**Total Curvature of Some Curves in 3-dimensional Lorentzian Warped Products**

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1 Introduction

Given a space of suitable curves $\Gamma$ in a semi-Riemannian manifold $(\bar{M}, \bar{g})$, let us consider the Total Curvature Lagrangian $$\mathcal{L} : \Gamma \rightarrow \mathbb{R}, \quad \mathcal{L}(\bar{\gamma}) = \int \tilde{\kappa} \, ds,$$
where $\tilde{\kappa}$ is the Frenet first curvature of $\bar{\gamma}$. The target is to study the critical points of $\mathcal{L}$. Let us review a few applications of this lagrangian.

1. In the Euclidean plane $\mathbb{R}^2$, if $\bar{\gamma}$ is closed, then $\mathcal{L}(\bar{\gamma}) = 2\pi i(\bar{\gamma})$, where $i(\bar{\gamma})$ is an integer number, often called the rotation number of $\bar{\gamma}$. Also, Whitney and Grauenstein proved that the rotation number characterizes the regular homotopy class of closed curves. Hence, the functional $\mathcal{L}$ is constant on any regular homotopy class of closed curves. This fact also holds true for curves connecting two fixed points of the Euclidean plane and with the same direction in these boundary points.

2. Fenchel’s Theorem, [3]: given any closed curve $\bar{\gamma}$ in the Euclidean 3-space, then $\mathcal{L}(\bar{\gamma}) \geq 2\pi$, and the equality holds if, and only if, $\bar{\gamma}$ is a convex plane curve.

3. Arroyo, Barros and Garay, [1] showed that if $\Gamma$ is a space of curves in $\mathbb{R}^n$ connecting two points, with the same tangent vectors at these points, then the critical points of $\mathcal{L}$ are plane curves.

In Physics, lagrangians are a very powerful tool to introduce Mathematical models. Let’s show some hints. A *particle* will be a one-dimensional critical point of a certain Lagrangian.

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1. In a $(2 + 1)$-dimensional space-time, the interest to the field models of particles called anyons, is due to their application to different planar physical phenomena: the fractional quantum Hall effect, high-$T_c$ superconductivity and description of the physical processes in the presence of cosmic strings [8].

2. M. S. Plyushchay [6] studied the lagrangian

\[ A_1 = -\int (m + \lambda \kappa) ds, \]

where $m$ and $\lambda$ are (real) parameters ($m$ mass, $\lambda$ dimensionless). The model of a particle described by the action $A_1$ with $m = 0$, [7, 9], corresponds to a massless particle moving classically at a superrelativistic velocity, which is essentially given by our total curvature $\mathcal{L}$ [5]. Nevertheless, this model has been shown not to contradict relativity [7].

3. In [2], massless particles in Generalized Robertson-Walker three space-times were studied. From a mathematical point of view, this means the study of critical points of the total curvature lagrangian in three-dimensional Lorentzian warped products with Riemannian fibers, being the curves contained in the fibers.

Thus, our target is to study critical points of $\mathcal{L}$ in a 3-dimensional warped manifold with 2-dimensional Lorentzian fibers, being the curves contained in the fibers.

2 General Settings

We consider an oriented Lorentzian surface $(M, g)$ and a three dimensional warped product

\[ \bar{M} = I \times_f M, \quad \bar{g} = \varepsilon dt^2 + f^2 g, \]

where $\varepsilon = \pm 1$, $dt^2$ is the metric on the open interval $I$, and $f : I \to (0, \infty)$ is a smooth function. Thus, $(\bar{M}, \bar{g})$ is a three-dimensional manifold of index 1 or 2, according to $\varepsilon$ be 1 or $-1$, respectively.

Given $t \in I$ and $\gamma(s)$ a unit space-like curve in $M$, we define a curve contained in the fiber $\{t\} \times M$ (associated with $\gamma$)

\[ \bar{\gamma}(s) = (t, \gamma(s)). \]

We consider two points $p_1, p_2 \in \{t\} \times M \subset \bar{M}$ and two space-like tangent vectors $v_i \in T_{p_i} \bar{M}$, $i = 1, 2$. Then, the space of clamped curves (associated with $p_i, v_i$, $i = 1, 2$) lying in a fiber $\{t\} \times M$, is

\[ \Gamma = \{ \bar{\gamma}_t : [0, L] \to \{t\} \times M \subset \bar{M} / \bar{\gamma}_t(0) = p_1, \bar{\gamma}_t(L) = p_2, \bar{\gamma}_t'(0) = v_1, \bar{\gamma}_t'(L) = v_2 \}. \]

In addition, if $p_1 = p_2$ and $v_1 = v_2$, $\Gamma$ is a space of closed curves. Let $\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}$ and $\bar{\tau}$ be the Frenet apparatus of $\bar{\gamma}_t$. We have the Frenet equations

\[
\begin{align*}
\nabla_{\bar{T}} \bar{T} &= \varepsilon_2 \bar{\kappa} \bar{N} \\
\nabla_{\bar{T}} \bar{N} &= -\bar{\kappa} \bar{T} + \varepsilon_3 \bar{\tau} \bar{B} \\
\nabla_{\bar{T}} \bar{B} &= -\varepsilon_2 \bar{\tau} \bar{N},
\end{align*}
\]

(2.1)
where $\nabla$ is the Levi-Civita connection of $\bar{g}$, $\varepsilon_1 = \bar{g}(\bar{T}, \bar{T}) = 1$, $\varepsilon_2 = \bar{g}(\bar{N}, \bar{N})$ and $\varepsilon_3 = \bar{g}(\bar{B}, \bar{B})$, $\varepsilon_i = \pm 1$, $i = 2, 3$.

**Problem** Given a Lorentzian surface $M$ and a three-dimensional warped product $\bar{M} = I \times_f M$, find those (clamped) Frenet space-like curves contained in the fibers $\{t\} \times M$, which are critical points of the total curvature lagrangian

$$\mathcal{L} : \Gamma \longrightarrow \mathbb{R}, \quad \mathcal{L}(\bar{\gamma}_t) = \int \bar{\kappa} \, ds.$$ 

**Note:** Time-like curves will be considered in Section 6.

### 3 Euler-Lagrange Equation

Given a space of clamped curves $\Gamma$, we choose a tangent vector at $\bar{\gamma}_t \in \Gamma$, which is nothing but a vector field, $W$, along the curve $\bar{\gamma}_t$. Also, $W$ defines in $\bar{M}$ a variation of $\bar{\gamma}_t$ in $\Gamma$, say $\Phi = \Phi(s, r) : [a, b] \times (-\varepsilon, \varepsilon) \to \bar{M}$ such that $\Phi(s, 0) = \bar{\gamma}_t(s)$, with variational field $W = W(s) = (\partial \Phi/\partial r)(s, 0)$. Thus, the critical points of the variational problem are those curves $\bar{\gamma}_t \in \Gamma$ vanishing the first derivative of $\mathcal{L}$,

$$\delta \mathcal{L}(\bar{\gamma}_t)[W] = 0, \quad \forall W \in T_{\bar{\gamma}_t} \Gamma.$$

By some computations,

$$\delta \mathcal{L}(\bar{\gamma}_t)[W] = \int_{\bar{\gamma}_t} \bar{g}(\Omega(\bar{\gamma}_t), W) \, ds + [B(\bar{\gamma}_t, W)]^b_a,$$

where

$$\Omega(\bar{\gamma}_t) = \nabla^2_{\bar{T}} \bar{N} + \varepsilon_1 \bar{\kappa}' \bar{T} + \varepsilon_1 \bar{\kappa} \nabla_{\bar{T}} \bar{T} + \bar{R}(\bar{N}, \bar{T}) \bar{T},$$

$$B(\bar{\gamma}_t, W) = \bar{g}(\nabla_{\bar{T}} W, \bar{N}) - \bar{g}(W, \bar{\nabla}_{\bar{T}} \bar{N}) - \varepsilon_1 \bar{\kappa} \bar{g}(W, \bar{T}),$$

$\bar{R}$ stands for the curvature operator of $\bar{g}$ and $\bar{\kappa}'$ is the derivative of the function $\bar{\kappa}$ with respect to the arc-length parameter. It is customary to call $\Omega(\bar{\gamma}_t)$ the Euler-Lagrange operator and $B(\bar{\gamma}_t, W)$ the Boundary operator. Now, if we recall that $\bar{\gamma}_t \in \Gamma$ is clamped, then it is easy to obtain

$$[B(\bar{\gamma}_t, W)]^b_a = 0.$$

By using the Frenet equations of $\bar{\gamma}_t$, (2.1), the Euler-Lagrange operator becomes

$$\Omega(\bar{\gamma}_t) = -\varepsilon_2 \varepsilon_3 \bar{\tau}' \bar{N} + \varepsilon_3 \bar{\tau}' \bar{B} + \bar{R}(\bar{N}, \bar{T}) \bar{T},$$

where $\bar{\tau}'$ is the derivative of $\bar{\tau}$ with respect to the arc-length. Finally, a curve $\bar{\gamma}_t$ in $\Gamma$ is a critical point of $\mathcal{L}$ if, and only if,

$$0 = \delta \mathcal{L}(\bar{\gamma}_t)[W] = \int_{\bar{\gamma}_t} \bar{g}(\Omega(\bar{\gamma}_t), W) \, ds,$$

for any $W \in T_{\bar{\gamma}_t} \Gamma$ if, and only if, it satisfies the Euler-Lagrange equation

(3.2) $$\bar{R}(\bar{N}, \bar{T}) \bar{T} = \varepsilon_2 \varepsilon_3 \bar{\tau}' \bar{N} - \varepsilon_3 \bar{\tau}' \bar{B}.$$
4 Solving the Euler-Lagrange Equation

The original curve $\gamma$ has its own Frenet apparatus $\{T, N\}$ with curvature function $\kappa$. We recall $\bar{\gamma}(t) = (t, \gamma(s)) \in \{t\} \times M \subset \bar{M}$. From these data and the fact that $\bar{M}$ is a warped product, \[4\], it is possible to compute $R(\bar{N}, \bar{T})\bar{T}$ and the Frenet apparatus of $\bar{\gamma}_t$,

$$\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau},$$
in terms of

$$N, T, \kappa, \partial_t, f,$$
and the Gaussian curvature of $M$ along $\gamma$, $G \circ \gamma$.

4.1 Case $\dot{f}(t) = 0$ for a certain $t \in I$

If $\gamma$ is a geodesic, $\gamma_t$ is also a geodesic on $\bar{M}$. Next, if $\dot{f}(t) = 0$ and $\kappa \neq 0$, by simple computations, the Euler-Lagrange equation reduces to

$$0 = R(\bar{N}, \bar{T})\bar{T} = (G \circ \gamma)\bar{N}.$$

**Theorem 1.** Assume that $t \in I$ is such that $\dot{f}(t) = 0$. Let $\gamma(s)$ be a unit space-like curve in $M$ such that either $\gamma$ is a geodesic or $G(\gamma(s)) = 0$ for any $s$ (that is, $\gamma$ is a curve of parabolic points). Then, $\bar{\gamma}_t$ is a critical point of $L$.

4.2 Case $\dot{f}(t) \neq 0$

4.2.1 Assume that $\gamma$ is a geodesic in the fiber $M$, i.e., $\kappa = 0$.

The Euler-Lagrange equation reduces to $0 = R(\bar{N}, \bar{T})\bar{T} = \varepsilon \dot{f} \partial_t$.

**Theorem 2.** Let $\gamma$ be a unit space-like geodesic in $M$. Let $t \in I$ such that $\dot{f}(t) \neq 0$ and $\ddot{f}(t) = 0$. Then, $\bar{\gamma}_t$ is a non-geodesic critical point of $L$.

4.2.2 We suppose that $\kappa > 0$ is a constant

Now, we obtain $0 = R(\bar{N}, \bar{T})\bar{T}$ and by using some formulae for warped products,

$$0 = R(\bar{N}, \bar{T})\bar{T} = \frac{1}{\sqrt{\varepsilon \dot{f}^2 - \kappa^2}} \left( \frac{\kappa \dot{f}^2 - G \circ \gamma}{\dot{f}} N - \dot{f} \partial_t \right).$$

Therefore, $\ddot{f}(t) = 0$ and $G(\gamma(s)) - \varepsilon \dot{f}^2(t) = 0$, for any $s$.

**Theorem 3.** Let us assume that the curvature function $\kappa$ of $\gamma$ is a positive constant. Let $t \in I$ such that $\dot{f}(t) \neq 0$ and $\ddot{f}(t) = 0$. In addition, if $G(\gamma(s)) = \varepsilon \dot{f}^2(t)$ for any $s$, then, $\bar{\gamma}_t$ is a critical point of $L$. 
4.2.3 The general case

The curvature $\kappa$ is not a constant and $\dot{f}(t) \neq 0$ for $t \in I$. Making computations,

\[(4.3) \quad \bar{R}(\bar{N}, \bar{T}) \bar{T} = \frac{\varepsilon_2 \kappa (\varepsilon \dot{f}^2 - G \circ \gamma)}{f^3 \sqrt{\varepsilon_2 (\varepsilon f^2 - \kappa^2)}} N + \frac{\varepsilon_2 \ddot{f}}{\sqrt{\varepsilon_2 (\varepsilon f^2 - \kappa^2)}} \partial_t. \]

By inserting equation (4.3) in the Euler-Lagrange equation, and taking components, we obtain

\[\frac{\varepsilon_2 \kappa (\varepsilon \dot{f}^2 - G \circ \gamma)}{f^2} = -\varepsilon \varepsilon_3 \dot{\kappa}^2 - \varepsilon_3 \dot{\kappa}', \quad \frac{\varepsilon_2 \dddot{f}}{f} = -\varepsilon \varepsilon_3 \dot{\kappa}^2 - \varepsilon_3 \kappa \dot{\kappa}'. \]

After some computations, we obtain:

\[(4.4) \quad (\kappa'(s))^2 = \left( \frac{\dot{f}(t)^2 - \varepsilon (G \circ \gamma)(s)}{\dot{f}(t)^2} \right) (\kappa(s)^2 - \varepsilon \dot{f}(t)^2) \]

\[(4.5) \quad k''(s) = 2 \left( \frac{\dot{f}(t)^2 - \varepsilon (G \circ \gamma)(s)}{\dot{f}(t)^2} \right) \kappa(s)^3 \]

\[+ \left( \frac{\varepsilon \ddot{f}^2 - (G \circ \gamma)(s)}{f(t)} - \varepsilon \dot{f}(t) - 2 \varepsilon f(t) \ddot{f} \right) \kappa(s). \]

If we regard everything depending on $t$ as a constant, the only part that we cannot control is function $G \circ \gamma$. Thus, a reasonable assumption might be that function $G \circ \gamma$ is a constant.

5 Rigidity conditions

5.1 The Gaussian curvature of $M$ along $\gamma$, $G \circ \gamma$, is a constant

If we assume that function $G \circ \gamma$ is a constant, equation (4.5) can be integrated by multiplying by $2\kappa'$,

\[(5.6) \quad (k')^2 = \frac{\dot{f}^2 - \varepsilon G \circ \gamma}{f^2} \kappa^4 + \left( \frac{\varepsilon \dot{f}^2 - G \circ \gamma}{f} - \varepsilon \ddot{f} - 2 \varepsilon \dot{f} \dddot{f} \right) \kappa^2 + A, \]

where $A$ is an integration constant. Now, by some computations, Eq. (4.4) compared with Eq. (5.6) gives

\[(5.7) \quad A = f \dddot{f} \dddot{f}, \quad \varepsilon G \circ \gamma = \dot{f}^2 - f \ddot{f}, \quad (k')^2 = \frac{f \dddot{f}}{f^2} (\kappa^2 - \varepsilon \dot{f}^2)^2. \]
Since we are assuming that the curvature function $\kappa$ of $\gamma$ is not constant, then $\ddot{f} > 0$. Next, we solve equation (5.7) by discussing the value of $\varepsilon$.

\begin{align*}
\varepsilon &= 1, \quad \kappa(s) = \dot{f} \tanh \left( s \sqrt{ff + \dot{f}c} \right), \\
\varepsilon &= -1, \quad \kappa(s) = \dot{f} \tan \left( s \sqrt{ff + \dot{f}c} \right),
\end{align*}

(5.8)  \hspace{1cm} (5.9)

where $c$ is an integration constant. Therefore, we have proved the following.

**Theorem 4.** Fix a point $t \in I$. We assume

1. $\dot{f}(t) \neq 0$, $\ddot{f}(t) > 0$;
2. $G \circ \gamma$ is a constant function and $\varepsilon G \circ \gamma = \dot{f}^2(t) - f(t)\dot{f}(t)$;
3. the curvature of $\gamma$ is given by either (5.8) or (5.9).

Then, the curve $\bar{\gamma}_t$ is a critical point of $\mathcal{L}$.

### 5.2 The Gaussian curvature $G$ of $M$ is constant on the whole surface

We assume $G = 0, 1, -1$. We recall:

- $dS^2(1)$ is the 2-dimensional de-Sitter space of Gaussian curvature $G = 1$,
- $AdS^2(-1)$ is the two-dimensional anti de-Sitter space of Gaussian curvature $G = -1$.

We denote by $M_\delta$ the space $dS^2(1)$ or $AdS^2(-1)$ according to $\delta = \pm 1$, respectively. Now, let $\bar{M}_\delta$ be the warped product of an open interval and $M_\delta$. From Theorems 1, 2, 3 and 4, we can describe completely the space-like critical points of $\mathcal{L}$ in the following results.

**Corollary 1.** A unit space-like curve $\bar{\gamma}_t \in \Gamma$ in $\bar{M}_\delta$ is a critical point of $\mathcal{L}$ if, and only if, one of the following statements holds:

1. $\dot{f}(t) = 0$ and $\gamma$ is a geodesic;
2. $\dot{f}(t) \neq 0$, $\ddot{f}(t) = 0$ and $\gamma$ is a geodesic;
3. $\dot{f}^2(t) = \varepsilon \delta > 0$, $\ddot{f}(t) = 0$ and $\kappa \neq 0$ is constant;
4. $\dot{f}(t) \neq 0$, $\ddot{f}(t) > 0$, $\varepsilon \delta = \dot{f}^2(t) - f(t)\dot{f}(t)$ and the curvature function is given by (5.8) or (5.9), according to $\varepsilon = 1$ or $\varepsilon = -1$, respectively.
Next, let $\mathbb{L}^2$ be the Lorentz-Minkowski 2-plane.

**Corollary 2.** A unit space-like curve $\bar{\gamma}_t \in \Gamma$ in $\bar{M} = I \times_f \mathbb{L}^2$ is a critical point of $\mathcal{L}$ if, and only if, one of the following statements holds:

1. $\dot{f}(t) = 0$, and $\kappa$ is any smooth function;

2. $\dot{f}(t) \neq 0$, $\ddot{f}(t) = 0$ and $\gamma$ is a geodesic;

3. $0 \neq \dot{f}^2(t) = f(t)\ddot{f}(t)$, $\dot{f}(t) > 0$ and the curvature function is given by (5.8) or (5.9), according to $\varepsilon = 1$ or $\varepsilon = -1$, respectively.

6 Time-like curves

We assume that $\gamma$ is a unit time-like curve with curvature $\kappa$ in the Lorentzian surface $(M, g)$. Next, we consider the Lorentzian metric $g^o = -g$ on $M$. Obviously, $\gamma$ is a space-like curve for $g^o$. Let $\nabla^o$, $N^o$, $\kappa^o$, $G^o$ denote the Levi-Civita connection of $g^o$, the normal vector along the curve, the curvature function of the curve and the Gaussian curvature of the surface, respectively. Now,

\begin{equation}
\nabla^o = \nabla, \quad N^o = N, \quad \kappa^o = -\kappa, \quad G^o = -G.
\end{equation}

Given $\varepsilon = \pm 1$, we consider the warped product manifold

\[ \bar{M}^o = I \times M, \quad \bar{g}^o = \varepsilon \, dt^2 + f^2 g^o. \]

In this way we reduce the case of time-like curves to that of space-like curves. This readily proves Theorems 1, 2, 3 and 4 for a time-like curve $\gamma$, with the changes $\varepsilon \rightarrow -\varepsilon$, $G^o \rightarrow -G$, $\kappa^o \rightarrow -\kappa$.

Next, we assume that the Gaussian curvature $G$ of the surface $M$ is constant. As in Section 5, we denote $M_\delta$ the spaces $dS^2(1)$ or $AdS^2(-1)$ according to $\delta = \pm 1$, respectively. Now, let $\bar{M}_\delta$ be the warped product of an open interval and $M_\delta$.

**Corollary 3.** A unit time-like curve $\bar{\gamma}_t \in \Gamma$ in $\bar{M}_\delta$ is a critical point of $\mathcal{L}$ if, and only if, one of the following statements hold:

1. $\dot{f}(t) = 0$ and $\gamma$ is a geodesic;

2. $\dot{f}(t) \neq 0$, $\ddot{f}(t) = 0$ and $\gamma$ is a geodesic;

3. $\dot{f}^2(t) = \varepsilon \delta > 0$, $\ddot{f}(t) = 0$ and $\kappa \neq 0$ is constant;
4. $\dot{f}(t) \neq 0$, $\ddot{f}(t) > 0$, $\varepsilon \delta = \dot{f}^2(t) - f(t)\ddot{f}(t)$ and the curvature function is given by

\begin{align}
(6.11) & \quad \varepsilon = 1, \quad \kappa(s) = -\dot{f} \tan \left( s \sqrt{\dot{f}^2 + \dot{f} c} \right), \\
(6.12) & \quad \varepsilon = -1, \quad \kappa(s) = -\dot{f} \tanh \left( s \sqrt{\dot{f}^2 + \dot{f} c} \right),
\end{align}

where $c$ is an integration constant.

Next, let $\mathbb{L}^2$ be the Lorentz-Minkowski 2-plane.

**Corollary 4.** A unit time-like curve $\bar{\gamma}_t \in \Gamma$ of $\mathcal{L}$ in $\bar{M} = I \times f \mathbb{L}^2$ is a critical point of $\mathcal{L}$ if, and only if, one of the following statements hold:

1. $\dot{f}(t) = 0$, and $\kappa$ is any smooth function;
2. $\dot{f}(t) \neq 0$, $\ddot{f}(t) = 0$ and $\gamma$ is a geodesic;
3. $0 \neq \dot{f}^2(t) = f(t)\ddot{f}(t)$, $\ddot{f}(t) > 0$ and the curvature function is given by Eq. (6.11) or (6.12), according to $\varepsilon = 1$ or $\varepsilon = -1$, respectively.

**References**