Minimal surfaces in $S^3$ and Yau’s conjecture

Jaigyoung Choe
Department of Mathematics, Seoul National University, Seoul, 151-742, Korea
e-mail: choe@math.snu.ac.kr

Abstract. We list some known facts and open problems about minimal surfaces in $S^3$. And we sketch a proof of Yau’s conjecture for Lawson’s minimal surfaces and Karcher-Pinkall-Sterling’s minimal surfaces.

1 Minimal Surfaces in $S^3$

The catenoid, the helicoid, Scherk’s surfaces, and some triply periodic minimal surfaces had been the only complete embedded minimal surfaces known to exist in $\mathbb{R}^3$ until Costa and Hoffman-Meeks constructed minimal surfaces of arbitrary genus in 1980’s. In the three-dimensional sphere $S^3$ Lawson [L1] constructed compact embedded minimal surfaces of arbitrary genus, and Karcher-Pinkall-Sterling [KPS] added some more examples. Both in $\mathbb{R}^3$ and in $S^3$, a paucity of examples has been a main obstacle to the study of embedded minimal surfaces. Still, we know some a priori properties of compact minimal surfaces in $S^3$ as follows.

1. An immersed minimal sphere in $S^3$ is totally geodesic. (Almgren)
2. The center of gravity of a compact minimal submanifold of $S^n$ is at the origin.
3. Two minimal hypersurfaces of $S^n$ must intersect each other. (Frankel [F])
4. For each integer $g$ there is a compact embedded minimal surface of genus $g$ in $S^3$. (Lawson [L1])
5. In $S^3$ there exist compact embedded minimal surfaces of genus 3, 5, 6, 7, 11, 17, 19, 73, and 601. (Karcher-Pinkall-Sterling [KPS])
6. To each complete minimal surface in $S^3$ there is a complete locally isometric surface of constant mean curvature in $\mathbb{R}^3$. (Lawson [L1])
7. Embedded minimal surfaces in $S^3$ cannot have knotted handles. (Lawson [L2])
8. If a compact branched minimal surface and a great circle in $S^3$ are disjoint, then they are linked. (Solomon [S])
9. The space of compact embedded minimal surfaces in $S^3$ is compact in $C^k$ topology. (Choi-Schoen [ChS])
10. The Morse index of compact minimal surfaces in $\mathbb{S}^3$ is 1 for the great sphere, 5 for the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$, and higher for the others. (Urbano [U])

11. If the boundary of a compact immersed orientable and stable minimal hyper-surface $\Sigma$ in $\mathbb{S}^n$ lies in a great sphere $\mathbb{S}^{n-1}$, then $\Sigma \subset \mathbb{S}^{n-1}$. (Ros [R])

12. If the boundary of a compact immersed orientable minimal hypersurface $\Sigma$ in $\mathbb{S}^n$ lies in a great sphere $\mathbb{S}^{n-1}$, then $\text{Vol}(\Sigma) \geq \frac{1}{2}\text{Vol}(\mathbb{S}^{n-1})$, with equality only if $\Sigma$ is a hemisphere. (Ros [R])

13. The only compact embedded orientable minimal surface in $\mathbb{S}^3$ that bounds a great circle is the hemisphere. (Hardt-Simon [HS])

14. If a compact embedded orientable minimal surface $\Sigma$ in $\mathbb{S}^3$ bounds two orthogonally intersecting great circles, then $\Sigma$ is a half of the Clifford torus. (Hardt-Rosenberg [HR])

15. In $\mathbb{S}^n$ any great sphere divides a compact embedded minimal hypersurface into two connected pieces. (Ros [R])

16. The Gauss map of a minimal surface $\Sigma \subset \mathbb{S}^3$ gives a branched minimal surface $\Sigma^*$ in $\mathbb{S}^3$. Moreover, $\Sigma^{**} = \Sigma$. (Lawson [L1])

17. If $\Sigma$ is a compact embedded minimal torus in $\mathbb{S}^3$, then its Gauss image $\Sigma^*$ is also embedded. (Ros [R])

18. For each conformal structure on a compact surface, there exists at most one metric admitting a minimal immersion into $\mathbb{S}^n$ on which the first eigenvalue of the Laplacian equals two. (Montiel-Ros [MR])

19. The only minimal torus in $\mathbb{S}^3$ on which the first eigenvalue of the Laplacian equals two is the Clifford torus. (Montiel-Ros [MR])

Now let’s consider some open problems and conjectures for minimal surfaces in $\mathbb{S}^n$:

1. Is there a complete immersed minimal surface in $\mathbb{S}^3$ which is disjoint from a great sphere $\mathbb{S}^2$? This is an $\mathbb{S}^3$-version of Calabi’s question which was solved affirmatively by Nadirashvili [N].

2. For any given integer $g$ there are only finitely many noncongruent minimal surfaces of genus $g$ in $\mathbb{S}^3$.

3. (Lawson’s conjecture) The only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus. Combining with (2), one may even conjecture that the only compact embedded minimal surfaces are the surfaces $\xi_{m,k}$ constructed by Lawson in [L1].

4. (Yau’s conjecture [Y]) The first eigenvalue of the Laplacian on a compact embedded minimal hypersurface $\Sigma^n$ in $\mathbb{S}^{n+1}$ is equal to $n$. 

Let $x_1, \ldots, x_m$ be the rectangular coordinates of $\mathbb{R}^m$ and let $X := (x_1, \ldots, x_m)$. Given a submanifold $M$ of $\mathbb{R}^n$, it is well known that

$$\Delta_M X = \vec{H},$$

where $\vec{H}$ is the mean curvature vector of $M$. Therefore $x_1, \ldots, x_m$ are harmonic functions on a minimal submanifold $\Sigma \subset \mathbb{R}^m$. If $\Sigma^n$ is minimal in $S^{m-1}$, then the cone $O \times \Sigma$ is also minimal in $\mathbb{R}^m$. Therefore $\Delta_\Sigma X$ must be perpendicular to $S^{m-1}$ and hence $\Delta_\Sigma X$ is parallel to $X$. Then it is not difficult to show that

$$\Delta_\Sigma X + nX = 0.$$ 

Therefore $x_1, \ldots, x_m$ are eigenfunctions of $\Delta$ with eigenvalue $n$ on the $n$-dimensional minimal submanifold $\Sigma$ of $S^{m-1}$.

Thus it was natural for Yau to propose his conjecture as above. Yau’s conjecture does not concern minimal surfaces with nonempty self intersection and minimal surfaces of high codimension because a minimal surface of revolution of large area in $S^3$ and the Veronese surface in $S^4$ have the first eigenvalue much smaller than two.

It may have been just out of curiosity that Yau made his conjecture. But Montiel-Ros [MR] showed that Yau’s conjecture has a geometric implication: If Yau’s conjecture is true, then the Clifford torus is the only embedded minimal torus in $S^3$, i.e., Lawson’s conjecture is true as well. It should be mentioned that Choi-Wang [CW] proved that the first eigenvalue on $\Sigma^n$ is at least $n/2$.

There is a well-known theorem by Courant that the first eigenfunction of $\Delta$ on $\Sigma$ has two nodal domains. In this regard it is very interesting to note that a compact embedded minimal surface in $S^3$ has two-piece property: Ros [R] proved that any great sphere in $S^3$ divides a compact embedded minimal surface $\Sigma$ into two connected pieces. However, if Yau’s conjecture is true, then Ros’s two-piece property follows from Courant’s theorem. Indeed, if $2$ is the first eigenvalue of $\Delta$, then Courant’s nodal theorem for the linear function $\phi = a_1 x_1 + \ldots + a_4 x_4$ with $\phi|_{\mathbb{S}^2} = 0$ implies the two-piece property.

Therefore, now that the two-piece property holds, one might presume that Yau’s conjecture should be true. As a matter of fact, the author and M. Soret [CS] found that by using Courant’s nodal theorem and Ros’s two piece property one can prove Yau’s conjecture for minimal surfaces in $S^3$ which are sufficiently symmetric (as much symmetric as Lawson’s surfaces and Karcher-Pinkall-Sterling’s surfaces).

2 Yau's Conjecture

In this section we briefly outline the arguments of our paper [CS].

**Lemma 1.** If the boundary of a compact immersed orientable and stable minimal hypersurface $\Sigma^n$ in $S^{n+1}$ lies in a great sphere, then $\Sigma$ is totally geodesic.
Proof. See Lemma 1 of [CS].

**Theorem 1.** Any great sphere in $\mathbb{S}^{n+1}$ divides a compact embedded minimal hypersurface $\Sigma$ of $\mathbb{S}^{n+1}$ into two connected pieces.

Proof. See Theorem 1 of [CS].

**Lemma 2.** Let $G$ be a group of reflections in $\mathbb{S}^3$. Assume that a minimal surface $\Sigma \subset \mathbb{S}^3$ is invariant under $G$. If the first eigenvalue of $\Delta$ on $\Sigma$ is less than 2, then the first eigenfunction must be invariant under $G$.

Proof. (Sketch) Let $\sigma \in G$ be the reflection across a great sphere $\Pi$ in $\mathbb{S}^3$ and let $\phi$ be an eigenfunction on $\Sigma$ corresponding to the first eigenvalue $\lambda_1$. Note that $\phi \circ \sigma$ is again an eigenfunction with eigenvalue $\lambda_1$. Consider

$$\psi(x) := \phi(x) - \phi \circ \sigma(x).$$

If $\psi$ is the null function then $\phi$ is invariant under $\sigma$. If $\psi \neq 0$ then $\psi$ itself is an eigenfunction with eigenvalue $\lambda_1$. Furthermore its nodal set contains $\Sigma \cap \Pi$. But Courant’s nodal theorem implies that $\psi$ vanishes only on $\Sigma \cap \Pi$. Let $D_1, D_2$ be the components of $\Sigma \setminus \Pi$ such that $\psi$ is positive on $D_1$ and negative on $D_2$. By Ros’s two-piece property $D_1, D_2$ are each connected. One can find a linear function of $\mathbb{R}^4$ $\xi = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$ that vanishes on $\Pi$ and is positive on $D_1$. Clearly $\xi$ is orthogonal to $\psi$ on $\Sigma$. But $\psi$ and $\xi$ have the same sign on $D_1 \cup D_2$, which contradicts the orthogonality of $\psi$ and $\xi$. Therefore $\psi$ must vanish on $\Sigma$. This completes the proof as $\sigma$ is an arbitrary element of $G$.

**Theorem 2.** Let $\Sigma$ be a minimal surface in $\mathbb{S}^3$ which is invariant under a group $G$ of reflections. Suppose that the fundamental domain of $G$ in $\mathbb{S}^3$ is a tetrahedron $T$. If the fundamental patch $S = \Sigma \cap T$ is simply connected and has four edges, then the first eigenvalue of the Laplacian on $\Sigma$ equals 2.

Proof. Suppose $\lambda_1 < 2$. Let $\phi$ be an eigenfunction with eigenvalue $\lambda_1$ on $\Sigma$ and $N \subset \Sigma$ the nodal set of $\phi$. From Lemma 2 it follows that $S \setminus N$ has at least two
connected components. Since $S$ is simply connected one can find a face $F$ of $T$ and a component $D$ of $S \setminus N$ such that $\partial D$ is disjoint from $F$. Let $\Pi$ be the great sphere containing $F$ and let $\hat{D}$ be the mirror image of $D$ across $\Pi$. Denote by $D_1, D_2, D_3$ the components of $\Sigma \setminus N$ containing $D, \hat{D}$ and intersecting $\Pi$, respectively. We claim that $D_1, D_2, D_3$ are all distinct. $D_2$ is the mirror image of $D_1$ and $D_3$ is nonempty and symmetric with respect to $\Pi$. See [CS] for the details. Therefore $\phi$ has at least three nodal domains, which contradicts Courant’s nodal theorem. Thus $\lambda_1 = 2$. □

**Lemma 3.** Lawson’s minimal surfaces $\xi_{m,k}$ can also be constructed in the same way as Karcher-Pinkall-Sterling’s surfaces are constructed.

**Proof.** See Section 2 of [CS]. □

**Corollary 1.** The first eigenvalue of the Laplacian on Lawson’s embedded minimal surfaces $\xi_{m,k}$ and Karcher-Pinkall-Sterling’s minimal surfaces in $S^3$ is equal to 2.

**Theorem 3.** Let $\Sigma$ be a compact embedded minimal surface in $S^3$ which is invariant under a group or reflections, and let $D \subset \Sigma$ be a fundamental patch in a tetrahedron of the tessellation. If $D$ is simply connected and has at most five edges, then $\lambda_1(\Sigma) = 2$.

**Proof.** See Theorem 3 of [CS]. □

**Remark.** If the fundamental patch $D$ has six edges, $\lambda_1$ may still equal two in case the genus of the minimal surface is sufficiently small. See Section 6 of [CS] for the details.

**References**


