R_X = R_X \circ R_\xi$.

For any $X, Y \in TM$ define the following operator

$$(2.11) R_{X,Y}(Z) = (1/2) \{ R(Z, X)Y + R(Z, Y)X \},$$

for any $Z$ tangent to $M$. As $R_{X,Y} = (1/2) \{ R_{X+Y} - R_X - R_Y \}$, all these operators commute with $R_\xi$. Note also that $R_{X,Y}(Y) = -(1/2)R_Y(X)$. Thus we have

$$(2.12) 0 = R_{X,\xi}(R_\xi(\xi)) = R_\xi(R_{X,\xi}(\xi)) = -(1/2)R_\xi^2(X)$$

for any $X$ tangent to $M$. Thus $0 = g(R_\xi(X), R_\xi(X))$ for any $X$ tangent to $M$. This yields $R_\xi = 0$.

Let $\alpha = g(A\xi, \xi)$. From (2.10) we get

$$(2.13) 0 = R_\xi(X) = \epsilon\{ X - \eta(X)\xi \} + \alpha AX - g(A\xi, X)A\xi. $$

Let $\pi = Span\{\xi, A\xi\}$. As $m \geq 2$, $\pi^\perp$ is non-trivial. If $X \in \pi^\perp$, then $0 = \epsilon X + \alpha AX$. Thus we conclude that $\alpha \neq 0$, $\pi^\perp$ is $A$-invariant and $AX = -\epsilon\alpha^{-1}X$, for any $X \in \pi^\perp$.

Suppose $M$ is Hopf, that is, $A\xi = \alpha\xi$. Let $X \in D$. Then $\phi X \in D$ and by Theorem 2.1, $-\epsilon\alpha^{-1} = (\alpha(-\epsilon\alpha^{-1}) + 2\epsilon)/(-2\epsilon\alpha^{-1} - \alpha)$. This yields $\epsilon = 0$ which is impossible.

Thus $A\xi = \alpha\xi + \beta U$, where $U$ is a suitably chosen unit vector field in $D$ and $\beta$ is a nonnull function on $M$. From (2.10) we have

$$(2.14) AU = \beta \xi + (\beta^2 - \epsilon)/\alpha U$$

and

$$(2.15) AX = -\epsilon/\alpha X$$

for any $X \in \pi^\perp$. 

If \( m \geq 3 \) we can take a unit \( X \in \pi^\perp \) orthogonal to \( \phi U \). Codazzi equation gives

\[
(\nabla_X A)\phi X - (\nabla_{\phi X} A)X = -2\epsilon \xi.
\]

Scalar product with \( \xi \) yields

\[
(2.16) \quad g([\phi X, X], U) = 2/\alpha^2 \beta
\]

and taking scalar product with \( U \) we obtain

\[
(2.17) \quad g([\phi X, X], U) = 2\epsilon/\beta.
\]

From (2.16) and (2.17) we have \( \epsilon \alpha^2 = 1 \). If \( \epsilon = -1 \) we arrive to a contradiction. If \( \epsilon = 1 \), \( \alpha^2 = 1 \) and changing \( \xi \) by \( -\xi \), if necessary, we have conditions of Theorem 2.2. Thus these real hypersurfaces do not exist and we have proved

**Theorem 2.3.** There exist no real hypersurfaces in \( \mathbb{C}M^m(4\epsilon), m \geq 3 \), with commuting structure Jacobi operator.

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