The Serre Duality for Holomorphic Vector Bundles over
Strongly Pseudo Convex CR Manifolds

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1 Main Theorem.

1.1 The Serre duality for a holomorphic vector bundle \( E \) and its dual vector bundle \( E^* \) holds over a compact, strongly pseudo convex CR manifold. This duality is a vector bundle generalization of the Serre duality for scalar valued harmonic forms over a strongly pseudo convex CR manifold given by N. Tanaka.

Let \( M \) be a compact, strongly pseudo convex CR \((2n-1)\)-dimensional manifold and \( E \) be a holomorphic vector bundle over \( M \). Assume \( 2n-1 \geq 5 \) in what follows.

Then we have

**Theorem A ([I-S])**. Let \( H^{p,q}(M;E) \) be the space of \( E \)-valued harmonic \((p,q)\)-forms over \( M \). Then

\[
H^{p,q}(M;E) \cong H^{n-p,n-q-1}(M;E^*)
\]

for any \((p,q), 0 \leq p \leq n, 0 \leq q \leq n-1\). Here \( E^* \) is the dual of \( E \).

**Theorem B ([I-S])**. The \( q \)-th cohomology group \( H^q(M;E \otimes \Lambda^p \hat{T}_M^*) \) satisfies

\[
H^q(M;E \otimes \Lambda^p \hat{T}_M^*) \cong H^{n-q-1}(M;E^* \otimes \Lambda^{n-p} \hat{T}_M^*)
\]

for any \((p,q), 0 \leq p \leq n, 0 \leq q \leq n-1\). Here \( \hat{T}_M \) is the holomorphic tangent bundle of \( M \) and \( H^q(M;F) \) denotes the \( \bar{\partial} \)-cohomology group of the \( \bar{\partial} \)-complex \( \{C^q(M;F);\bar{\partial} = \bar{\partial}_f\} \) associated with a holomorphic vector bundle \( F \).

Moreover,

\[
H^q(M;E) \cong H^{n-q-1}(M;E^* \otimes \hat{K}_M), \quad 0 \leq q \leq n-1,
\]

where \( \hat{K}_M = \Lambda^n \hat{T}_M^* \) denotes the canonical complex line bundle of \( M \).

1.2 Over a compact complex \( n \)-dimensional complex manifold \( M \) the following duality

\[
H^q(M^n;\Omega^p(F)) \cong H^{n-q}(M;\Omega^{n-p}(F^*))
\]
holds for a holomorphic vector bundle $F$ and $0 \leq p, q \leq n$, where $\Omega^p(F)$ denotes the sheaf of holomorphic $F$-valued $p$-forms. This is called Serre duality due to J.P. Serre ([S]), which is a fundamental machinery in study of the $\overline{\partial}$-cohomology of holomorphic vector bundles over a complex manifold. See for this [K] and also [O-S-S], for examples.

1.3 In his famous lecture note [T] N.Tanaka established the following duality over a strongly pseudo convex CR manifold in scalar valued harmonic $(p, q)$-forms:

$$ H^{p,q}(M) \cong H^{n-p,n-q-1}(M) $$

for any $(p, q)$, $0 \leq p \leq n$, $0 \leq q \leq n - 1$. Here $H^{p,q}(M)$ is the space of harmonic $(p, q)$-forms on $M$.

2 Strongly Pseudo Convex CR Manifolds

2.1 Let $M$ be a $(2n - 1)$-dimensional manifold. If $M$ carries the following CR structure $(S, \theta, P, I, g)$, we call $M$ or $M$ with $(S, \theta, P, I, g)$ a strongly pseudo convex CR manifold (by abbreviation, s.p.c. CR manifold):

(i) $\theta$ is a contact form of $M$ so that $P$ is a subbundle of $TM$ defined by $P = \text{Ker} \theta$,

(ii) $S$ is a complex subbundle of the tangent vector bundle $T\mathbb{C}M$ such that $S \cap \overline{S} = \{0\}$,

and

$$ [\Gamma(S), \Gamma(S)] \subset \Gamma(S) $$

(iii) Further

$$ P^\mathbb{C} = S \oplus \overline{S} $$

and $I$ is an endomorphism of $P$ satisfying $I^2 = -\text{id}_P$ and

$$ S = \{X \in P^\mathbb{C} | IX = -\sqrt{-1}X\} $$

(iv) The symmetric bilinear form, called Levi form, $g : P \times P \rightarrow \mathbb{R}$

$$ g(X, Y) = -d\theta(IX, Y) $$

is positive definite. So the Levi form $g$ together with the contact form $\theta$ yields a smooth Riemannian metric $h = g \oplus \theta \otimes \theta$ on $M$.

Since a s.p.c. CR manifold $M$ is contact, $M$ admits a smooth vector field $\xi$, called basic field (or Reeb field).
We call a s.p.c. CR manifold normal, when it holds
\[ [\xi, \Gamma(S)] \subset \Gamma(S). \]

2.2 EXAMPLE 1. The following are typical examples of s.p.c. CR manifolds.
(a) a strictly convex real hypersurface \( M \) in a complex manifold \( N \),
(b) a Sasakian manifold,
(c) an \( S^1 \)-bundle over a complex Hodge manifold,
(d) link of zero locus of a complex weighted homogenious polynomial
\[ f(z_1, \ldots, z_{n+1}) = \sum c_{i_1i_2\cdots i_{n+1}} z_1^{a_{i_1}} \cdots z_{n+1}^{a_{i_{n+1}}}, \]
namely, the link \( K_f \) is defined
\[ K_f = S^{2n+1} \cap \{ z \in \mathbb{C}^{n+1} | f(z) = 0 \}. \]
Notice that examples (b), (c) and (d) are normal s.p.c. manifolds

3 Holomorphic Vector Bundles

3.1 Let \( E \longrightarrow M \) be a complex vector bundle over a s.p.c. CR manifold \( M \).

DEFINITION. A holomorphic structure of \( E \) is a first order differential operator
\( \bar{\partial} = \bar{\partial}_E : \Gamma(M; E) \longrightarrow \Gamma(M; E \otimes \mathbb{S}^*) \) satisfying
\[ (\bar{\partial}(fu)) = f \bar{\partial}(u) + u \otimes d'' f, \]
(here \( d'' \) denotes the CR operator, that is, the \( \mathbb{S} \)-component of the ordinary exterior derivative \( d \)), namely, if we set \( \bar{\partial}_X(u) = \bar{\partial}(u)(X), X \in S \), then, equivalently
\[ (\bar{\partial}_X fu) = (\bar{\partial}_X f) u + X f u, \]
and
\[ (\bar{\partial}_X \bar{\partial}_Y u - \bar{\partial}_Y \bar{\partial}_X u - \bar{\partial}_{[X,Y]}u) = 0, \]
u \( \in \Gamma(M; E) \) and \( X, Y \) are smooth sections of \( S \). We call the bundle \( (E, \bar{\partial}) \), or \( E \) holomorphic.

The condition (ii) means the vanishing of \( (0,2) \)-component of the curvature form \( R_{\bar{D}} \), when \( E \) admits a connection \( D \) whose \( (0,1) \)-part is the operator \( \bar{\partial} \).

A smooth section \( u \) of \( E \) is called holomorphic when it satisfies \( \bar{\partial} u = 0 \).

3.2 EXAMPLE 2. The quotient complex vector bundle \( \hat{T}_M = T^C M / \mathbb{S} \) of \( T^C M \) is holomorphic. It is called the holomorphic tangent bundle of \( M \), when \( M \) is s.p.c. CR. The holomorphic structure \( \bar{\partial} \) is defined
\[ \bar{\partial}_X u = \pi([X, Z]), \]
where $Z$ is a section of $\mathcal{T}^{\mathbb{C}}M$ such that $\pi(Z) = u$ and $\pi : \mathcal{T}^{\mathbb{C}}M \to \hat{T}_M$ is the projection.

**Notice.** If $M$ is normal, then the basic field $\xi$, more precisely $\pi(\xi)$, is a holomorphic section of $\hat{T}_M$.

### 3.3 Similarly to holomorphic vector bundles over a complex manifold, we have the following basic facts.

The tensor product $E \otimes F$ of holomorphic vector bundles $E$ and $F$ over a s.p.c. CR manifold is also holomorphic. Moreover the dual bundle $E^*$ and the exterior product bundle $\Lambda^k E$ are holomorphic. Furthermore, if $U$ is a holomorphic subbundle of a holomorphic vector bundle $E$, then it induces the quotient bundle $E/U$ which is holomorphic:

\[
0 \to U \to E \to E/U \to 0
\]

**Example 3.** Let $\mathcal{E} \to \hat{M}$ be a holomorphic vector bundle over a Hodge manifold $\hat{M}$ and $M$ be a s.p.c. CR manifold whose CR structure comes from the $S^1$-bundle over $\hat{M}$. Then it is not hard to show that the pull-back of $\mathcal{E}$ over $M$ is holomorphic.

**Example 4.** If $M$ is a s.p.c. CR manifold $M$ which is a real hypersurface in a complex manifold $\hat{M}$, then any holomorphic vector bundle $\mathcal{E}$ over $\hat{M}$ restricts to $M$ holomorphic.

Notice that if $\dim M \geq 7$, any holomorphic vector bundle $E$ over a s.p.c. CR manifold fulfills the local holomorphic frame property, i.e., around any point of $M$ $E$ admits a local holomorphic frame $(s_1, \cdots, s_r)$. It still open whether this property holds even when $\dim \leq 5$.

### 4 Cohomologies $H^{p,q}(M; E)$

#### 4.1 Let $E$ be a holomorphic vector bundle over a s.p.c. CR manifold $M$. We define a complex $(C^q(M; E), \overline{\partial} = \overline{\partial}^q)$, called the $\overline{\partial}$-complex as follows: $C^q(M; E) = \Gamma(M; E \otimes \Lambda^q \mathbb{S}^\circ)$, $q = 0, 1, \cdots, n-1$ and the operator $\overline{\partial} : C^q(M; E) \to C^{q+1}(M; E)$ by

\[
(\overline{\partial}^q \varphi)(Y_1, \cdots, Y_{q+1}) = \sum_{j} (-1)^{j+1} \overline{\partial}_{Y_j} (\varphi(Y_1, \cdots, \hat{Y}_j, \cdots, Y_{q+1})) + \sum_{i,j} (-1)^{i+j} \varphi([Y_i, Y_j], Y_1, \cdots, \hat{Y}_i, \cdots, \hat{Y}_j, \cdots, Y_{q+1}),
\]

where $\varphi \in C^q(M; E)$, $Y_j \in \Gamma(M; S), j = 1, \cdots, q + 1$.

It holds

\[
\overline{\partial}^{q+1} \circ \overline{\partial}^q = 0.
\]
Hence we have the cohomology group $H^q(M; E)$, $q = 0, \cdots, n-1$ as

$$H^q(M; E) = \text{Ker} \bar{\partial} / \text{Im} \bar{\partial}^{-1}.$$  \hfill (4.17)

### 4.2

To define the $(p, q)$-cohomology group $H^{p, q}(M; E)$ we take the holomorphic vector bundle $E \otimes \Lambda^p (\hat{T}_M^*)$ by tensoring $E$ with the holomorphic exterior product bundle $\Lambda^n (\hat{T}_M^*)$, and by $C^{p, q}(M; E)$ we denote $\Gamma(M; E \otimes \Lambda^p \hat{T}_M^* \otimes \Lambda^q S^*)$, the space of all smooth sections of $E \otimes \Lambda^p \hat{T}_M^* \otimes \Lambda^q S^*$.  \hfill (4.18)

**Definition.** The $(p, q)$-cohomology $H^{p, q}(M; E)$ is defined

$$H^{p, q}(M; E) = H^q(M; E \otimes \Lambda^p \hat{T}_M^*).$$  \hfill (4.19)

### 5 Harmonic $E$-valued $(p, q)$-forms

#### 5.1

Let $M$ be, same as before, a s.p.c. CR manifold. Then $M$ admits the canonical volume form $dv = \theta \wedge (d\theta)^{n-1}$ and also the Riemannian metric $h = g \oplus \theta \otimes \theta$ induced from the Levi form $g$ and the contact form $\theta$ of $M$.  

In order to get the notion of harmonic forms taking values in $E \otimes \Lambda^p \hat{T}_M^*$ we need to define the formal adjoint $\bar{\partial}^*$ of the operator $\bar{\partial}$. For this, just like the complex manifold case as in [K-M], we define the Hodge star operator

$$\sharp : E \otimes \Lambda^p \hat{T}_M^* \otimes \Lambda^q S^* \longrightarrow E^* \otimes \Lambda^{n-p} \hat{T}_M^* \otimes \Lambda^{n-q-1} S^*,$$  \hfill (5.20)

defined by $\psi \longrightarrow \sharp(\psi) = \overline{\psi}$, by exploiting the identification (18). Then the formal adjoint $\tilde{\partial}^*$ is defined

$$\tilde{\partial}^* = (-1)^{n-k} \sharp \circ \bar{\partial} \circ \sharp$$  \hfill (5.21)

on $E \otimes \Lambda^p \hat{T}_M^* \otimes \Lambda^q S^*$, $k = p + q$. It is not difficult to show that $\tilde{\partial}^*$ is the $L^2$-adjoint of the holomorphic operator $\bar{\partial}$:

$$\langle \bar{\partial}\varphi, \psi \rangle = \langle \varphi, \tilde{\partial}^* \psi \rangle$$  \hfill (5.22)

for $\varphi \in \mathcal{C}^{p,q}(M; E)$ and $\psi \in \mathcal{C}^{p,q+1}(M; E^*)$. The inner product is

$$\langle \varphi, \psi \rangle = \int_M (\varphi, \psi) \ dv.$$  

A section $\varphi$ of $\mathcal{C}^{p,q}(M; E)$ is called harmonic when it satisfies

$$\langle \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^* \rangle \varphi = 0$$  \hfill (5.23)
and denote by $H^{p,q}(M;E)$ the space of all $E$-valued harmonic $(p,q)$-forms.

5.2 By Kohn’s theorem (K), if $\dim M \geq 5$, then $H^{p,q}(M;E)$ is finite dimensional and $C^{p,q}(M;E)$ is endowed with the Hodge-decomposition property in terms of the Laplace operator $\bar{\partial} \partial + \partial \bar{\partial}$, (for the details, see [T], or the original paper [K]), so that we get the first part of THEOREM B from THEOREM A.

**Proof of Theorem A.** Let $\psi$ be an element of $H^{p,q}(M;E)$. Then we have by definition

$$\bar{\partial}_E \psi = 0, \quad \text{and} \quad \bar{\partial}^*_E \psi = 0.$$ 

Since $\psi \in C^q(M; E \otimes \Lambda^p \hat{T}_M)$, \hfill (5.24)

$$\sharp \psi \in C^{n-q-1}(M; E^* \otimes \Lambda^{n-p} \hat{T}_M).$$

Note that $\sharp \psi$ is an $E^*$-valued form.

Then by exploiting the definition of the formal adjoint, we have

$$\bar{\partial}_{E^*} (\sharp \psi) = (-1)^{n-k} \sharp (\bar{\partial}_E)^* \psi = 0,$$

where $k = p + q$ and

$$\bar{\partial}_{E^*} (\sharp \psi) = (-1)^{n-k} \sharp E^*_E (\sharp \psi) = (-1)^{n-k} \sharp \bar{\partial}_E \psi = 0.$$

Thus, we have

$$\sharp \psi \in H^{n-p,n-q-1}(M; E^*)$$

and from that $\sharp^* = \sharp^{-1}$, $\sharp$ induces the isomorphism

$$\sharp: H^{p,q}(M; E) \longrightarrow H^{n-p,n-q-1}(M; E^*)$$

(5.26)

to obtain Theorem A.

6 **Remarks**

6.1 To show the last half part of THEOREM B, we observe the following.

(6.27) \hfill \begin{align*}
H^q(M; E) & \cong H^{0,q}(M; E) \\
& \cong H^{n,n-q-1}(M; E^*) \\
& \cong H^{0,n-q-1}(M; \Lambda^n \hat{T}_M \otimes E^*) \\
& \cong H^{n-q-1}(M; \Lambda^n \hat{T}_M \otimes E^*),
\end{align*}

which is just $H^{n-q-1}(M; E^* \otimes \hat{K}_M)$.
6.2 Let $M$ be a s.p.c. CR $(2n - 1)$-dimensional manifold which is normal and $\xi$ be a basic field of $M$. Then $\xi$ is a holomorphic section of the holomorphic tangent bundle $\hat{T}_M$ which vanishes nowhere.

Then this field $\xi$ induces the inner product which satisfies for $\alpha \in \mathcal{C}^\infty(M; E \otimes \Lambda^p\hat{T}_M) = \Gamma(M; E \otimes \Lambda^p\hat{T}_M^* \otimes \Lambda^q\hat{S})$

$$\overline{\partial} i_\xi \alpha = i_\xi \overline{\partial} \alpha$$

so that this induces a linear map between the cohomology groups:

$$i_\xi : H^{p,q}(M; E) \longrightarrow H^{p-1,q}(M; E) : [\psi] \mapsto [i_\xi \psi]$$

which fulfills

$$i_\xi \circ i_\xi = 0.$$  

6.3 Example 5 Let $M = K_f$ be a link of zero locus in $\mathbb{C}^4$ of the weighted homogeneous polynomial $f = z_1^6 + z_2^6 + z_3^6 + z_4^2$. Then $\dim M = 5$ and the weight of $f$ is $(1, 1, 1, 3)$ and the degree $d = 6$. So the Hodge numbers are $h^{2,0}(M) = 1$ and $h^{1,1}(M) = 19$ by computing in terms of the Milnor algebra associated to the $f$, where $h^{p,q}(M) = \dim \mathbb{C} H^{p,q}(M)$. See [I] for the counting formula of $h^{p,q}(M)$.

By applying the Serre duality in the trivial bundle case, $M$ admits a non-trivial holomorphic 3-form $\psi$, since $H^0(M; \Lambda^3\hat{T}_M^*) \cong \mathbb{C}$. This is because

$$h^{2,0} = h^{0,2} = h^{3,0} = \dim \mathbb{C} H^{3,0}(M).$$

Here the second equality is from the Serre duality and then

$$H^{3,0}(M) \cong H^0(M; \Lambda^3\hat{T}_M^*).$$

Moreover $i_\xi \psi \in H^0(M; \Lambda^2\hat{T}_M^*)$ is a holomorphic 2-form on $M$ which is also non-trivial so that this form yields a base of $H^{2,0}(M) \cong \mathbb{C}$.

References


