Symmetries on unit tangent bundles

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Abstract. For a Riemannian manifold $M$, its unit tangent bundle $T_1M$ admits a standard
contact metric structure induced from an almost Kähler structure of its tangent bundle $TM$. In this article, we deal with several symmetries on $T_1M$ concerned with its contact
metric structures.

1 Introduction

The local symmetry ($\nabla R = 0$) is one of the fundamental notions in Riemannian
geometry. D. E. Blair ([4]) proved that the standard contact Riemannian structure
($\eta, g, \varphi, \xi$) of the unit tangent bundle $T_1M$ is locally symmetric if and only if the base
manifold $M$ is of constant curvature 0 or 2-dimensional and of constant curvature
1. Quite recently, for general contact Riemannian manifolds the first author and
E. Boeckx [11] have proved that a locally symmetric contact metric space is either
Sasakian and of constant curvature 1 or locally isometric to the unit tangent bundle
of a Euclidean space (with its standard contact metric structure). (For Sasakian
manifolds, the situation had been clear already for a long time by work of Okumura
[23].) These results mean that local symmetry is too strong a condition to impose in
contact geometry. In this context, T. Takahashi ([26]) introduced Sasakian locally
$\varphi$-symmetric spaces, which may be considered as the analogues of locally Hermi-
tian symmetric spaces. He calls a Sasakian manifold locally $\varphi$-symmetric if the
Riemannian curvature tensor $R$ satisfies ($*$): $g((\nabla_X R)(Y, Z)V, U) = 0$ for all vector
fields $X, Y, Z, V$ and $U$ orthogonal to $\xi$. He proves that this condition is equiva-
lent to having $\varphi$-geodesic symmetries which are local automorphisms. Later, it was
proved in [6] that the isometry property of the $\varphi$-geodesic symmetry is already suf-
sufficient. For the broader class of contact metric spaces, we have two generalizations for the notion of local \( \varphi \)-symmetry. In [2] a contact Riemannian manifold is called locally \( \varphi \)-symmetric if it satisfies the condition \((\ast)\). In [7] the authors give a different definition for a locally \( \varphi \)-symmetric contact Riemannian manifold: they require the characteristic reflections (i.e., the reflections with respect to the integral curves of \( \xi \)) to be local isometries. This geometric definition leads to an infinite number of curvature conditions, including \((\ast)\). The symmetry (on a contact Riemannian manifold) defining in [2] is therefore called local \( \varphi \)-symmetry in the weak sense and the one in [7] is called local \( \varphi \)-symmetry in the strong sense. They ([7]) proved that \( T_1M \) is of strong local \( \varphi \)-symmetry if and only if \( M \) is of constant curvature. This yields that a unit tangent bundle \( T_1M \) of a space of constant curvature is weakly locally \( \varphi \)-symmetric.

Apart from the defining structure tensors \( \eta, g, \varphi \) and \( \xi \), two other operators play a fundamental role in contact Riemannian geometry, namely the structural operator \( h = \frac{1}{2} \mathcal{L}_\xi \varphi \) and the characteristic Jacobi operator \( \ell = R(\cdot, \xi)\xi \), where \( \mathcal{L}_\xi \) denotes Lie differentiation in the characteristic direction \( \xi \). Special properties for the geometry of \((M, G)\) will be reflected in special properties for the contact structure on \( T_1M \) and vice versa. In particular, the characteristic vector field \( \xi \) on \( T_1M \) contains crucial information about \( M \). In fact, all the geodesics in \( M \) are controlled by the geodesic flow on \( T_1M \) which is precisely given by \( \xi \). An important topic in the study of the contact metric structure on unit tangent bundles has been to determine those Riemannian manifolds \((M, G)\) for which the corresponding contact structure on \( T_1M \) enjoys symmetry properties related to the geodesic flow.

A first symmetry type for the contact metric structure occurs when the geodesic flow, generated by \( \xi \), leaves some structure tensors invariant. This is always the case for \( \xi \) and \( \eta \) since \( \mathcal{L}_\xi \xi = 0 \) and \( \mathcal{L}_\xi \eta = 0 \). The metric \( g \) is left invariant by the flow of \( \xi \) (or equivalently, the flow consists of local isometries or \( \xi \) is a Killing vector field) if and only if the structural operator \( h \) vanishes. By definition, this corresponds precisely to \( \mathcal{L}_\xi \varphi = 0 \), i.e., also \( \varphi \) is preserved under the geodesic flow. Y. Tashiro proved in [27] that this happens for a unit tangent bundle \((T_1M; \eta, g, \varphi, \xi)\) if and only if \((M, G)\) has constant curvature \( c = 1 \). In [12], we proved that \( T_1M \) satisfies the condition \( \mathcal{L}_\xi h = 0 \) (\( \mathcal{L}_\xi h' = 0 \), respectively) if and only if \((M, g)\) is of constant curvature \( c = 1 \) (\( c = -1, 0, 1 \), respectively, where \( h' = \nabla_\xi h \) denotes the characteristic derivative) or \( T_1M \) satisfies the condition \( \mathcal{L}_\xi \ell = 0 \) if and only if \((M, g)\) is of constant curvature \( c = 0 \) or \( 1 \).

A second type of symmetry occurs when some structure tensors are covariantly parallel along the integral curves of \( \xi \). On a contact metric space, it always holds \( \nabla_\xi \xi = \nabla_\xi \eta = \nabla_\xi g = \nabla_\xi \varphi = 0 \), but the other structure tensors need not be parallel in the \( \xi \)-direction. Recently, it was proved that \( T_1M \) satisfies the condition \( \nabla_\xi h = 0 \) or, equivalently, \( \nabla_\xi \ell = 0 \), if and only if \((M, G)\) is of constant curvature \( c = 0 \) or \( c = 1 \) ([24], [25]). This result can easily be verified from the formulas in this paper.
A final type of symmetry on contact Riemannian manifolds is the notion of \( \eta \)-parallelity. We call a \((1, 1)\)-tensor \( T \) \( \eta \)-parallel if \( g((\nabla_X T)Y, Z) = 0 \) for all vector fields \( X, Y, Z \) orthogonal to \( \xi \). A weak local \( \varphi \)-symmetry mentioned above means the \( \eta \)-parallelity of the curvature tensor \( R \). In particular, the tensor \( \varphi \) is \( \eta \)-parallel if and only if the contact structure is CR-integrable, \( (\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \). On a unit tangent bundle \( T_1M \), this occurs if and only if \( (M, G) \) is of constant curvature, as can easily be verified from the formulas further on. Contact metric spaces with \( \eta \)-parallel structural operator \( h \) were completely classified by the first author and E. Boeckx in [10]. As a corollary of that result, we obtain also that the \( \eta \)-parallel \( h \) of a unit tangent bundle \( T_1M \) has \( \eta \)-parallel \( h \) if and only if \( (M, G) \) is of constant curvature. In the second part of Section , we determine the base manifold when the characteristic Jacobi operator \( \ell \) on \( T_1M \) is \( \eta \)-parallel.

One may consider the Ricci-parallel condition \( (\nabla \rho = 0) \) which is deduced by the local symmetry. In fact, in [7] they determined the unit tangent bundle with the parallel Ricci tensor. There are two natural generalizations of a Ricci-parallel space, one is a space of Codazzi-type Ricci tensor \( ((\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z) \) for all vector fields \( X, Y, Z \) on the manifold) the other is a space of cyclic parallel Ricci tensor \( ((\nabla_X \rho)(Y, Z) + (\nabla_Y \rho)(Z, X) + (\nabla_Z \rho)(X, Y) = 0 \) for all vector fields \( X, Y, Z \) on the manifold). In [8] they dealt with the two conditions for \( T_1M(c) \) of a space of constant curvature \( c \). In Section , we do the corresponding study for \( T_1M^C(4c) \) of an \( n \)-dimensional complex space form \( M^C(4c) \) of constant holomorphic sectional curvature \( 4c \). Indeed, we prove that \( T_1M^C(4c) \) \( (n \geq 2) \) has Codazzi-type Ricci tensor if and only if \( c = 0 \), in this case \( T_1M^C(4c) \) is locally symmetric. Also, we prove that \( T_1M^C(4c) \) \( (n \geq 2) \) has a cyclic parallel Ricci tensor if and only if \( c = 0 \) or \( c = 1 \). This yields that \( T_1CP_n(4) \) of a complex projective space \( CP_n(4) \) with the Fubini-Study metric is a proper example whose Ricci tensor is cyclic parallel and is not Ricci-parallel.

2 Preliminaries

All manifolds in the present paper are assumed to be connected and of class \( C^\infty \). We start by collecting some fundamental material about contact metric geometry. We refer to [3] for further details. A \((2n + 1)\)-dimensional manifold \( M^{2n+1} \) is said to be a contact manifold if it admits a global 1-form \( \eta \) such that \( \eta \wedge (d\eta)^n \neq 0 \) everywhere. Given a contact form \( \eta \), we have a unique vector field \( \xi \), the characteristic vector field, satisfying \( \eta(\xi) = 1 \) and \( d\eta(\xi, X) = 0 \) for any vector field \( X \). It is well-known that there exists a Riemannian metric \( g \) and a \((1, 1)\)-tensor field \( \varphi \) such that

\[
\begin{align*}
\eta(X) &= g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi \\
\end{align*}
\]
where \( X \) and \( Y \) are vector fields on \( M \). From (1) it follows that

\[
(2) \quad \nabla \xi = 0, \quad \eta \circ \nabla = 0, \quad g(\nabla X, \nabla Y) = g(X, Y) - \eta(X)\eta(Y).
\]

A Riemannian manifold \( M \) equipped with structure tensors \((\eta, g, \nabla, \xi)\) satisfying (1) is said to be a contact Riemannian manifold and is denoted by \( M = (M; \eta, g, \nabla, \xi) \).

Given a contact Riemannian manifold \( M \), we define the structural operator \( h \) by

\[
h = \frac{1}{2} L_{\xi} \nabla,
\]

where \( L \) denotes Lie differentiation. Then we may observe that \( h \) is symmetric and satisfies

\[
(3) \quad h\xi = 0 \quad \text{and} \quad h\varphi = -\varphi h,
\]

\[
(4) \quad \nabla_X \xi = -\varphi X - \varphi h X
\]

where \( \nabla \) is the Levi-Civita connection. From (3) and (4) we see that each trajectory of \( \xi \) is a geodesic. We denote by \( R \) the Riemannian curvature tensor defined by

\[
R(X, Y)Z = \nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) - \nabla_{[X,Y]} Z
\]

for all vector fields \( X, Y, Z \). Along a trajectory of \( \xi \), the Jacobi operator \( \ell = R(\cdot, \xi) \xi \) is a symmetric \((1, 1)\)-tensor field. We call it the characteristic Jacobi operator. We have

\[
(5) \quad \nabla_\xi h = \varphi - \varphi \ell - \varphi h^2,
\]

\[
(6) \quad \ell = \varphi \ell \varphi - 2(h^2 + \varphi^2).
\]

A contact Riemannian manifold for which \( \xi \) is Killing is called a \( K \)-contact manifold. It is easy to see that a contact Riemannian manifold is \( K \)-contact if and only if \( h = 0 \) or, equivalently, \( \ell = I - \eta \otimes \xi \).

3 The standard contact metric structure for the unit tangent bundles

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [18], [21], [29]). We only briefly review some notations and definitions. Let \( M = (M, G) \) be an \( n \)-dimensional Riemannian manifold and let \( TM \) denote its tangent bundle with the projection \( \pi : TM \to M, \pi(x, u) = x \). For a vector \( X \in T_x M \), we denote by \( X^H \) and \( X^V \), the horizontal lift and the vertical lift, respectively. Then we can define a Riemannian metric \( \tilde{g} \), the Sasaki metric on \( TM \), in a natural way by

\[
\tilde{g}(X^H, Y^H) = \tilde{g}(X^V, Y^V) = G(X, Y) \circ \pi, \quad \tilde{g}(X^H, Y^V) = 0
\]

for all vector fields \( X \) and \( Y \) on \( M \). Also, a natural almost complex structure tensor \( J \) of \( TM \) is defined by \( JX^H = X^V \) and \( JX^V = -X^H \). Then we easily see that \( (TM; \tilde{g}, J) \) is an almost Hermitian manifold. We note that \( J \) is integrable if and only if \((M, G)\) is locally flat ([18]).
Now we consider the unit tangent bundle \((T_1M, \bar{g})\), which is an isometrically embedded hypersurface in \((TM, \tilde{g})\) with unit normal vector field \(N = uV\). For \(X \in T_2M\), we define the tangential lift of \(X\) to \((x, u) \in T_1M\) by

\[
X^T_{(x,u)} = X^V_{(x,u)} - G(X, u)N_{(x,u)}.
\]

Clearly, the tangent space \(T_{(x,u)}T_1M\) is spanned by vectors of the form \(X^H\) and \(X^T\) where \(X \in T_xM\). We put \(\bar{\xi} = -JN\), \(\bar{\varphi} = J - \bar{\eta} \otimes N\).

Then we find \(\bar{g}(X, \bar{\varphi}Y) = 2d\bar{\eta}(X, Y)\). By taking \(\xi = 2\bar{\xi}\), \(\eta = \frac{1}{2}\bar{\eta}\), \(\varphi = \bar{\varphi}\), and \(g = \frac{1}{4}\bar{g}\), we get the standard contact Riemannian structure \((\varphi, \xi, \eta, g)\). Indeed, we easily check that these tensors satisfy (1). Here we notice that \(\xi\) determines the geodesic flow. The tensors \(\xi\) and \(\varphi\) are explicitly given by

(7) \(\xi = 2u^H\), \(\varphi X^T = -X^H + \frac{1}{2}G(X, u)\xi\), \(\varphi X^H = X^T\)

where \(X\) and \(Y\) are vector fields on \(M\). From now on, we consider \(T_1M = (T_1M; \eta, g)\) with the standard contact Riemannian structure. We list the fundamental formulae which we need for the proof of our theorems. They are derived in, e.g., [4], [7], [9], [24], [27]. The Levi-Civita connection \(\nabla\) of \((T_1M, g)\) is given by

(8) \(\nabla_{X^T}Y^T = -G(Y, u)X^T\),
\(\nabla_{X^T}Y^H = \frac{1}{2}(K(u, X)Y)^H\),
\(\nabla_{X^u}Y^T = (D_XY)^T + \frac{1}{2}(K(u, Y)X)^H\),
\(\nabla_{X^u}Y^H = (D_XY)^H - \frac{1}{2}(K(X, Y)u)^T\).

For the Riemann curvature tensor \(R\), we give only the two expressions we need for
the characteristic Jacobi operator $\ell$:

\begin{align}
R(X^T, Y^T)Z^T &= -g(X^T, Z^T)Y^T + g(Z^T, Y^T)X^T, \\
R(X^T, Y^T)Z^H &= \left\{ K(X - G(X, u)u, Y - G(Y, u)u)Z \right\}^H \\
&\quad + \frac{1}{4} \left\{ [K(u, X), K(u, Y)]Z \right\}^H \\
R(X^H, Y^T)Z^T &= -\frac{1}{2} \left\{ K(Y - G(Y, u)u, Z - G(Z, u)u)X \right\}^H \\
&\quad - \frac{1}{4} \left\{ K(u, Y)K(u, Z)X \right\}^H \\
R(X^H, Y^T)Z^H &= \frac{1}{2} \left\{ K(X, Z)(Y - G(Y, u)u) \right\}^T - \frac{1}{4} \left\{ K(X, K(u, Y)Z)u \right\}^T \\
&\quad + \frac{1}{2} \left\{ (D_X K)(u, Y)Z \right\}^H, \\
R(X^H, Y^H)Z^T &= \left\{ K(X, Y)(Z - G(Z, u)u) \right\}^T \\
&\quad + \frac{1}{4} \left\{ K(Y, K(u, Z)X - K(X, K(u, Z)u)Y \right\}^T \\
&\quad + \frac{1}{2} \left\{ (D_X K)(u, Z)Y - (D_Y K)(u, Z)X \right\}^H, \\
R(X^H, Y^H)Z^H &= (K(X, Y)Z)^H + \frac{1}{2} \left\{ K(u, K(X, Y)u) \right\}^H \\
&\quad - \frac{1}{4} \left\{ K(u, K(Y, Z)u)X - K(u, K(X, Z)u)Y \right\}^H \\
&\quad + \frac{1}{2} \left\{ (D_Z K)(X, Y)u \right\}^T
\end{align}

for all vector fields $X$, $Y$ and $Z$ on $M$.

In the above, we denote by $D$ the Levi-Civita connection and by $K$ the Riemannian curvature tensor associated with $G$. From (7) and (8), it follows

\begin{align}
\nabla_X T \xi &= -2\varphi X^T - (K_u X)^H, \\
\nabla_X h \xi &= -(K_u X)^T
\end{align}

where $K_u = K(\cdot, u)u$ is the Jacobi operator associated with the unit vector $u$. From (4) and (10), it follows that

\begin{align}
hX^T &= X^T - (K_u X)^T, \\
hX^H &= -X^H + \frac{1}{2} G(X, u)\xi + (K_u X)^H.
\end{align}

Using the formulae (9), we get

\begin{align}
\ell X^T &= (K_u^2 X)^T + 2(K_u X)^H, \\
\ell X^H &= 4(K_u X)^H - 3(K_u^2 X)^H + 2(K_u X)^T
\end{align}
where \( K'_u = (D_u K)(\cdot, u) u \) and \( K''_u = K(K(\cdot, u) u, u) u \). By using (5), (7) and (9) we obtain

\[
\begin{align*}
h'X^T &= -2(K_u X)^H + 2(K''_u X)^H - 2(K'_u X)^T, \\
h'X^H &= -2(K_u X)^T + 2(K''_u X)^T + 2(K'_u X)^H
\end{align*}
\]

where we put \( h' = \nabla_{\xi} h \).

The above formulae (10)–(13) are also found in [9]. Finally, from (8) and (12) we compute

\[
\begin{align*}
\ell'X^T &= 4(K'_u K_u X + K_u K'_u X)^T + 4(K''_u X + K''_u X + K''_u X)^H, \\
\ell'X^H &= 8(K'_u X - K'_u K_u X - K_u K'_u X)^H + 4(K''_u X + K''_u X - K''_u X)^T
\end{align*}
\]

where \( \ell' = (\nabla_{\xi} R)(\cdot, \xi) \).

4 Invariance under the geodesic flow and the \( \eta \)-parallelity

By using the fundamental formulae (10)–(14), we have the following theorems ([12]):

**Theorem 1.** Let \( T_1 M \) be the unit tangent bundle with the standard contact Riemannian structure \((\eta, g, \varphi, \xi)\). Then \( T_1 M \) satisfies \( \mathcal{L}_\xi h = 0 \) if and only if \((M, G)\) is of constant curvature \( c = 1 \).

**Theorem 2.** Let \( T_1 M \) be the unit tangent bundle with the standard contact Riemannian structure \((\eta, g, \varphi, \xi)\). Then \( T_1 M \) satisfies \( \mathcal{L}_\xi \ell = 0 \) if and only if \((M, G)\) is of constant curvature \( c = 0 \) or \( c = 1 \).

**Theorem 3.** Let \( T_1 M \) be the unit tangent bundle with the standard contact Riemannian structure \((\eta, g, \varphi, \xi)\). Then \( T_1 M \) satisfies \( \mathcal{L}_\xi h' = 0 \) if and only if \((M, G)\) is of constant curvature \( c = -1 \), \( c = 0 \) or \( c = 1 \).

We omit the detailed proofs of the above theorems. Instead of it, we note here that when \((M, G)\) is of constant curvature \( c \) the following explicit expressions for \( h, \ell, h' \) and \( \ell' \) from (11)–(14) are obtained:

\[
\begin{align*}
hX^T &= (1 - c)X^T, & hX^H &= (c - 1)(X^H - \frac{1}{2} G(X, u) \xi), \\
\ell X^T &= c^2 X^T, & \ell X^H &= (4c - 3c^2)(X^H - \frac{1}{2} G(X, u) \xi), \\
h'X^T &= 2(c^2 - c)(X^H - \frac{1}{2} G(X, u) \xi), & h'X^H &= 2(c^2 - c)X^T, \\
\ell'X^T &= 4(c^2 - c^3)(X^H - \frac{1}{2} G(X, u) \xi), & \ell'X^H &= 4(c^2 - c^3)X^T
\end{align*}
\]

for vector fields \( X \) on \( M \).
If the characteristic Jacobi operator \( \ell \) of a given contact Riemannian manifold satisfies \( g((\nabla_X \ell) Y, Z) = 0 \) for all vector fields \( X, Y \) and \( Z \) orthogonal to \( \xi \), then we say that \( \ell \) is \( \eta \)-parallel. Now, we determine the unit tangent bundles with \( \eta \)-parallel characteristic Jacobi operator by giving a short proof of:

**Theorem 4.** ([12]) Let \( T_1 M \) be the unit tangent bundle with the standard contact Riemannian structure \((\eta, g, \varphi, \xi)\). Then the characteristic Jacobi operator \( \ell \) of \( T_1 M \) is \( \eta \)-parallel if and only if \((M, G)\) is of constant curvature.

**Proof.** From (8) and (12), we first compute the covariant derivatives of the characteristic Jacobi operator \( \ell \):

\[
(\nabla_X \ell)^Y = \left( \frac{1}{2} K(u, K^2 Y) X + 2(D_X K)(Y, u) T + G(X, u) \left( 12(K^2 Y)^H - 8(K_u Y)^H - 6(K_u Y)^T \right) \right),
\]

\[
(\nabla_X \ell)^T = \left( \frac{1}{2} K(u, K^2 Y) X + 2(D_X K)(Y, u) T + G(X, u) \left( 12(K^2 Y)^H - 8(K_u Y)^H - 6(K_u Y)^T \right) \right),
\]

\[
(\nabla_X \ell)^H = \left( \frac{1}{2} K(u, K^2 Y) X + 2(D_X K)(Y, u) T + G(X, u) \left( 12(K^2 Y)^H - 8(K_u Y)^H - 6(K_u Y)^T \right) \right),
\]
\[ (\nabla_X \ell) Y^T = \left( 2(D_X K)(Y, u) u + 2(D_u K)(Y, X) u 
+ 2(D_u K)(Y, u) X + K(u, X) K_u^r Y \right)^H 
+ \left( K_u(K(Y, X) u) + K_u(K(Y, u) X) 
+ K(K_u Y, X) u + K(K_u Y, u) X \right)^T 
- G(X, u) \left( 4(K_u^2 Y)^T + 6(K_u^r Y)^H \right) 
+ G(Y, u) \left( (K_u^2 X)^T + 2(K_u^r X)^H \right). \]

Now, we suppose that the Jacobi operator \( \ell \) of \( T_1 M \) is \( \eta \)-parallel. Then from (16)–(19), we obtain several equations containing the curvature tensor \( K \) and its covariant derivatives. Then combining the equations obtained and using the 1st Bianchi identity, we can derive

\[ 0 = 2G((D_X K)(Y, u) u, Z) + 2G((D_u K)(Y, X) u, Z) 
+ 4G((D_u K)(Y, u) X, Z) - G(K(K_u^r Y, u) X, Z) 
+ 2G((D_u K)(X, u) Y, Z). \]

We suppose that \( X = Y = Z \) are orthogonal to \( u \). Then from (20), we find that \( (D_X K)(\cdot, X)X = 0 \) for all tangent vectors \( X \). From this, we conclude that the base manifold is locally symmetric. In the case when \( \dim M = 2 \), we at once see that \( M \) is of constant curvature. Furthermore, using the fact that the base manifold \( M \) is locally symmetric, we obtain

\[ 0 = 8G(K(Y, X) u, Z) + 8G(K(Y, u) X, Z) 
- G(K(K_u^2 K(Z, Y) u), X) - G(K(K_u^2 K(X, Y), Z). \]

In (21), we put \( Y = Z \). Then \( G(K(Y, X) Y, u) = 0 \) for any orthogonal triple \( u, X, Y \). By Cartan’s theorem ([14]), the base manifold \( (M, G) \) must have constant curvature if \( \dim M \geq 3 \). We conclude that \( M \) is of constant curvature for all dimensions.

Conversely, we can use the expressions (15) to show that \( T_1 M \) has \( \eta \)-parallel characteristic Jacobi-operator \( \ell \) when the manifold \( M \) is of constant curvature \( c \).

5 Einstein-like unit tangent bundles

By making use of the decomposition of the covariant derivative \( \nabla_\rho \) of the Ricci \((0,2)\)-tensor \( \rho \), A. Gray([20]) introduced two interesting classes \( \mathfrak{A} \) and \( \mathfrak{B} \) of Riemannian manifolds which lie between the class of Ricci-parallel Riemannian manifolds and the one of Riemannian manifolds of constant scalar curvature, namely,
1. the class $\mathfrak{A}$ of Riemannian manifolds whose Ricci tensors are Codazzi tensors, i.e., $(\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z)$ for all vector fields $X, Y, Z$ on the manifold.

2. the class $\mathfrak{B}$ of Riemannian manifolds whose Ricci tensors are Cyclic parallel (or Killing tensors), i.e., $(\nabla_X \rho)(Y, Z) + (\nabla_Y \rho)(Z, X) + (\nabla_Z \rho)(X, Y) = 0$ for all vector fields $X, Y, Z$ on the manifold.

In ([7]) it was proved that the unit tangent bundles $T_1 M(c)$ of spaces of constant curvature $c$ has a Codazzi-type Ricci tensor if and only if $c = 0$ or the base manifold is 2-dimensional and $c = 1$, i.e., if and only if $T_1 M(c)$ is locally symmetric. Also, they proved that $T_1 M(c)$ has a cyclic parallel Ricci tensor if and only if $M$ is 2-dimensional or $c \in \{0, 1\}$. In the former paper ([17]), we prove the corresponding results in the case that the base manifold is a complex space form. But, among the proofs of main Theorems A and B, we have some mistakes including the dimensional argument. Here, we state the corrected results:

**Theorem 5.** Let $M_n^C(4c)$ be an $n(\geq 2)$-dimensional complex space form with constant holomorphic sectional curvature $4c$. Then $T_1 M_n^C(4c)$ is of class $\mathfrak{A}$ if and only if $c = 0$, in this case $M_n^C(4c)$ is locally symmetric.

**Theorem 6.** Let $M_n^C(4c)$ be an $n(\geq 2)$-dimensional complex space form with constant holomorphic sectional curvature $4c$. Then $T_1 M_n^C(4c)$ is of class $\mathfrak{B}$ if and only if $c = 0$ or $c = 1$.

In the above Theorem 6, $T_1 CP_n(4)$ of a complex projective space $CP_n(4)$ with the Fubini-Study metric is a proper example whose Ricci tensor is cyclic parallel and is not Ricci-parallel.

**Remark 1.** In the case of (real) 2-dimensional base manifold $M$, we already know (cf. [15]) that $T_1 M \in \mathfrak{A}$ if and only if $M$ is of constant curvature 0 or 1, or that $T_1 M \in \mathfrak{B}$ if and only if $M$ is of constant curvature.

From now, we right down the corrected version the proof. Let $M_n$ be a complex $n$-dimensional Kählerian manifold with almost complex structure $J$ and metric $G$. For each 2-plane $p$ in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined by

$$K(p) = G(K(X, Y)Y, X),$$

where $\{X, Y\}$ is an orthonormal basis for $p$. If $p$ is invariant by $J$, then $K(p)$ is called the holomorphic sectional curvature determined by $p$. The holomorphic sectional curvature $K(p)$ is given by

$$K(p) = G(K(X, JX)JX, X),$$

where $X$ is a unit vector in $p$. If $K(p)$ is a constant for all holomorphic planes $p$ in $T_x(M)$ and for any point $x \in M$, then $M$ is called a space of constant holomorphic sectional curvature or simply, a complex space form. (Sometimes, a complex space form is defined as a simply connected and complete one.) Then it is well-known that
the curvature tensor of a complex space form is expressed in a nice form. Namely, we have (cf. [30])

**Proposition 7.** A Kählerian manifold $M^n_C(4c)$ is of constant holomorphic sectional curvature $4c$ (i.e., $M^n_C(4c)$ is a complex space form of constant holomorphic sectional curvature $4c$) if and only if

\[
K(X,Y)Z = c\{G(Y,Z)X - G(X,Z)Y + G(JY,Z)JX - G(X,JZ)JY + 2G(X,JY)JZ\}
\]

for any vector fields $X, Y$ and $Z$ on $M$.

Let $(M^n_C(4c), G)$ be a $n$-dimensional complex space form of constant holomorphic sectional curvature $4c$. We consider the unit tangent bundle $(T_1M^n_C(4c), g)$ of a complex space form $M^n_C(4c)$. We compute the Levi Civita connection $\nabla$ and the Riemannian curvature tensor $R$ of $(T_1M^n_C(4c), g)$. Namely, from (8), (9) and (22), we obtain

\[
\begin{align*}
\nabla^T_{X^T}Y^T &= -G(Y,u)X^T, \\
\nabla^T_{X^T}Y^H &= c \left\{ \frac{1}{2} G(X,Y)\xi - G(Y,u)X^H + G(JX,Y)(Ju)^H \\
&\quad + G(JY,u)(JX)^H + 2G(JX,u)(JY)^H \right\}, \\
\nabla^H_{X^T}Y^T &= (DX)^T + c \left\{ \frac{1}{2} G(X,Y)\xi - G(X,u)Y^H + G(X,JY)(Ju)^H \\
&\quad + 2G(JY,u)(JX)^H + G(JX,u)(JY)^H \right\}, \\
\nabla^H_{X^T}Y^H &= (DX)^H - c \left\{ G(Y,u)X^T - G(X,u)Y^T + 2G(X,JY)(Ju)^T \\
&\quad + G(JY,u)(JX)^T - G(JX,u)(JY)^T \right\}
\end{align*}
\]

and further we have

\[
R(X^T,Y^T)Z^T = (G(Y,Z) - G(Y,u)G(Z,u))X^T \\
- (G(X,Z) - G(X,u)G(Z,u))Y^T,
\]
\[
R(X^T, Y^T)Z^H = \left( c - \frac{c^2}{4} \right) \left\{ (G(Y, Z) - G(Y, u)G(Z, u)) \left( X^H - \frac{1}{2} G(X, u)\xi \right) \right.
\]
\[
- (G(X, Z) - G(X, u)G(Z, u)) \left( Y^H - \frac{1}{2} G(Y, u)\xi \right) \right.
\]
\[
+ (G(JY, Z) + G(Y, u)G(JZ, u))(JX)^H - G(X, u)(Ju)^H \}
\]
\[
+ 2G(JX, Y)G(JY, u) + G(X, u)G(JX, u) \right\} \{ JZ \}^H
\]
\[
R(X^H, Y^T)Z^T = \left( \frac{c^2}{4} - \frac{c}{2} \right) \left\{ (G(X, Z) - G(X, u)G(Z, u)) \left( Y^H - \frac{1}{2} G(Y, u)\xi \right) \right.
\]
\[
+ \frac{c}{2} (G(X, Y) - G(X, u)G(Y, u)) \left( Z^H - \frac{1}{2} G(Z, u)\xi \right) \right.
\]
\[
+ \left( \frac{c^2}{4} - \frac{c}{2} \right) \left\{ (G(X, JZ) + G(JX, u)G(Z, u))(JY)^H - G(Y, u)(Ju)^H \}
\]
\[
+ \frac{c}{2} (G(X, JY) + G(JX, u)G(Y, u))(JZ)^H - G(Z, u)(Ju)^H \}
\]
\[
+ \frac{c^2}{8} (G(X, u)(G(Y, Z) - G(Y, u)G(Z, u))\xi
\]
\[
- \frac{c^2}{4} G(JX, u)(G(Y, Z) - G(Y, u)G(Z, u))(Ju)^H
\]
\[
- cG(Y, JZ) + G(JY, u)G(Z, u) - G(Y, u)G(JZ, u)(JX)^H \}
\]
\[
- \frac{c^2}{4} \left\{ 1 + 3G(X, JZ)G(JY, u) + G(JX, u)G(Y, JZ) \right\}
\]
\[
+ 2G(JX, Y)G(JY, u) + G(X, u)G(JX, u) \right\} \{ JZ \}^H
\]
\[
- 4G(JY, Z)G(JZ, u)X^H + 3G(JX, u)G(JZ, u)Y^H
\]
\[
+ (3G(X, Z)G(JY, u) - G(JY, Z)G(X, u) + 2G(X, Y)G(JZ, u)) (Ju)^H
\]
\[
- 3G(X, u)G(JZ, u)(JY)^H - 2G(JX, u)G(JY, u)Z^H
\]
\[
- 2G(X, u)G(JY, u)(JZ)^H \right\}
\[ R(X^H, Y^T)Z^H = \left( \frac{c}{2} - \frac{c^2}{4} \right) (G(Y, Z) - G(Y, u)G(Z, u))X^T \]
\[ - \frac{c^2}{4} G(X, u)G(Z, u)Y^T \]
\[ - \frac{c}{2} G(X, Y) - G(X, u)G(Y, u))Z^T \]
\[ + \left( \frac{c}{2} - \frac{c^2}{4} \right) (G(Y, JZ) - G(Y, u)G(JZ, u)) (JX)^T \]
\[ - \frac{c^2}{4} G(JX, u)G(Z, u)(JY)^T \]
\[ - \frac{c}{2}(G(JX, Y) - G(JX, u)G(Y, u)) (JZ)^T \]
\[ - \frac{3c^2}{4} G(JY, u)G(JZ, u)X^T - \frac{c^2}{4} G(JX, u)G(JZ, u)Y^T \]
\[ - \frac{c}{2} G(JX, u)G(JY, u)Z^T + \frac{3c^2}{4} G(JY, u)G(Z, u)(JX)^T \]
\[ + \left( c G(X, JZ) + \frac{c^2}{4} G(X, u)G(JZ, u) \right) (JY)^T \]
\[ + \frac{c^2}{2} G(X, u)G(JY, u)(JZ)^T - c G(X, JZ)G(Y, u)(Ju)^T \]
\[ + \frac{c^2}{4} \left\{ 2G(X, Y)G(JZ, u) + 4G(X, Z)G(JY, u) \right. \]
\[ + 3G(Y, Z)G(JX, u) + 2G(X, JY)G(Z, u) \]
\[ + 3G(JY, Z)G(X, u) \left\} (Ju)^T \right. \]
\[ R(X^H, Y^H)Z^T = \left( c - \frac{c^2}{4} \right) \left\{ (G(Y, Z) - G(Y, u)G(Z, u))X^T \right. \]
\[ - (G(X, Z) - G(X, u)G(Z, u))Y^T \} \]
\[ + \left( c - \frac{c^2}{4} \right) \left\{ (G(JY, Z) - G(JY, u)G(Z, u)) (JX)^T \right. \]
\[ - (G(JX, Z) - G(JX, u)G(Z, u)) (JY)^T \} \]
\[ + 2c G(X, JY) (JZ)^T - 2c G(X, JY) G(Z, u) (Ju)^T \]
\[ + \frac{c^2}{4} \left\{ G(JX, u)G(JZ, u) Y^T - G(JY, u)G(JZ, u) X^T \right. \]
\[ - G(X, u)G(JZ, u) (JY)^T + G(Y, u)G(JZ, u) (JX)^T \]
\[ - (G(X, Z)G(JY, u) - G(Y, Z)G(JX, u) \]
\[ + G(X, JZ)G(Y, u) - G(Y, JZ)G(X, u) (Ju)^T \]
\[ + 2G(X, u)G(JY, u) - G(Y, u)G(JX, u) (JZ)^T \} \]
\[ R(X^H, Y^H)Z^H = \left( c G(Y, Z) - \frac{3c^2}{4} G(Y, u) G(Z, u) \right) X^H \\
- \left( c G(X, Z) - \frac{3c^2}{4} G(X, u) G(Z, u) \right) Y^H \\
+ \frac{3c^2}{8} \left( G(X, Z) G(Y, u) - G(Y, Z) G(X, u) \right) \xi \\
+ \left( c G(JY, Z) - \frac{3c^2}{4} G(JY, u) G(Z, u) \right) (JX)^H \\
- \left( c G(JX, Z) - \frac{3c^2}{4} G(JX, u) G(Z, u) \right) (JY)^H \\
+ \frac{3c^2}{4} \left( G(Y, Z) G(JX, u) - G(X, Z) G(JY, u) \right) (Ju)^H \\
- \frac{3c^2}{4} G(Jy, u) G(JZ, u) X^H + \frac{3c^2}{4} G(JX, u) G(JZ, u) Y^H \\
- \left\{ \frac{c^2}{2} G(Jy, u) G(Z, u) - \frac{5c^2}{4} G(Y, u) G(JZ, u) \\
- c^2 G(Y, JZ) \right\} (JX)^H \\
+ \left\{ \frac{c^2}{2} G(JX, u) G(Z, u) - \frac{5c^2}{4} G(X, u) G(JZ, u) \\
- c^2 G(X, JZ) \right\} (JY)^H \\
- \left\{ \frac{5c^2}{2} G(X, u) G(JY, u) - \frac{5c^2}{2} G(JX, u) G(Y, u) \\
- (2c - 2c^2) G(X, JY) \right\} (JZ)^H \\
- \left\{ \frac{5c^2}{8} G(JY, Z) G(JX, u) - \frac{5c^2}{8} G(JX, Z) G(JY, u) \\
- \frac{5c^2}{4} G(JX, Y) G(JZ, u) \right\} \xi \\
- \left\{ \frac{5c^2}{4} G(JY, Z) G(X, u) - \frac{5c^2}{4} G(JX, Z) G(Y, u) \\
- \frac{5c^2}{2} G(JX, Y) G(Z, u) \right\} (Ju)^H \]

Next, we determine the Ricci tensor \( R \) of \((T_1 M_n^c(4c), g)\) and its first covariant derivative. To calculate these tensors at the point \((x, u) \in T_1 M_n(4c)\), let \(E_1, \cdots, E_n = u, JE_1, \cdots, JE_n = Ju\) be an orthonormal basis of \(T_x M\). Then \(2E_1^T, \cdots, 2E_{n-1}^T, 2(JE_1)^T, \cdots, 2(JE_n)^T, 2E_1^H, \cdots, 2E_n^H = \xi, 2(JE_1)^H, \cdots, 2(JE_n)^H\) is an orthonor-
mal basis for $T_{(x,u)}T_1M$. Then $\rho$ is given by

\begin{equation}
\rho(X, Y) = \sum_{i=1}^{n} R(2E_i^T, X, Y, 2E_i^T) + \sum_{i=1}^{n} R(2(JE_i)^T, X, Y, 2(JE_i)^T) \\
+ \sum_{i=1}^{n} R(2E_i^H, X, Y, 2E_i^H) + \sum_{i=1}^{n} R(2(JE_i)^H, X, Y, 2(JE_i)^H)
\end{equation}

Thus by using (24) we see that

\begin{equation}
\rho(X^T, Y^T) = (2n - 2 + c^2)(G(X, Y) - G(X, u)G(Y, u)) \\
+ c^2(2n + 5)G(JX, u)G(JY, u), \\
\rho(X^H, Y^H) = c(2n + 2 - 3c)G(X, Y) - c^2(n + 4)G(X, u)G(Y, u) \\
- c^2(n + 4)G(JX, u)G(JY, u), \\
\rho(X^T, Y^H) = 0.
\end{equation}

Furthermore, by using (2) we obtain

\begin{equation}
(\nabla_{Z^T}\rho)(X^T, Y^T) = c^2(2n + 5)\{(G(JX, Z) - G(JX, u)G(Z, u) \\
+ G(X, u)G(JZ, u))G(JY, u) \\
+ (G(JY, Z) - G(JY, u)G(Z, u))G(JX, u) \\
+ G(Y, u)G(JZ, u))G(JX, u)\},
\end{equation}

\begin{equation}
(\nabla_{Z^H}\rho)(X^H, Y^H) = \frac{1}{2} c^2(c - 2)(n + 4)\{(G(X, Z) - G(X, u)G(Z, u))G(Y, u) \\
+ (G(Y, Z) - G(Y, u)G(Z, u))G(X, u) \\
+ (G(JX, Z) - G(JX, u)G(Z, u))G(JY, u) \\
+ (G(JY, Z) - G(JY, u)G(Z, u))G(JX, u)\},
\end{equation}

\begin{equation}
(\nabla_{Z^u}\rho)(X^T, Y^H) = \frac{c^3}{2} (n + 6)((G(X, Z) - G(X, u)G(Z, u))G(Y, u) \\
- c^3(G(X, Y) - G(X, u)G(Y, u))G(Z, u) \\
+ \frac{c^3}{2} (n + 6)((G(X, JZ) - G(X, u)G(JZ, u))G(JY, u) \\
- c^3(G(X, JY) - G(X, u)G(JY, u))G(JZ, u) \\
+ \frac{c^3}{2} (7n + 22)\{G(JX, u)G(Y, u)G(JZ, u) \\
- G(JX, u)G(JY, u))G(Z, u)\} \\
+ c\{c^2 - (n + 1)c + (n - 1)\}\{G(X, Z)G(Y, u) - G(X, Y)G(Z, u) \\
- G(JX, Z)G(JY, u) + G(JX, Y)G(JZ, u)\} \\
- c\{2(n + 9)c^2 - 2(n + 1)c + 2(n - 1)\}G(JY, Z)G(JX, u),
\end{equation}
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Proof of Theorems 5 and 6

Before we prove Theorem 6, we remark in general that the polarization and the symmetry of \( \nabla \rho \) yields that the cyclic parallel Ricci tensor condition is equivalent to \( (\nabla Z \rho)(Z, Z) = 0 \) for all vector field \( Z \). Next, we prove Theorem 6. By using (27) and \( X = X^T + Y^H \), we obtain

\[
(\nabla_{X^T + Y^H} \rho)(X^T + Y^H, X^T + Y^H) \\
= (\nabla_{X^T} \rho)(X^T, X^T) + (\nabla_{X^T} \rho)(X^T, Y^H) + (\nabla_{X^T} \rho)(Y^H, X^T) + (\nabla_{X^T} \rho)(Y^H, Y^H) \\
+ (\nabla_{Y^H} \rho)(X^T, X^T) + (\nabla_{Y^H} \rho)(X^T, Y^H) + (\nabla_{Y^H} \rho)(Y^H, X^T) + (\nabla_{Y^H} \rho)(Y^H, Y^H) \\
= 2c^2(c - 1)(n + 4) \{ G(X, Y) - G(X, u)G(Y, u) \} G(Y, u) \\
+ (G(X, JY) - G(X, u)G(JY, u))G(JY, u) .
\]

Then, taking account of (27), we have at once the following

**Theorem 8.** \( T_1M_n^c(4c) \) \( (n \geq 2) \) is Ricci-parallel \( (\nabla \rho = 0) \) if and only if \( M \) is flat \( (c = 0) \).

**Proof of Theorems 5 and 6**

We first prove Theorem 5. Then, together with (27), we have

\[
0 = (\nabla_{X^T} \rho)(X^T, Y^T) - (\nabla_{Y^H} \rho)(X^T, Y^T) \\
= c^2(2n + 5) \{ 2G(JX, Z)G(JY, u) + G(JY, Z)G(JX, u) + G(JX, Y)G(JZ, u) \\
+ 3(G(X, u)G(JZ, u) - G(JX, u)G(Z, u)G(JY, u)) \}.
\]

Suppose that \( T_1M_n^c(4c) \) \( (n \geq 2) \) have Codazzi-type Ricci tensors. Then from (29) we have \( c^2(2n + 5) = 0 \). Hence, we see that \( c = 0 \). Conversely, we easily can check that \( T_1M_n(0) \) satisfies all possibilities:

\[
(\nabla_{X^T} \rho)(X^T, Y^T) - (\nabla_{Y^H} \rho)(X^T, Y^T) = 0, \\
(\nabla_{X^T} \rho)(X^T, Y^H) - (\nabla_{Y^H} \rho)(X^T, Y^H) = 0, \\
(\nabla_{X^T} \rho)(X^H, Y^H) - (\nabla_{Y^H} \rho)(X^H, Y^H) = 0, \\
(\nabla_{Y^H} \rho)(X^H, Y^T) - (\nabla_{X^T} \rho)(X^H, Y^T) = 0, \\
(\nabla_{Y^H} \rho)(X^H, Y^H) - (\nabla_{X^T} \rho)(X^H, Y^H) = 0.
\]
From (31), we see that it is necessary and sufficient condition for $T^c_1M_n^c (4c)$ ($n \geq 2$) to have cyclic parallel Ricci tensors that $c = 0$ or $c = 1$. This has completed the proof of Theorem 6.

Reference


