Pseudo-Z symmetric space-times with divergence-free Weyl tensor and $pp$-waves

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Dedicated to the memory of Professor Witold Roter

In this paper we present some new results about $n(\geq 4)$-dimensional pseudo-Z symmetric space-times. First we show that if the tensor $Z$ satisfies the Codazzi condition then its rank is one, the space-time is a quasi-Einstein manifold, and the associated 1-form results to be null and recurrent. In the case in which such covector can be rescaled to a covariantly constant we obtain a Brinkmann-wave. Anyway the metric results to be a subclass of the Kundt metric. Next we investigate pseudo-Z symmetric space-times with harmonic conformal curvature tensor: a complete classification of such spaces is obtained. They are necessarily quasi-Einstein and represent a perfect fluid space-time in the case of time-like associated covector; in the case of null associated covector they represent a pure radiation field. Further if the associated covector is locally a gradient we get a Brinkmann-wave space-time for $n > 4$ and a $pp$-wave space-time in $n = 4$. In all cases an algebraic classification for the Weyl tensor is provided for $n = 4$ and higher dimensions. Then conformally flat pseudo-Z symmetric space-times are investigated. In the case of null associated covector the space-time reduces to a plane wave and results to be generalized quasi-Einstein. In the case of time-like associated covector we show that under the condition of divergence-free Weyl tensor the space-time admits a proper concircular vector that can be rescaled to a line like vector of concurrent form and is a conformal Killing vector. A recent result then shows that the metric is necessarily a generalized Robertson–Walker space-time. In particular we show that a conformally flat $(PZS)_n$, $n \geq 4$, space-time is conformal to the Robertson–Walker space-time.

Keywords: Pseudo-Z symmetric space-times; conformal curvature tensor; Weyl compatible manifolds; recurrent conformal 2-forms; Petrov types; Bel–Debever conditions in
1. Introduction

Recently the present authors in [79, 84] introduced and studied a type of pseudo-Riemannian manifold satisfying the following relation:

\[ \nabla_k Z_{jl} = 2A_k Z_{jl} + A_j Z_{kl} + A_l Z_{kj}, \]  

where \( Z_{kl} \) is a symmetric tensor defined as

\[ Z_{kl} = R_{kl} + \phi g_{kl} \]

being \( \phi \) a scalar function, and \( A_k \) is a non-null covector called associated 1-form, \( \nabla \) is the operator of covariant differentiation with respect to the metric \( g_{kl} \) and the manifold is called \( n \)-dimensional pseudo-\( Z \) symmetric and denoted by \((PZS)_n\). The Ricci tensor is defined as

\[ R_{kl} = -\frac{1}{2}g_{mj} R_{jklm} \]  

and the scalar curvature as

\[ R = g^{ij} R_{ij} \].

If \( \phi = 0 \) we recover a pseudo-Ricci symmetric manifold introduced by Chaki [17], Arslan et al. [7]. It should be noted that if \( Z_{kl} = 0 \), then we have \( R_{kl} = -\phi g_{kl} \) and thus \( R = -n \phi \) so that \( R_{kl} = \frac{R}{n} g_{kl} \) and the manifold is an Einstein space. Throughout the paper we will avoid this condition. In [79] the present authors considered conformally harmonic and conformally flat pseudo-\( Z \) symmetric Riemannian manifolds, i.e. \((PZS)_n\) on which the conformal curvature tensor

\[ C_{jklm} = R_{jklm} + \frac{1}{n-2} \left( \delta^m_j R_{kl} - \delta^m_k R_{jl} + R^m_j g_{kl} - R^m_k g_{jl} \right) \]

satisfies the condition \( \nabla_m C_{jklm} = 0 \) or vanishes, that is \( C_{jklm} = 0 \): we recall that the corresponding \((0,4)\) curvature tensor is defined by \( C_{jklm} = g_{mp} C_{jklm} \) and that the conformal curvature tensor vanishes identically for \( n = 3 \) [102]. In the same reference it was shown that an \( n \)-dimensional conformally flat \((PZS)_n\) pseudo-Riemannian manifold with non-singular \( Z_{kl} \) tensor admits a proper concircular vector and the local form of the metric was presented; moreover a sufficient condition for a \((PZS)_n\) manifold to be Ricci pseudo-symmetric in the sense of Deszcz was shown [41]. On the other hand in [74] the authors defined and studied a weakly-\( Z \) symmetric Riemannian manifold, i.e. a differential structure on which the tensor \( Z_{kl} \) satisfies the condition

\[ \nabla_k Z_{jl} = A_k Z_{jl} + B_j Z_{kl} + D_l Z_{kj}, \]  

being \( A_k, B_k, D_k \) non-null associated covectors. Here we set the following.

**Definition 1.1.** A pseudo-\( Z \) symmetric space-time is an \( n(\geq 4) \)-dimensional Lorentzian manifold satisfying (1.1) for which \( Z_{kl} = \kappa T_{kl} \) being \( \kappa = \frac{8\pi G}{c^4} \) the
Einstein’s gravitational constant and $T_{\mu\nu}$ the energy–momentum tensor (see [34, 63]) describing the matter content of the space-time.

In this case the condition $\nabla^\mu T_{\mu\nu} = 0$ gives $\varphi = -\frac{R}{2} + \Lambda$ and the term can be viewed as a cosmological constant and Einstein’s equations take the form $R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ (see for example [34, 63, 115–117]). However hereafter we will set $\Lambda = 0$ for simplicity. In view of the above considerations Eq. (1.1) is really a condition on the energy–momentum tensor, namely:

$$\nabla_k T_{jl} = 2A_k T_{jl} + A_j T_{kl} + A_l T_{kj}. \quad (1.4)$$

In general this is not sufficient to determine completely the tensor $T_{\mu\nu}$. In fact in [84] we studied 4-dimensional pseudo-Z symmetric space-times (see also the paper [19] for covariantly constant energy–momentum tensor or [112, 60]) specifying the nature of the energy–momentum tensor and then studying consequences of restriction (1.4). In this way investigations of perfect fluid and scalar field $(PZS)_4$ space-times were pursued and interesting properties were pointed out, including the Petrov [116] classification of the Weyl tensor. In the same paper [84] we gave some examples of $(PZS)_4$ space-times.

In this paper we generalize the results obtained in our papers [79, 84] showing some new interesting facts about $n(\geq 4)$-dimensional pseudo-Z symmetric space-times. We will show that a specific curvature restriction on the conformal curvature tensor gives rise to a completely determined $T_{\mu\nu}$ and allows an algebraic classification of such space-times. In Sec. 2 some known properties of $(PZS)_4$ space-times [84] are reviewed and readily extended to $n(\geq 4)$ dimensions; we prove also that the covector $A_k$ satisfying (1.1) is unique. Further we show that on a $(PZS)_4$ space-time with $Z_{\mu\nu}$ satisfying the Codazzi equation [36, 79], i.e. $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ the vector $A_k$ is null and recurrent, i.e. satisfies the condition $\nabla_j A_k = p_j A_k$ for a suitable 1-form $p_j$, and the rank of tensor $Z_{\mu\nu}$ is one; the space-time can be described by Walker coordinates [85, 14, 70, 109, 68, 69]. In the case in which $A_k$ is locally a gradient such null covector can be rescaled to a covariant constant one and the space is a Brinkmann-wave [14, 68, 69]. Anyway the metric results to be a subclass of the Kundt metric [100, 101, 26, 95].

In Secs. 3 and 4 we study $(PZS)_n$ space-times with harmonic conformal curvature tensor, i.e. with the property $\nabla^A C_{ijkl}^n = 0$. In Sec. 3 the case of time-like associated covector, i.e. $A_j A^j < 0$ is investigated: it is shown that such space time is necessarily a perfect fluid space-time; the form of the Ricci tensor and a state equation are provided. The Weyl tensor results to be purely electric in $n \geq 4$ (see [95] and references therein) and vanishes in $n = 4$. In Sec. 4 the case of light-like associated covector, i.e. $A_j A^j = 0$ is considered and the space reduces to a pure radiation field (see [58, 116, 71]) or null dust field; the local form of the Ricci tensor is again provided. The Weyl tensor results to be of Petrov type III (or more special) for $n = 4$ and the Weyl type is at least as special as type $II_{ab}$ for $n > 4$ (see [90, 93, 95]).
Moreover, if the associated covector is locally a gradient the space-time reduces to Brinkmann-wave according to the definitions of \cite{70, 109, 68, 69}. In this case the Weyl tensor is of Petrov type \( N \) with respect to \( A_k \) in \( n = 4 \) and it is at least as special as type \( II_{a b d} \) with respect to \( A_k \) in \( n > 4 \); further in \( n = 4 \) the space results a pp-wave according to \cite{70, 109, 68, 69}. In Sec. 5 we take into consideration conformally flat (PZS) \( n \): in the case of null associated covector the space-time reduces to a plane wave \cite{see [56]} and results to be generalized quasi-Einstein \cite{16, 15}. In the case of time-like associated covector we show that under the condition \( \nabla_m C_{j k l} = 0 \) the space-time admits a proper concircular vector \cite{see [120, 121]} that can be rescaled to a time like vector of the form \( \nabla_k X_j = \rho g_{kj} \) and it is a conformal Killing vector \cite{116}.

A recent result achieved in \cite{20} then shows that the metric is necessarily a generalized Robertson–Walker space-time \cite{5, 107, 108}. In particular we show that a conformally flat (PZS) \( n \) space-time is conformal to the Robertson–Walker space-time \cite{116, 10}. Finally in Sec. 6 we expose some further examples of metrics that are pseudo-Z symmetric. In detail it is shown that under particularly simple conditions the pp-wave space-times are pseudo-Z symmetric with divergence-free Weyl tensor; we find a curvature condition of differential type for the Weyl tensor that holds for arbitrary wave potential; finally we give sufficient conditions for the conformal 2-form to be recurrent \cite{80, 85}. Throughout the paper, all manifolds under consideration are assumed to be connected Hausdorff manifolds endowed with a non-degenerate metric of arbitrary signature, i.e. \( n \)-dimensional pseudo-Riemannian manifolds. When necessary we will restrict to \( n \)-dimensional Lorentzian manifolds, i.e. with metrics of signature \( s = n - 2 \) \cite{63}.

In this context it is possible to introduce a null frame (see for example \cite{95}) \( l_a, n_a, m(i) a, i = 2, \ldots, n - 1 \) with two null vectors and \( n - 2 \) space-like vectors such that \( l^n a = n^a n_a = 0, l^a n_a = 1, m^a(i) m_a(i) = \delta_{ij} \). The corresponding form of the metric results to be \( g_{ab} = 2l_a n_b + \delta_{ij} m(i) a m(j) b \). Finally we will assume also a vanishing cosmological constant, i.e. \( \Lambda = 0 \).

2. General Properties of \((PZS)_n\) Space-Time

In this section elementary properties of a \((PZS)_n\) space-time are reviewed and some new interesting properties are introduced. Some of them were studied by the present authors in \cite{79, 84}. Let \( M \) be a non-flat \( n \geq 4 \)-dimensional \((PZS)_n\) Lorentzian manifold with metric \( g_{ij} \), the Levi-Civita connection \( \nabla \) and endowed with the Einstein’s equation \( Z_{kl} = \kappa T_{kl} \). As \( \Lambda = 0 \) we get \( \varphi = -R \) and from \( Z_{kl} = R_{kl} + \varphi g_{kl} \) we get \( Z = R + n \varphi \) so that \( Z = \frac{(2-n)R}{2} \), where \( Z = g^{kl} Z_{kl} \) denotes the trace of the symmetric tensor \( Z_{kl} \) and \( R = g^{kl} R_{kl} \) the scalar curvature of \( M \) respectively. First we show that if the one form \( A_k \) satisfies condition \( (1.1) \) then such form is unique: this fact holds generally on an \( n \)-dimensional \((PZS)_n\) pseudo-Riemannian manifold. Let us suppose that another covector \( B_k \) satisfies condition \( (1.1) \), namely

\[
\nabla_k Z_{jl} = 2B_k Z_{jl} + B_j Z_{kl} + B_l Z_{kj},
\]

(2.1)
then we get immediately

\[ 0 = 2\omega_k Z_{jl} + \omega_j Z_{kl} + \omega_l Z_{kj}, \quad (2.2) \]

being \( \omega_j = A_j - B_j \). Interchanging indices \( k \) and \( j \) in (2.2) we get

\[ 0 = 2\omega_j Z_{kl} + \omega_k Z_{jl} + \omega_l Z_{jk}, \]

and subtracting this from (2.2) we infer easily:

\[ \omega_j Z_{kl} = \omega_k Z_{jl}. \quad (2.3) \]

In this way Eq. (2.2) takes the form

\[ 0 = 2\omega_k Z_{jl} + \omega_j Z_{kl} + \omega_l Z_{kj} = 2\omega_k Z_{jl} + 2\omega_j Z_{jl} = 4\omega_k Z_{jl} \]

from which we infer \( \omega_k = 0 \).

**Proposition 2.1.** Let \( M \) be an \( n \geq 4 \)-dimensional \((PZS)_n\) pseudo-Riemannian manifold, then the covector \( A_j \) satisfying \((1.1)\) is unique.

Second we recall two fundamental relations (see [79, 84]) coming from definition \((1.1)\). Transvecting \((1.1)\) with \( g^{kl} \) and \( g^{kl} \) recalling the condition \( \nabla^l Z_{kl} = 0 \) coming from the stress–energy tensor we get:

\[ A^l Z_{kl} = -\frac{A_k}{3} Z, \quad \nabla_k Z = \frac{4}{3} A_k Z, \quad (2.4) \]

being \( Z = g^{kl} Z_{kl} \). From this we note that \( \nabla_k Z \neq 0 \) if and only if \( Z \neq 0 \) (and thus if \( R \neq 0 \)) and consequently we have proven the following.

**Proposition 2.2.** Let \( M \) be an \( n \geq 4 \)-dimensional \((PZS)_n\) space-time. If \( Z \) is a non-zero at a point \( p \) of \( M \), then on a some coordinate domain \( U \) of \( p \), \( \nabla_k A_l = \nabla_l A_k \), i.e. the covector \( A_k \) is locally a gradient. Moreover, the covector \( A_k \) is an eigenvector of \( Z_{kl} \) with eigenvalue \( -\frac{2}{3} \).

Now from \((1.1)\) it is easily inferred that:

\[ \nabla_k Z_{jl} - \nabla_j Z_{kl} = A_k Z_{jl} - A_j Z_{kl}. \quad (2.5) \]

In recent papers (see [80–82, 85]) the present authors introduced the notion of recurrent forms on pseudo-Riemannian manifolds. In particular the \( Z \) form \( \Lambda_{(Z)l} = Z_{kl} dx^k \) is defined to be recurrent if the following relation is satisfied:

\[ D\Lambda_{(Z)l} = \beta \land \Lambda_{(Z)l}, \quad (2.6) \]

being \( D \) the exterior covariant derivative and \( \beta = \beta_i dx^i \) an associated 1-form. From [85, Theorem 6; 80, Theorem 2.4] it turns out that Eq. (2.5) is a necessary and sufficient condition to have \( D\Lambda_{(Z)l} = \Lambda \land \Lambda_{(Z)l} \). So we have the following.

**Proposition 2.3.** Let \( M \) be an \( n \geq 4 \)-dimensional \((PZS)_n\) pseudo-Riemannian manifold, then the form \( \Lambda_{(Z)l} = Z_{kl} dx^k \) is recurrent, i.e. satisfies the condition \( D\Lambda_{(Z)l} = \Lambda \land \Lambda_{(Z)l} \).

In [84] we showed that on an \( n \geq 4 \)-dimensional \((PZS)_n\) space-time with closed one form \( A_k \) the Ricci tensor results to be Weyl compatible; this was
recently introduced in [75–77] and then investigated in detail by some authors (see [44, 50, 75–77, 83, 85]): it is of some importance also in General Relativity as shown in [28, 26]. First the following relation was proved:

\[ R_{im} R_{jkl} - R_{jm} R_{kil} + R_{km} R_{ijl} = 0. \] (2.7)

A pseudo-Riemannian manifold on which the Ricci tensor satisfies the previous relation is called \textit{Riemann compatible} manifold (see [44, 50, 75–77, 84]). In terms of the local components of the Weyl tensor (1.2) Eq. (2.7) reads:

\[ R_{im} C_{jkl} + R_{jm} C_{kilm} + R_{km} C_{ijlm} = 0. \] (2.8)

A pseudo-Riemannian manifold on which the Ricci tensor satisfies relation (2.8) is called \textit{Weyl compatible} manifold (see [44, 50, 76, 77, 83, 85]). It should be remarked here that some authors achieved Eq. (2.7) in their investigations about pseudo-Riemannian manifolds endowed with differential structures as [6, 33, 43, 103]. However a study of its consequence on the structure of the Riemann and Weyl tensors and on Weyl’s scalars and thus on the classification of space-times was pursued in [75–77, 83, 85]. In terms of the stress–energy tensor condition (2.8) reads:

\[ T_{im} C_{jkl} + T_{jm} C_{kilm} + T_{km} C_{ijlm} = 0. \] (2.9)

Then the result of [84] was stated as follows.

\textbf{Proposition 2.4 ([84])}. Let \( M \) be an \( n \) (\( \geq 4 \))-dimensional \((PZS)\) space-time. If \( Z \neq 0 \), then the space-time is a Weyl compatible manifold and the stress–energy tensor results to be Weyl compatible.

Equation (2.9) is of great importance in the classification of perfect fluid and scalar field \((PZS)\) space-times as shown in [83–85]. In general Weyl compatibility arises from an identity involving the divergence of the Weyl tensor: it follows from the Lovelock’s identity for the Riemann tensor [72] as exposed in [73, 76, 77] i.e.:

\textbf{Lemma 2.5 ([73, 76, 77])}. Let \( M \) be an \( n \) (\( \geq 4 \))-dimensional pseudo-Riemannian manifold. On any coordinate domain of \( M \) the following identity is satisfied:

\[ \nabla_i \nabla_m C_{jkl} + \nabla_j \nabla_m C_{kil} + \nabla_k \nabla_m C_{ijl} = -\frac{n-3}{n-2} (R_{im} R_{jkl} + R_{jm} R_{kil} + R_{km} R_{ijl}). \] (2.10)

The previous equation, in view of \( R_{im} R_{jkl} + R_{jm} R_{kil} + R_{km} R_{ijl} = R_{im} C_{jkl} + R_{jm} C_{kil} + R_{km} C_{ijl} \) and of Einstein’s field equation, takes the form

\[ \nabla_i \nabla_m C_{jkl} + \nabla_j \nabla_m C_{kil} + \nabla_k \nabla_m C_{ijl} = -\frac{n-3}{n-2} (T_{im} C_{jkl} + T_{jm} C_{kil} + T_{km} C_{ijl}), \] (2.11)

as shown in [76]. The divergence of the Weyl tensor depends only locally on the sources of gravitational fields. The left-hand side of (2.11) is, up to a multiplicative
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factor, the exterior covariant derivative of the vector valued form $\Pi_{(C)j} = \nabla_m C_{jkl}^m dx^l \wedge dx^k$, i.e. $D\Pi_{(C)j} = 2(\nabla_i \nabla_m C_{jkl}^m + \nabla_j \nabla_m C_{kli}^m + \nabla_k \nabla_m C_{ijl}^m)$. The following result was stated in [77].

**Proposition 2.6 ([77, Theorem 2.3]).** Let $M$ be an $n(\geq 4)$-dimensional space-time manifold, then the energy momentum is Weyl compatible if and only if the curvature condition $D\Pi_{(C)j} = 0$ holds.

**Corollary 2.7.** Let $M$ be an $n(\geq 4)$-dimensional $(PZS)_n$ space-time. If $R \neq 0$ then $D\Pi_{(C)j} = 0$.

For example if $\nabla_m C_{jkl}^m = 0$ Eq. (2.7) is then easily recovered. In Secs. 3 and 4 we will take into consideration $(PZS)_n$ space-times with the property $\nabla_m C_{jkl}^m = 0$: in this case the manifold is again Weyl compatible and Eqs. (2.7)–(2.9) are still satisfied independently from the closedness of the form $A_k$.

We recall here that a covariant tensor $T_{i...i}$ is said to be **recurrent** if $\nabla_k T_{i...i} = p_k T_{i...i}$ for some non-vanishing 1-form $p_k$.

We present now some important properties that hold in the case in which the tensor $Z_{kl}$ satisfies the Codazzi condition [36, 79], i.e. $\nabla_k Z_{jl} = \nabla_j Z_{kl}$. From (2.5) we get:

$$A_j Z_{kl} = A_k Z_{jl}. \quad (2.12)$$

From this and (1.1) using the same way as in Proposition 2.1 it follows immediately that:

$$\nabla_j Z_{kl} = 4A_j Z_{kl}, \quad (2.13)$$

and the tensor $Z_{kl}$ results to be recurrent. Further from the condition $\nabla^l Z_{kl} = 0$ we get $A^l Z_{kl} = 0$; thus on multiplying Eq. (2.12) by $A^l$ it is easily seen that:

$$(A^l A_j) Z_{kl} = 0 \quad (2.14)$$

and $A_k$ results to be a null vector, i.e. $A^l A_j = 0$. Now let $\theta^k$ a vector such that $\theta^k A_k = 1$: from (2.12) we have $Z_{kl} = A_k \theta^j Z_{jl}$ and by symmetry also $A_k \theta^l Z_{jl} = A_j \theta^l Z_{jk}$ and thus $\theta^l Z_{jl} = A_l (\theta^k \theta^l Z_{kj})$ from which finally:

$$Z_{kl} = \psi A_k A_l, \quad (2.15)$$

being $\theta^m \theta^l Z_{mj} = \psi$ a scalar function. The rank of the tensor $Z_{kl}$ is thus one. Contracting Eq. (2.12) with $g^{kl}$ we get also $A_j Z = 0$ from which $Z = 0$ in this case, so that $R = 0$ and thus the Ricci tensor is given by $R_{kl} = Z_{kl} = \psi A_k A_l$ and its rank is one. In view of (2.13) the Ricci tensor is also recurrent, i.e. $\nabla_j R_{kl} = 4A_j R_{kl}$ and the following proposition may be stated.

**Proposition 2.8.** Let $M$ be an $n(\geq 4)$-dimensional $(PZS)_n$ space-time. If the condition $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ holds on some coordinate domain $U$, then on this set the tensors $Z_{kl}$ and $R_{kl}$ are recurrent, i.e. $\nabla_k Z_{jl} = 4A_k Z_{jl}$ and $\nabla_k R_{jl} = 4A_k R_{jl}$, the vector $A_k$ is null and the rank of the tensors $Z_{kl}$ and $R_{kl}$ is one, i.e. $Z_{kl} = R_{kl} = \psi A_k A_l$.  

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We follow now a method due to Roter in [105, Theorem 1]. Inserting (2.15) in (2.13) after a straightforward calculation we infer:

\[(\nabla_j A_k)A_l + A_k(\nabla_j A_l) = [4A_j - \nabla_j \ln|\psi||]A_k A_l.\] (2.16)

On multiplying the previous result by \(\theta^l\) we get easily:

\[\nabla_j A_k + A_k \theta^l(\nabla_j A_l) = [4A_j - \nabla_j \ln|\psi||]A_k.\] (2.17)

Again a multiplication by \(\theta^k\) gives:

\[(\nabla_j A_k)\theta^k = \frac{1}{2}[4A_j - \nabla_j \ln|\psi||]A_k \theta^k.\] (2.18)

and inserting back in (2.17) the covector \(A_j\) results to be recurrent, i.e.

\[\nabla_j A_k = \frac{1}{2}[4A_j - \nabla_j \ln|\psi||]A_k = p_j A_k.\] (2.19)

If the covector \(A_j\) is closed, then from the recurrence relation we get \(p_j A_k = p_k A_j\) and transvecting this with \(\theta^k\) it is easily seen that \(p_j = \gamma A_j\) for some function \(\gamma\) and thus:

\[\nabla_j A_k = \gamma A_j A_k.\] (2.20)

Now let us suppose that the one form \(A_k\) is locally a gradient, i.e. \(A_j = \nabla_j h\) for some scalar function \(h\) on the manifold; it can be easily seen that the rescaled null covector \(\bar{A}_k = A_k e^{-\frac{1}{2}[4h - \ln|\psi||]}\) is covariantly constant, i.e. \(\nabla_\theta \bar{A}_k = 0\); we have proved the following.

**Proposition 2.9.** Let \(M\) be an \(n(\geq 4)\)-dimensional \((PZS)_n\) space-time. If the condition \(\nabla_k Z_{jl} = \nabla_j Z_{kl}\) holds on a some coordinate domain \(U\) then on this set we have: the null covector \(A_k\) is recurrent, i.e. \(\nabla_j A_k = p_j A_k\) for some 1-form \(p_j\); further if the same covector is locally a gradient, then it can be rescaled to a null covariant constant.

Lorentzian manifolds with recurrent null vectors were studied for a long time (see for example [118, 14, 109, 68, 69]). In particular Walker [118] found a set of canonical coordinates for the metric in such case. Here we refer to [69, Proposition 1].

**Proposition 2.10 ([118, 14, 109, 68, 69]).** Let \(M\) be a Lorentzian manifold of dimension \(n(\geq 4)\) admitting a recurrent null vector field \(X\) and let \(\nabla_k X_j = p_k X_j\) holds on a coordinate domain \(U\).

(1) This is equivalent to the existence of coordinates \((v, x_1, \ldots, x_n, u)\) in which the metric has the following local shape:

\[ds^2 = 2du dv + a_i(x_1, \ldots, x_n, u)dx^i du + H(v, x_1, \ldots, x_n, u)du^2 + g_{ij}(x_1, \ldots, x_n, u)dx^i dx^j,\] (2.21)
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with $\frac{\partial \rho}{\partial v} = \frac{\partial \rho}{\partial u} = 0, H \in C^\infty(M)$. To these coordinates we refer as Walker coordinates.

(2) $\nabla_k X_j = 0$ on a some coordinate domain $U$ if and only if $H$ does not depend on $v$, i.e., $\frac{\partial H}{\partial v} = 0$. To these coordinates we refer as Brinkmann coordinates.

A Lorentzian manifold with null covariantly constant vector field is named Brinkmann-wave after [14]. In [71] an $n$-dimensional pseudo-Riemannian manifold on which the Ricci tensor has the form $R_{kl} = \psi X_k X_l$ and the null vector $X_k$ is recurrent, i.e. $\nabla_k X_j = p_k X_j$, is named pure radiation metric with parallel rays or aligned pure radiation metric. In view of Proposition 2.10 we can thus state the following.

**Proposition 2.11.** Let $M$ be an $n(\geq 4)$-dimensional $(PZS)_n$ space-time. If the condition $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ is satisfied then the metric assumes the local shape $(2.21)$ in Walker coordinates; further if the null vector $A_k$ is locally a gradient, then the manifold is a Brinkmann-wave.

**Remark 2.12.** The metric $(2.21)$ belongs to a subclass of the Kundt metrics (see [101, 100, 24, 26, 95] for a compendium). Kundt space-times are defined as space-times admitting a null geodesic vector field $l_a$ with vanishing shear, expansion and twist. The local shape of the metric can be written in the form

$$ds^2 = 2dudv + 2W_i(v, x_1, \ldots, x_n, u)dx^i du$$

$$+ 2H(v, x_1, \ldots, x_n, u)du^2 + g_{ij}(x_1, \ldots, x_n, u)dx^i dx^j,$$  (2.22)

where in an adapted null frame the covariant derivative of $l_a = \nabla_a u$ is written in the form $\nabla_b \tau_a = L_{11} l_a b + \tau_1 (l_a m_1^b + l_b m_1^a)$, being $L_{11} = \frac{\partial H}{\partial u}$ and $\tau_1 = 0 \Leftrightarrow \frac{\partial \psi}{\partial u} = 0$. The subclass $\tau_1 = 0$ can be also defined by the presence of a null recurrent vector field. Usually in literature a pp-wave space-time is defined as a space-time equipped with a null covariant constant vector field (see for example [95]); in this way a pp-wave space-time results belong to the subclass $L_{11} = 0, \tau_1 = 0$ of the Kundt metric. In this paper, however we follow the definitions contained in [109, 68–70], and a pp-wave will be defined as a Brinkmann-wave with some other curvature restrictions.

3. Pseudo Z-Symmetric Space-Times with Harmonic Conformal Curvature Tensor and Time-Like Vector $A_k$

In this section $(PZS)_n$, $n \geq 4$, space-times manifold with the property $\nabla_m C_{ijkl}^m = 0$ [12], i.e. with harmonic conformal curvature tensor are considered. The case of time-like associated vector, i.e., $A_j A^j < 0$ is investigated. It is well known that the divergence of the conformal tensor satisfies the relation:

$$\nabla_m C_{ijkl}^m = \frac{n-3}{n-2} \left[ \nabla_m R_{ijkl}^m + \frac{1}{2(n-1)}(\nabla_j R_{gkl} - \nabla_k R_{glj}) \right].$$  (3.1)
From the contracted second Bianchi identity and from the definition of the $Z$ tensor $Z_{kl} = R_{kl} + \varphi g_{kl}$ the following equation can be written as

$$\nabla_m R_{jklm} = \nabla_k Z_{jl} - \nabla_j Z_{kl} + (\nabla_k \varphi g_{jl} - \nabla_j \varphi g_{kl}).$$  \hfill (3.2)

In this way recalling Eqs. (2.5) and (3.2) one obtains an expression for the divergence of the Weyl tensor on a ($PZS_n$) pseudo-Riemannian manifold:

$$\nabla_m C_{jklm} = \frac{n-3}{n-2} \left\{ A_k Z_{jl} - A_j Z_{kl} + \frac{1}{2(n-1)} \left[ \nabla_j (R + 2(n-1)\varphi) g_{kl} - \nabla_k (R + 2(n-1)\varphi) g_{jl} \right] \right\}. \hfill (3.3)$$

Transvecting the previous result with $g^{kl}$ it is easily inferred that:

$$A_j Z - A^m Z_{jm} = \frac{1}{2} \nabla_j (R + 2(n-1)\varphi). \hfill (3.4)$$

Inserting this in (3.3) we get:

$$\nabla_m C_{jklm} = \frac{n-3}{n-2} \left\{ A_k Z_{jl} - A_j Z_{kl} + \frac{1}{(n-1)} \left[ g_{kl} (A_j Z - A^m Z_{jm}) - g_{jl} (A_k Z - A^m Z_{km}) \right] \right\}. \hfill (3.5)$$

We have thus the following proposition.

**Proposition 3.1.** Let $M$ be an $n(\geq 4)$-dimensional ($PZS_n$) pseudo-Riemannian manifold, then the divergence of the Weyl tensor is given by Eq. (3.5).

Now in particular for a ($PZS_n$) space-time it is $A^m Z_{jm} = -\frac{2}{3(n-1)} Z$ and Eq. (3.5) becomes:

$$\nabla_m C_{jklm} = \frac{n-3}{n-2} \left\{ A_k \left( Z_{jl} - \frac{4Z}{3(n-1)} g_{jl} \right) - A_j \left( Z_{kl} - \frac{4Z}{3(n-1)} g_{kl} \right) \right\}. \hfill (3.6)$$

**Proposition 3.2.** Let $M$ be an $n(\geq 4)$-dimensional ($PZS_n$) space-time, then the divergence of the Weyl tensor is given by Eq. (3.6).

It is worth to notice that the divergence of the Weyl tensor depends only locally on the sources of gravitational fields.

This is the starting point for the proofs of the most important properties of a ($PZS_n$) space-time having harmonic conformal curvature tensor. We set $\nabla_m C_{jklm} = 0$ on any coordinate domain $M$ and obtain:

$$A_k \left( Z_{jl} - \frac{4Z}{3(n-1)} g_{jl} \right) = A_j \left( Z_{kl} - \frac{4Z}{3(n-1)} g_{kl} \right). \hfill (3.7)$$
Let $A_k A^k \neq 0$ at every point of some coordinate domain $U$ of $M$; on multiplying the previous equation by $A^k$ after straightforward calculations it is inferred that:

$$Z_{jl} = \frac{4Z}{3(n-1)}g_{jl} - \frac{A_j A_l}{(A_k A^k)} \left( \frac{n+3}{n-1} \right) Z \frac{3}{3}.$$  

(3.8)

Consequently the energy–momentum tensor of $(PZS)_n$ space-times with divergence-free Weyl tensor takes on $U$ the form:

$$T_{jl} = \frac{4T}{3(n-1)}g_{jl} - \frac{A_j A_l}{(A_k A^k)} \left( \frac{n+3}{n-1} \right) T \frac{3}{3}.$$  

(3.9)

being $Z = \kappa T$. If we consider a time-like vector field $A_k$, i.e. $A_k A^k < 0$ at every point of some coordinate domain $U$ of $M$ and define $u_j = \frac{A_j}{\sqrt{|A_k A^k|}}$ on $U$, we have $u_j u^j = -1$ and consequently on this set:

$$T_{jl} = \frac{4T}{3(n-1)}g_{jl} + u_j u_l \left( \frac{n+3}{n-1} \right) T \frac{3}{3}.$$  

(3.10)

This is the expression of a perfect fluid energy–momentum tensor (see [58, 63, 115; 116, p. 61]) $T_{jl} = \left( \mu + p \right) u_j u_l + pg_{jl}$ where $\mu = \frac{T}{4}$ is the energy density, $p = \frac{4T}{3(n-1)}$ is the isotropic pressure and $u_j$ is the fluid flow velocity.

**Theorem 3.3.** Let $M$ be an $n(\geq 4)$-dimensional $(PZS)_n$ space-time with the property $\nabla_m C_{ijkl}^m = 0$. If $A_k A^k < 0$ then the energy–momentum tensor is of a perfect fluid with $\mu = \frac{T}{4}$ and $p = \frac{4T}{3(n-1)}$.

In Sec. 5 we will show that $u_j$ is irrotational. The following remarks are now to be pointed out.

**Remark 3.4.** Usually in a perfect fluid $p$ and $\mu$ are related by an equation of state of the form $p = p(\mu, T)$ being $T$ the absolute temperature. However the main interest is focused on situations in which $T$ is a constant, the equation of state reduces to $p = p(\mu)$ [63] and the fluid is named isotropic. From Theorem 3.3 the state equation of an $n(\geq 4)$-dimensional $(PZS)_n$ space-time with the property $\nabla_m C_{ijkl}^m = 0$ results to be $p = \frac{\mu}{4}$. This generalizes the results obtained previously in [84].

**Remark 3.5.** We recall that a pseudo-Riemannian manifold $M$, $n \geq 3$ is said to be a quasi-Einstein manifold if at every point of $M$ we have rank $(R_{kl} - \alpha g_{kl}) \leq 1$ for some $\alpha \in \mathbb{R}$. From (3.8) it is easily inferred that:

$$R_{kl} = \alpha g_{kl} + \beta u_k u_l,$$  

(3.11)

where $\alpha = \frac{4Z}{4(n-1)} - \varphi = \frac{-2(n-5)}{2(n-1)} R$ and $\beta = \frac{-Z(n+3)}{4(n-1)} = \frac{-(n-2)(n+3)}{4(n-1)} R$ are the associated scalars and $u_k = \frac{A_k}{\sqrt{|A_j A^j|}}$ is a unit time-like covector in the case of time-like $A_k$, i.e. $A_j A^j < 0$: the space is thus quasi-Einstein [18, 33]. The Ricci tensor
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takes the form:

\[ R_{kl} = \frac{(n-5)}{6(n-1)} R g_{kl} + u_k w \frac{(n-2)(n+3)}{6(n-1)} R. \]  \quad (3.12)

It should be noted that for \( n = 5 \) the rank of the Ricci tensor is one. This generalizes the results due to Ray-Gua [104]. For instance, in the Riemannian case quasi-Einstein spaces were investigated in [18, 33, 42]; in the pseudo-Riemannian case they arose during the study of exact solutions of Einstein’s equations and during the investigations of quasi-umbilical hypersurfaces of pseudo-Euclidean spaces [43, 49], see also [40, 47, 66]. For example, the Robertson–Walker space-times are quasi-Einstein (see [10, 116] and references therein for details on these space-times). For recent results on quasi-Einstein manifolds we refer to [22, 51, 52, 50]. An example of a warped product manifold of dimension \( \geq 4 \), with 1-dimensional fiber, is presented in [43, Example 4.1]. The Ricci tensor of that manifold is of rank one. Moreover, it is easy to check that a condition of the form (4.13) is satisfied. In [47] an example of a warped product manifold, with \((n-1)\)-dimensional fiber, \( n \geq 3 \), is given. That manifold is quasi-Einstein. Moreover, a condition of the form (4.13) is satisfied. In addition, both warped products are non-conformally flat manifolds. We also mention that quasi-Einstein conformally flat pseudo-Riemannian manifolds were investigated in [40].

According to [35], a pseudo-Riemannian manifold \((M, g)\), \( n \geq 3 \), is said to be \textit{Ricci-simple} if the rank of the Ricci tensor is equal to one at all points of \( M \) at which this tensor is non-zero.

A different notion of quasi-Einstein manifolds emerges from some generalizations of Ricci solitons: it was defined in [16, 56] and references therein. From the second of Eqs. (2.4) we have readily \( \nabla_j \nabla_k Z = \frac{4}{3} [\nabla_j A_k] Z + A_k (\nabla_j Z) \); interchanging indices \( j \) and \( k \) and subtracting we infer \( A_j (\nabla_k Z) = A_k (\nabla_j Z) \) and from the definitions of \( u_k \) and \( \beta \) finally:

\[ u_j (\nabla_k \beta) = u_k (\nabla_j \beta). \]  \quad (3.13)

This equation will be useful in Sec. 5.

Conversely if the tensor \( Z_{kl} \) has the form (3.8) then it is easily seen that \( \nabla_m C_{jkl}^{\mu} = 0 \). So we may state the following.

**Proposition 3.6.** Let \( M \) be an \( n(\geq 4) \)-dimensional \((PZS)_n \) space-time. If the energy–momentum tensor is a perfect fluid with \( \mu = \frac{T}{3} \) and \( p = \frac{4T}{3(n-1)} \) then \( \nabla_m C_{jkl}^{\mu} = 0 \) on any coordinate domain of \( M \).

Theorem 3.3 and Proposition 3.6 generalize the result of [4] for pseudo-Ricci symmetric Riemannian manifolds. From Sec. 2 we know that a \((PZS)_n \) spacetime with non-zero scalar \( Z \) or subjected to the condition \( \nabla_m C_{jkl}^{\mu} = 0 \) is a \textit{Weyl compatible manifold} [44, 76, 77], i.e. (2.8) is satisfied, and the stress–energy tensor
is Weyl compatible, i.e. Eq. (2.9) is satisfied too. From (3.10) it is easily inferred:

\[ u_i u_m C_{jkl}^m + u_j u_m C_{kli}^m + u_k u_m C_{ijl}^m = 0. \]

(3.14)

The symmetric tensor \( u_i u_j \) is thus Weyl compatible. We recall here that the Weyl tensor as well as any other tensor may be decomposed in its electric and magnetic part. In \( n = 4 \) (see [96, 87, 88, 11, 116]) given a normalized velocity vector \( u_j \) (i.e. \( u_j u^j = -1 \)) the following tensors are defined:

\[ E_{kl} = u^j u^m C_{jkl}^m, \quad H_{kl} = \frac{1}{4} u^j u^m \left( \varepsilon_{\alpha\betajl} C_{k\alpha\beta}^{jm} + \varepsilon_{\alpha\betajl} C_{k\alpha\beta}^{jm} \right), \]

where the tensor \( C_{\alpha\beta}^{jm} \) is defined by \( C_{jkl}^m = g_{\alpha m} g_{\beta l} C_{\alpha\beta}^{jk} \) and \( \varepsilon_{ijkl} \) is the completely skew-symmetric Levi-Civita symbol [11, 116].

The tensor \( E_{kl} \) is named electric part of the Weyl tensor, while the tensor \( H_{kl} \) is known as the magnetic part of the Weyl tensor. On an \( n \)-dimensional Lorentzian manifold the decomposition of the Weyl tensor was obtained in the papers [110, 111, 64, 95]. More precisely in [64] the electric and magnetic parts are defined respectively as follows:

\[
(C_+)^{jk}_{\text{mt}} = h^{jp} h^{kq} h^r_m h^s_l C_{pqr}^{js} + 4 u^j u^m C_{pqr}^{js} u_p u_q,
\]

\[
(C_-)^{jk}_{\text{mt}} = 2 h^{jp} h^{kq} C_{pqr}^{js} u_m u^q + 2 u^j u^k C_{pqr}^{js} h_{mp} h_{lq},
\]

(3.15)

where \( h_{ij} = g_{ij} + u_i u_j \) is the projector orthogonal to \( u_j \). In [64] (see also [95]) the following definition was stated:

**Definition 3.7 ([64, 95])**. The Weyl tensor on a region of an \( n \geq 4 \)-dimensional space-time is named purely electric if \( C_- = 0 \) and it is called purely magnetic if \( C_+ = 0 \). The corresponding space-times are also called purely electric and purely magnetic.

The following theorem stated in [64, 95] gives a necessary and sufficient condition for the Weyl tensor to be purely electric.

**Theorem 3.8 ([64, 95])**. The Weyl tensor of a Lorentzian manifold \( M, n \geq 4 \), is purely electric if and only if on any coordinate domain we have \( u_i u_m C_{jkl}^m + u_j u_m C_{kli}^m + u_k u_m C_{ijl}^m = 0 \).

In view of the previous results we can state the following.

**Theorem 3.9.** Let \( M \) be a non-conformally flat \( n \geq 4 \)-dimensional (PZS) space-time with the property \( \nabla_m C_{jkl}^m = 0 \), then the space-time is purely electric.

In the following remarks some other known results in \( n = 4 \) are collected.

**Remark 3.10.** In [114] the solutions of Einstein equations are studied under the following assumptions:

(1) the space is a perfect fluid 4-dimensional space-time;

(2) the divergence of the conformal curvature vanishes, i.e. \( \nabla_m C_{jkl}^m = 0 \);
(3) the space is equipped with a state equation $p = p(\mu)$. It was thus proved that the space-time is conformally flat and the metric is a Robertson–Walker metric. The flow is irrotational, shear free and geodesic. A similar result was found also in [113, Corollary, p. 3584]: if a general relativistic perfect fluid space-time with divergence-free Weyl tensor admits a proper conformal symmetry then it is conformally flat.

In this work the assumptions (1) and (3) are substituted by the restriction on the energy–momentum tensor (1.4) while (2) is retained: then the space results naturally to be a perfect fluid and a state equation follows too. In Sec. 5 we will show that n-dimensional perfect fluid space-time with divergence-free Weyl tensor admits a conformal Killing vector.

**Remark 3.11.** In [76, 83, 84] it was shown that on any 4-dimensional space-time having a Weyl compatible energy-momentum tensor of the form $T_{jl} = \alpha g_{jl} + \beta u_j u_l$ the magnetic part of the Weyl tensor vanishes.

In passing we note that a more general result than Theorem 3.9 can be pointed out when considering Eq. (2.11) written for an energy–momentum tensor of perfect fluid form $T_{jl} = \alpha g_{jl} + \beta u_j u_l$, i.e.:

$$\nabla_i \nabla_m C_{jkl}^m + \nabla_j \nabla_m C_{kli}^m + \nabla_k \nabla_m C_{ijl}^m = -\kappa \beta \frac{n-3}{n-2} \left( u_i u_m C_{jkl}^m + u_j u_m C_{kli}^m + u_k u_m C_{ijl}^m \right).$$

(3.16)

It is easy to realize that an $n(\geq 4)$-dimensional perfect fluid space-time is purely electric if and only if the Weyl tensor is subjected to the curvature restriction specified by $D\Pi_{(C)j} = 0$.

**Theorem 3.12.** Let $M$ be a non-conformally flat perfect fluid space-time, then the Weyl tensor is purely electric if and only if $D\Pi_{(C)j} = 0$.

Finally we take a look at the Petrov classification for $n = 4$ (see [58, 98, 99, 116]) and at the Weyl types for (see [28]) of $(PZS)_n$ space-times subjected to the condition $\nabla_m C_{jkl}^m = 0$ and with time-like associated vector. In $n = 4$ it is well known that purely electric space-times are of Petrov type $I, D$, or $O$ (conformally flat) [116]. In view of Remark 3.11 we realize that a $(PZS)_4$ space-time with the property $A_k A^k < 0$ is conformally flat and the Petrov type is $O$. On the other hand for $n > 4$ [95, Proposition 8.8] states that a purely electric Weyl tensor can be only of algebraic types $G, I_1, D$ or $O$.

**Theorem 3.13.** Let $M$ be a non-conformally flat $n(\geq 4)$-dimensional $(PZS)_n$ space-time with the property $\nabla_m C_{jkl}^m = 0$. If $A_k A^k < 0$ then the Weyl types can be only $G, I_1, D$ or $O$. 

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4. Pseudo Z-Symmetric Space-Times with Harmonic Conformal Curvature Tensor and Null Vector $A_k$

In this section we study in some detail pseudo-Z symmetric space-times with harmonic conformal curvature tensor and null covector $A_k$. Here we follow the definition of a pp-wave and related properties as stated in [70, 68, 69].

Definition 4.1. A Brinkmann-wave is called pp-wave if its curvature tensor satisfies the trace condition

$$R_{jk}R_{kilm} = 0.$$  \hfill (4.1)

In [109] a coordinate description and equivalences are proved. Here we remand to [70, 68, 69]:

Lemma 4.2 ([70, 68, 69]). A Lorentzian manifold $(M, g)$ of dimension $n \geq 4$ is a pp-wave if and only if there exist coordinates $(v, x_1, \ldots, x_n, u)$ in which the metric has the following local shape:

$$ds^2 = 2dvdu + H(x_1, \ldots, x_n)du^2 + dx_jdx^j,$$  \hfill (4.2)

where $H(x_1, \ldots, x_n)$ is an arbitrary smooth function with the property $\frac{\partial H}{\partial v} = 0$, usually called the potential function of the pp-wave.

Lemma 4.3 ([109, 70, 68, 69]). A Lorentzian manifold $(M, g)$ of dimension $n \geq 4$ with parallel null vector field $\nabla_k X_j = 0$ is a pp-wave if and only if one of the following conditions is satisfied:

$$X_i R_{jklm} + X_j R_{kim} + X_k R_{ijm} = 0,$$  \hfill (4.3)

$$R_{pjkq}R_{pqlm} = \chi X_j X_k X_l X_m,$$  \hfill (4.4)

being $D_{ij}$ a symmetric tensor and $\chi$ a suitable scalar function. The Ricci tensor of a pp-wave is given by $R_{kl} = \psi X_k X_l$ for a smooth function $\psi$, i.e. considered pp-wave as a Ricci-simple manifold. In dimension $n = 4$ this is even equivalent to $R_{jk} R_{kilm} = 0$ (see [70]).

Now we consider $A_k A^k = 0$: Eq. (3.7) is thus multiplied by $A^k$ to obtain

$$A_j A_l \left( n + 3 \frac{Z}{n-1} \right) = 0,$$

from which we infer $Z = 0$ and thus $A_k Z_{jl} = A_j Z_{kl}$; consequently from Eq. (2.5) $Z_{kl}$ results to be a Codazzi tensor [36]. Propositions 2.9–2.11 provide the following facts: the null vector $A_k$ is recurrent, i.e. $\nabla_j A_k = p_j A_k$, the metric has local shape (2.21) in Walker coordinates; further we have

$$Z_{kl} = R_{kl} = \psi A_k A_l$$  \hfill (4.5)

and both tensors are recurrent. Moreover, $(M, g)$ is a Ricci-simple manifold. In terms of the energy–momentum tensor this result reads:

$$T_{jl} = \Phi A_j A_l,$$  \hfill (4.6)
being \( \Phi = \theta^k \theta^\rho T_{kj} \). If \( \Phi > 0 \) this is the expression of a pure radiation field \([58; 116, p. 61]\) (or a null dust field); in the terminology of \([71]\) the space is a pure radiation metric with parallel rays or aligned pure radiation metric. Conversely if the energy–momentum tensor has the form \((4.6)\) we have easily \( Z_{jl} = \psi A_j A_l \) for a suitable scalar \( \psi \), from which \( Z = 0 \) and \( A_k Z_{jl} = A_j Z_{kl} \); consequently from Eq. \((3.6)\) it is easily seen that \( \nabla_m C_{jkl}^m = 0 \). The following theorem collects all these results.

**Theorem 4.4.** Let \( M \) be an \( n \) (\( \geq 4 \))-dimensional \((PZS)_n \) space-time with the property \( \nabla_m C_{jkl}^m = 0 \). If \( A_k A^k = 0 \) then the tensor \( Z_{kl} \) satisfies the Codazzi condition, the null vector \( A_k \) is recurrent, the metric has local shape \((2.21)\) in Walker coordinates, \( Z_{kl} = R_{kl} = \psi A_k A_l \), the rank of the energy-momentum tensor is one, i.e. \( T_{jl} = \Phi A_j A_l \) and if \( \Phi > 0 \) then the energy–momentum tensor is of a pure radiation field. Conversely if the energy–momentum tensor is of a pure radiation field then \( \nabla_m C_{jkl}^m = 0 \).

The Petrov classification of the Weyl tensor on a 4-dimensional Lorentzian manifold in terms of Weyl’s scalars is exposed for example in \([116]\). It is well known (see \([9, 32]\)) that an equivalent classification arises also from Bel and Debever criteria which are based on null vectors \( k \) satisfying increasingly restricted conditions as follows:

(a) type I \( k_{[l}C_{a]rsq}k^r k^s = 0 \),

(b) type II, D \( k_{[l}C_{a]rsq}k^r k^s = 0 \),

(c) type III \( k_{[l}C_{a]rsq}k^r = 0 \),

(d) type N \( C_{arsq}k^r = 0 \),

(e) type O \( C_{arsq} = 0 \).

The algebraic classification of the conformal curvature tensor on a Lorentzian manifold was extended to higher dimensions by some authors (see for example \([23, 28]\)). Moreover also the Bel–Debever criteria were extended to \( n \)-dimensional Lorentzian manifolds in the papers \([90, 93, 95]\). Here we refer to the classification given in \([93, Table 1]\).

In \([90; 95, Proposition 8.23]\) space-times which admit a recurrent null vector field were classified: the Weyl type depends on the matter content and resulted to be of type \( II_{bd} \) if the space is Einstein, and \( II'_{abd} \) if the space is Ricci flat. In four dimensions this corresponds to Petrov types \( III \) and \( II \) if the space is Einstein, and to type \( N \) in vacuum, as shown for example in \([62, 92]\).

**Remark 4.5.** It is worth to notice that the existence of a null recurrent vector field enables us at least to a rough classification independent from the matter content. We recall that a non-null recurrent vector can be always rescaled to a covariantly constant vector. A skew symmetrization of the covariant derivative of the recurrence condition \( \nabla_k A_l = p_k A_l \) gives \((\nabla_j \nabla_k - \nabla_k \nabla_j) A_l = F_{jk} A_l \) being \( F_{jk} = \nabla_j p_k - \nabla_k p_j \).
and from the Ricci identity it is \( R_{ijkl}^m A_m = F_{jk} A_l \) [92, 62]. The first Bianchi identity now implies \( F_{jk} A_l + F_{kl} A_j + F_{lj} A_k = 0 \). From this we get

\[
A_i A^m R_{ijkl} + A_j A^m R_{iklm} + A_k R_{ijlm} = A_l (A_i F_{jk} + A_j F_{ki} + A_k F_{ij}) = 0
\]

and the tensor \( A_i A_j \) results to be Riemann compatible and thus Weyl compatible, i.e.

\[
A_i A^m C_{jklm} + A_j A^m C_{iklm} + A_k A^m C_{ijlm} = 0.
\] (4.8)

It is readily seen that Eq. (4.8) with the choice \( A_j = k_j \) matches with the type \( II_d \) space-time of [93, Table 1] for an \( n \)-dimensional Lorentzian manifold. On the other hand, regarding the case \( n = 4 \) Eq. (4.8) is multiplied by \( A^k \) to obtain that the space-time is of Petrov type \( II \) or \( D \) and the Weyl tensor is algebraically special (see [106, 115, 116]).

**Remark 4.6.** Let \( (M, g) \) be an \( n(\geq 4) \)-dimensional Lorentzian manifold equipped with a null recurrent vector, then the Weyl tensor is at least as special as type \( II_d \) independently from the matter content. This applies also to the subclass of Kund metric specified by \( \tau_i = 0 \).

In this case, however, the conditions (4.5) \( R_{kl} = \psi \pounds A_k A_l \) and \( \nabla_j A_k = p_j A_k \) allow a more stringent and refined algebraic classification for the Weyl tensor. First from the definition of the conformal curvature tensor and from the local form of the Ricci tensor the following relation is displayed immediately:

\[
A_i C_{jklm} + A_j C_{iklm} + A_k C_{ijlm} = A_l R_{ijklm} + A_j R_{iklm} + A_k R_{ijlm}.
\] (4.9)

Transvecting the previous equation by \( g^{im} \) and taking account of \( R_{kl} = \psi \pounds A_k A_l \) we easily get:

\[
A^m C_{jklm} = A^m R_{ijklm}.
\] (4.10)

The recurrence relation \( \nabla_j A_k = p_j A_k \) gives \( R_{ijkl}^m A_m = F_{jk} A_l \) and thus \( C_{jklm} A_m = F_{jk} A_l \) and consequently \( A_i A^m C_{jklm} = A_i F_{jk} A_l = A_l A^m C_{jklm} \). This result is rewritten in the form:

\[
A_l [C_{ijkl} A^m] = 0.
\] (4.11)

In four dimensions with the choice \( A_j = k_j \) this matches with Petrov type \( III \) for the Weyl tensor and with the same choice this matches with the type \( II_d \) of the \( n \)-dimensional Bel–Debever criteria [93, Table 1] for the Weyl tensor on a Lorentzian manifold. From the above discussion and Theorem 4.4 we get:

**Theorem 4.7.** Let \( M \) be a non-conformally flat \( n(\geq 4) \)-dimensional (PZS)\textsubscript{\(n\)} space-time with the property \( \nabla_m C_{ijkl}^m = 0 \). If \( A_k A^k = 0 \) then the Weyl tensor is of Petrov type \( III \) (or more special) for \( n = 4 \) and the Weyl type is at least as special as type \( II_d \) for \( n > 4 \).

We focus now on the case in which the null covector \( A_k \) is locally a gradient, i.e. \( A_j = \nabla_j h \). From Propositions 2.9–2.11 it follows that \( A_k \) is a null recurrent
vector that can be rescaled to a covariant constant null covector $\bar{A}_k$ given by $\bar{A}_k = A_k e^{\frac{1}{2}h_{\mu
u}h^{\mu
u}}$, i.e. $\nabla_j \bar{A}_k = 0$. The space-time is a Brinkmann-wave [14] and the metric has the local shape (2.21). Moreover form the Ricci identity we have easily Lemma 4.9 ([72, p. 128] or [97]).

Theorem 4.8. Let $M$ be a non-conformally flat $n(\geq 4)$-dimensional $(PZS)_n$ space-time with the property $\nabla_k C_{jklm} = 0$. If $A_k A^k = 0$ and if $A_k$ is locally a gradient, then the Weyl tensor is of Petrov type $N$ with respect to $A_k$ in $n = 4$ and it is at least as special as type $II_{\text{ab}}$ with respect to $A_k$ in $n > 4$.

We show that the case $n = 4$ is special and reduces to a $pp$-wave metric. We recall the following lemma in four dimensions.

Lemma 4.9 ([72, p. 128] or [97]). Let $M$ be a 4-dimensional pseudo-Riemannian manifold, then the following identity involving the conformal curvature tensor holds:

$$\delta^i_r C^{ij}_{\text{tr}} + \delta^i_k C^{ij}_{rs} + \delta^i_j C^{ij}_{rt} + \delta^i_s C^{ij}_{st} + \delta^k_j C^{ji}_{rs} + \delta^s_i C^{ki}_{rs} + \delta^s_i C^{ki}_{rt} = 0.$$  

(4.12)

If the previous identity is multiplied by $\bar{A}_j$ taking account of the condition $\bar{A}_m C_{jklm} = 0$ we get immediately (see also [58, Theorem 7.4]):

$$\bar{A}_j C_{jklm} + \bar{A}_k C_{jklm} + \bar{A}_k C_{ijklm} = 0.$$  

(4.13)

Thus in view of Eq. (4.9) we get the condition (4.2), i.e. $\bar{A}_j R_{jklm} + \bar{A}_k R_{jklm} + \bar{A}_k R_{ijklm} = 0$; together with $\nabla_j \bar{A}_k = 0$ and Lemma 4.3 ensure that the local shape of the metric is (4.1) and thus a $pp$-wave space-time. For a review of $pp$-waves in $n = 4$ see [58, p. 248; 116, pp. 383–384; 61]. The metric of such space-time is of the form given in [58, Eq. (8.11)]; the recurrence properties of the Weyl curvature tensor of $pp$-waves are reported in [116]: under certain conditions they are complex recurrent space-times (see [59, 61, 89]) with closed recurrence parameter.

Theorem 4.10. Let $M$ be a non-conformally flat 4-dimensional $(PZS)_4$ space-time with the property $\nabla_m C_{jklm} = 0$. If $A_k A^k = 0$ and if $A_k$ is locally a gradient, then conditions (4.13) and (4.2) are satisfied and the metric is of a $pp$-wave (4.1).

The condition $\bar{A}_j C_{jklm} + \bar{A}_k C_{jklm} + \bar{A}_k C_{ijklm} = 0$ on a pseudo Riemannian manifold has been extensively studied among others in [33, 45, 46, 48, 78, 85]. Here
we quote the following:

**Lemma 4.11** ([85, 78]). Let \( M \) be an \( n(\geq 4) \)-dimensional non-conformally flat pseudo Riemannian manifold. If \( A_i C_{ijkl} + A_j C_{kilm} + A_k C_{ijlm} = 0 \) then: (1) \( A^i A_i = 0 \), (2) there is a symmetric tensor \( E_{kl} \) such that

\[
C_{ijkl} = A_j A_m E_{kl} - A_j A_t E_{mk} - A_k A_m E_{jl} + A_k A_t E_{jm},
\]

(4.14)

(3) \( C_{lmj}^k C_{pqk}^{ij} = 0 \), (4) \( C_{lmjk} C^{lmjk} = 0 \), (5) \( C_{lmab} C_{pqbc} b C_{rscd} d C_{tuda} = 0 \).

**Remark 4.12.** We can find equation (4.14) for instance, in [33] for a pseudo-Riemannian manifold. Results (3) and (4) were also proved in the Lorentzian case in [58, Theorem 7.4; 30, 25, 27].

**5. Conformally Flat Pseudo-Z Symmetric Space-Times**

In this section we study in some detail conformally flat \((PZS)\) space-times. We will show that in case of null covector the space reduces to a subclass of \( pp \)-wave,
named plane waves (see [56] and references therein). It results also a generalized quasi-Einstein space-time in the sense of [15, 16]. In the case of time-like associated covector, i.e. $A_j A^j < 0$ on a non-empty open subset of $M$, the existence of a proper concircular vector is ensured [120, 121]. This vector can be rescaled to a time-like vector of the form $\nabla_k X_j = \rho p_j$ and it is a conformal Killing vector [116]. A recent result achieved in [20] then shows that the metric is necessarily a generalized Robertson–Walker space-time [5, 107, 108]. In particular we show that a conformally flat $(PZS)_n$ space-time is conformal to the Robertson–Walker space-time [10, 116].

First we deal with a null covector $A_j$, i.e. $A_j A^j = 0$. Again from (3.7) we have $A_k Z_{jl} = A_j Z_{kl}$ as in Sec. 4; consequently from (2.5) for a symmetric tensor $Z_{kl}$ results to be a Codazzi tensor and again Propositions 2.9–2.11 provide the following facts: the null vector $A_j$ is a constant vector $\bar{A}_j$ and thus the covector $A_j$ is ensured [120, 121]. This vector can be rescaled to a time-like vector of the form $\nabla_k X_j = \psi A_i$. Here these conditions are employed to show that the covector can be rescaled to a covariant constant.

Since the space is conformally flat we have $A_k R_{jklm} + A_j R_{klm} + A_l R_{ijlm} = 0$ from (4.9) and $A^m R_{jklm} = 0$ from (4.10). A skew symmetrization of the covariant derivative of the recurrent condition $\nabla_k A_i = p_k A_i$ and the Ricci identity give $R_{jklm} A_m = (\nabla_j p_k - \nabla_k p_j) A_i$. This result ensures that, at least locally, $p_k$ (see [58, pp. 242–243]) is a gradient, i.e. $p_k = \nabla_k \eta$. Now the rescaled vector $A_j = A_j e^{-\eta}$ has the property that

$$\nabla_k A_j = (\nabla_k A_j) e^{-\eta} - A_j e^{-\eta} (\nabla_k \eta) = e^{-\eta} (p_k A_j - \nabla_k p_k) = 0.$$ 

In this way $A_k R_{jklm} + A_j R_{klm} + A_l R_{ijlm} = 0$. Again Lemma 4.3 ensures that the metric is (4.1) and thus a $pp$-wave metric. Moreover, by (2.9) it is $p_k = \frac{1}{2}[4A_k - \nabla_k \ln|\psi|]$ and thus the covector $A_j$ is locally a gradient.

**Theorem 5.1.** Let $M$ be a conformally flat $n(\geq 4)$-dimensional $(PZS)_n$ space-time. If $A_k A^k = 0$ then $A_j$ is locally a gradient and can be rescaled to a covariantly constant vector $\bar{A}_j$, the relation $\bar{A}_k R_{jklm} + \bar{A}_j R_{klm} + \bar{A}_l R_{ijlm} = 0$ holds and the space is thus a $pp$-wave metric (4.1).

It is well known that such metrics are conformally flat if and only if the potential function takes the form

$$H(x_1, \ldots, x_n, u) = a(u) \sum_{i=1}^n \dot{x}_i^2 + \sum_{i=1}^n b_i(u) x_i + c(u)$$

being $a, b_1, \ldots, b_n, c$ arbitrary functions of $u$ (see for example [56]); moreover they define a plane wave if $H(x_1, \ldots, x_n, u) = \sum_{i=1}^n a_{ij}(u) x_i x_j$ where $a_{ij}(u)$ are arbitrary matrix functions. Further, by a suitable change of coordinates the quadratic form of the conformally flat $pp$-wave can be reduced to a plane wave.

**Corollary 5.2.** Let $M$ be a conformally flat $n(\geq 4)$-dimensional $(PZS)_4$ space-time. If $A_k A^k = 0$ then there exists a suitable coordinate system in which metric (4.1) becomes a plane wave.
Divergence-free Weyl tensor and pp-waves

It is worth to notice that, if $A_j = \nabla_j h$, then the recurrence relation becomes
\[ \nabla_j A_k = \gamma A_j A_k \]
and thus $\nabla_j \nabla_k h = \gamma (\nabla_j h)(\nabla_k h)$ so that the Ricci tensor may be written as:
\[ R_{kl} + \nabla_k \nabla_l h = (\psi + \gamma)(\nabla_k h)(\nabla_l h). \]  
(5.1)

The space-time is thus generalized quasi-Einstein in the sense reported in [16, 56].

Second we consider the case $A_j A^j < 0$. We have already mentioned the results of [114]: A 4-dimensional perfect fluid space-time with $\nabla_m C_{jkl}^m = 0$ and subjected to a state equation is conformally flat and the metric is a Robertson–Walker metric. Here we generalize this result to the $n$-dimensional case. The following theorem was already proven elsewhere (see [74; 79, Theorem 5.1; 80, 83]).

**Theorem 5.3.** Let $M$ be an $n(\geq 4)$-dimensional manifold whose Ricci tensor is given by $R_{kl} = \alpha g_{kl} + \beta T_k T_l$ where $T_k$ is a unit vector. If the manifold is conformally flat and the condition $T_j (\nabla_k \beta) = T_k (\nabla_j \beta)$ is satisfied, then $T_k$ is a proper concircular vector.

However on noting that starting (5.1) in the proof of [79, Theorem 5.1] is equivalent to the condition $\nabla_m C_{jkl}^m = 0$ we are able to rephrase a slightly different result.

**Theorem 5.4.** Let $M$ be an $n(\geq 4)$-dimensional manifold whose Ricci tensor is given by $R_{kl} = \alpha g_{kl} + \beta u_k u_l$ where $u_k$ is a unit time-like vector, i.e. $u_k u^k = -1$. If the Weyl tensor is divergence-free, i.e. $\nabla_m C_{jkl}^m = 0$ and the condition $u_j (\nabla_k \beta) = u_k (\nabla_j \beta)$ is satisfied, then $u_k$ is a proper concircular vector. Moreover it can be rescaled to a time-like vector $X_j$ of the form $\nabla_k X_j = \rho g_{kj}$ and it is a conformal Killing vector.

In fact, following step by step the proof of [79, Theorem 5.1] it is possible to obtain:
\[ \nabla_k u_l = f u_k u_l + f g_{kl} \]  
(5.2)
being $f$ a suitable scalar function. So we conclude that $u_k$ is a concircular vector [120, 121]. Moreover, $u^k \nabla_k u_l = 0$ and the integral curves of $u_k$ are geodesics. Further it is possible to show that (see [79]) $\nabla_j f = \mu u_j$, thus the one form $\pi_k = f u_k$ is closed and consequently
\[ \nabla_k u_l = \pi_k u_l + f g_{kl}, \]  
(5.3)
so that $u_k$ is a proper concircular vector [120, 121], see also [66, 67]. Now if $\pi_j$ is closed it is locally a gradient of a suitable scalar function, i.e. $\pi_j = \nabla_j \sigma$ (see [58, pp. 242–243]); setting $X_j = u_j e^{-\sigma}$ we have
\[ \nabla_k X_j = e^{-\sigma} (\nabla_k u_j - u_j \nabla_k \sigma) = e^{-\sigma} [ (\nabla_k \sigma) u_j + f g_{kj} - u_j (\nabla_k \sigma) ] = (e^{-\sigma} f) g_{kj}, \]
and consequently
\[ \nabla_k X_j = \rho g_{kj} \]  
(5.4)
being \( \rho = e^{-\sigma} f \) a scalar function and \( X_j X^j = -e^{-2\sigma} < 0 \) is a time-like vector. The previous equation can be written in the form \( \nabla_k X_j + \nabla_j X_k = 2\rho g_{kj} \), i.e. \( X_j \) becomes a conformal Killing vector [116]. We recall now the definition of a generalized Robertson–Walker space-time [5, 108, 107].

**Definition 5.5 ([5, 108, 107]).** An \( n(\geq 3) \)-dimensional Lorentzian manifold is named generalized Robertson–Walker space-time if the metric takes the local shape:

\[
ds^2 = -(dt)^2 + q(t)^2 g^{\alpha\beta}dx^\alpha dx^\beta ,
\]

where \( g^{\alpha\beta} = g^{\alpha\beta}(x^\gamma) \) are functions of \( x^\gamma \) only (\( \alpha, \beta = 2, 3, \ldots, n \)) and \( q \) is a function of \( t \) only.

The generalized Robertson–Walker space-time is thus the warped product \(-I \times M^* \) (see [5, 107, 108]), where \( M^* \) is an \( n-1 \) Riemannian manifold. If \( M^* \) is a 3-dimensional Riemannian manifold of constant curvature, the space-time is called Robertson–Walker space-time. The following deep result was recently proved in paper [20] (for similar results see [121] and the recent paper [31]).

**Theorem 5.6 ([20]).** Let \( M \) be an \( n(\geq 3) \)-dimensional Lorentzian manifold. Then the space-time is a generalized Robertson–Walker space-time if and only if it admits a time-like vector of the form \( \nabla_k X_j = \rho g_{kj} \).

In view of Theorems 5.4 and 5.6 we have thus the following.

**Theorem 5.7.** Let \( M \) be an \( n(\geq 3) \)-dimensional space-time whose Ricci tensor is given by \( R_{kl} = \sigma g_{kl} + \beta u_k u_l \) where \( u_k \) is a unit time-like vector. If the Weyl tensor is divergence-free, i.e. \( \nabla_m C_{kl}^m = 0 \) and the condition \( u_j (\nabla_k \beta) = u_k (\nabla_j \beta) \) is satisfied, then the metric has the local shape (5.5) and is a generalized Robertson–Walker space-time.

Now (5.5) may be written in the form

\[
ds^2 = q(t)^2 (-q^{-2}(t)(dt)^2 + g^{\alpha\beta}dx^\alpha dx^\beta) = q^2(\tau) (-d\tau^2 + g^{\alpha\beta}dx^\alpha dx^\beta),
\]

where the variable \( t \) is changed by \( \tau = \frac{q(t)}{q(0)} \) (see [107]). So the metric (5.5) is conformal to a direct product metric \( ds^2 = (-d\tau^2 + g^{\alpha\beta}dx^\alpha dx^\beta) \) (see [95, Definition 8.2]). Recently, Robertson–Walker space-times were studied among others in [3, 20, 22, 52, 86].

Now we suppose that metric (5.5) is conformally flat the following results in [54] (quoted in [95, Proposition 8.13]) are useful (see also [57] for warped products).

**Proposition 5.8 ([54, 95]).** A direct product of spaces is conformally flat if and only if both product spaces are of constant curvature and their curvature scalars satisfy \( n_2(n_2-1)R_1(n_1) + n_1(n_1-1)R_2(n_2) = 0 \).

In our case since metric (5.5) is conformally flat and since \( n_1 = 1 \) (this makes the latter condition of Proposition 5.8 identically satisfied), we get that \( g^{\alpha\beta}dx^\alpha dx^\beta \) is a
space of constant curvature and thus the product $-1 \times q^2 M^*$ is the usual Robertson–Walker space-time [10, 116]. Thus a conformally flat generalized Robertson–Walker space-time is conformal to the usual Robertson–Walker space-time. From the above results we have the following.

**Theorem 5.9.** Let $M$ be an $n(\geq 3)$-dimensional space-time whose Ricci tensor is given by $R_{kl} = \alpha g_{kl} + \beta u_k u_l$ where $u_k$ is a unit time-like vector. If the space-time is conformally flat and the condition $u_j(\nabla_k \beta) = u_k(\nabla_j \beta)$ is satisfied, then the metric is conformal to the Robertson–Walker space-time.

Now if $PZS_n$ space-time has divergence-free Weyl tensor, i.e. $\nabla_m C_{jkl}^m = 0$ from Eq. (3.11) we have that the Ricci tensor is of the form $R_{kl} = \alpha g_{kl} + \beta u_k u_l$ and from Eq. (3.13) we have that the condition $u_j(\nabla_k \beta) = u_k(\nabla_j \beta)$ satisfied. In view of Theorems 5.4, 5.7, 5.9 one can state the following.

**Proposition 5.10.** Let $M$ be an $n(\geq 4)$-dimensional $PZS_n$ space-time with the property $\nabla_m C_{jkl}^m = 0$. The following facts hold.

(a) If the associated covector $A^k$ is time-like, then the manifold admits a proper concircular vector that can be rescaled to time-like vector $X_j$ of the form $\nabla_k X_j = \rho g_{kj}$ and it is a conformal Killing vector;
(b) the space-time is a generalized Robertson–Walker space-time;
(c) if the $(PZS)_n$ space-time is conformally flat then the metric is conformal to a Robertson–Walker space-time.

Now it is well known [1] that if a conformally flat space admits a proper concircular vector, then this space is subprojective in the sense of Kagan (see, e.g. [66]). In this way the following holds.

**Proposition 5.11.** Let $M$ be an $n(\geq 4)$-dimensional conformally flat $(PZS)_n$ space-time. If the associated covector $A^k$ is time-like, then the manifold is a subprojective space.

Very recently manifolds admitting concircular vector fields were studied in [21].

### 6. Examples of Pseudo-Z Symmetric Space-Times and Their Curvature Properties

In this section we collect some further examples of pseudo-Z symmetric space-times, and investigate some of their curvature properties.

First we consider an $n(\geq 4)$-dimensional pp-wave space-time according to the definition of [68–70]. This is given in a simply connected domain of $\mathbb{R}^{n+2}$ with coordinates $(v, x_1, \ldots, x_n, u)$ endowed with the metric shape (4.1), i.e. a pseudo-Riemannian metric written as:

$$ds^2 = 2dudv + H(x_1, \ldots, x_n, u)du^2 + dx_j dx^j,$$  \hspace{1cm} (6.1)
being $H = H(x_1, \ldots, x_n, u)$ the potential of the $pp$-wave. The Ricci tensor is given by:

$$R_{ab} = -\frac{1}{2}(\nabla^2 H)k_ak_b = \psi(x_1, \ldots, x_n, u)k_ak_b,$$

(6.2)

where $\nabla^2 H = \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_i^2}$ is the standard Euclidean Laplacian of the function $H$ and $k$ is a covariantly constant vector, i.e. $\nabla_a k_a = 0$, being $k_a = \nabla_a u$ [100].

The scalar curvature results to be $R = 0$ so that $\varphi = 0$ and thus $Z_{ab} = R_{ab} = \psi(x_1, \ldots, x_n, u)k_ak_b$, which means that the metrics defined by (6.1) are Ricci-simple metrics. It should be noted again that the previous metric is not the most general $pp$-wave metric studied in literature: we recall that usually a $pp$-wave is defined as a space-time endowed with a null covariantly constant vector.

As previously said in Remark 2.12 it belongs to the subclass $L_{11} = 0$, $\tau_i = 0$ of the Kundt metric. Such space-times have been extensively studied and classified in higher dimensions, see for example [29, 26, 93, 94, 101; 95, Sec. 7.1.3, Proposition 7.3]. However we take into consideration metric (6.1): for these it is well known that the Weyl tensor is of type $N$ in all dimensions: from (4.2) and (4.9) written with the null vector $k_a$ it follows $k_aC_{bcde} + k_bC_{cade} + k_cC_{abde} = 0$ (see [93, Table 1]). Hereafter we study some other curvature conditions related to (6.1). First it is inferred easily (see [61] for the case $n = 4$) that the Ricci tensor (see [65, 96]) and thus the tensor $Z_{ab}$ are recurrent, in fact we have:

$$\nabla_c Z_{ab} = \nabla_c R_{ab} = \nabla_c \psi(x_1, \ldots, x_n, u)k_ak_b = \frac{\nabla_c \psi}{\psi} k_ak_b = \frac{\nabla_c \psi}{\psi} Z_{ab} = \beta_c Z_{ab}.$$  

(6.3)

It is possible to find sufficient conditions for the potential function of $pp$-waves to have a pseudo-$Z$ symmetric space. We make the assumption that the scalar function $\psi(x_1, \ldots, x_n, u)$ could be written in the factorized form $\psi(x_1, \ldots, x_n, u) = \psi_1(x_1, \ldots, x_n)\psi_2(u)$. We have then:

$$\nabla_c Z_{ab} = \nabla_c R_{ab} = \nabla_c \psi_1(x_1, \ldots, x_n)\psi_2(u)k_ak_b + \psi_1(x_1, \ldots, x_n)\psi_2'(u)(\nabla_c u)k_ak_b$$

$$= \nabla_c \psi_1(x_1, \ldots, x_n)\psi_2(u)k_ak_b + \psi_1(x_1, \ldots, x_n)\psi_2'(u)k_ck_ka,$$

(6.4)

being $\psi'(u) = \partial_u \psi$. Under the same assumptions on $\psi$ in view of Eq. (3.1) we get:

$$\nabla^d C_{abc} = \frac{n-3}{n-2} \nabla^d R_{abc} = \frac{n-3}{n-2}(\nabla_b R_{ac} - \nabla_a R_{bc})$$

$$= \frac{n-3}{n-2} \psi_2'(u)k_c[\nabla_b \psi_1(x_1, \ldots, x_n)k_a - \nabla_a \psi_1(x_1, \ldots, x_n)k_b].$$

(6.5)

Now suppose that $\nabla^d C_{abc} = 0$ and that $\psi_2'(u) \neq 0$: it is easily inferred that $\nabla_b \psi_1(x_1, \ldots, x_n)k_a = \nabla_a \psi_1(x_1, \ldots, x_n)k_b$. On multiplying this by a suitable vector $\theta^a$ such that $\theta^a k_a = 1$ we get easily:

$$\nabla_b \psi_1(x_1, \ldots, x_n) = (\theta^a \nabla_a \psi_1(x_1, \ldots, x_n))k_b = \lambda(x_1, \ldots, x_n)k_b.$$  

(6.6)
Conversely if $\nabla_b \psi_1(x_1, \ldots, x_n) = \lambda(x_1, \ldots, x_n)k_b$, it is trivially $\nabla_d C_{abc}^d = 0$. Thus if the factorized form $\psi(x_1, \ldots, x_n, u) = \psi_1(x_1, \ldots, x_n)\psi_2(u)$ is supposed, then $\nabla_d C_{abc}^d = 0$ if and only if (6.6) is satisfied. On the other hand it is easily seen that if $\nabla_b \psi_1(x_1, \ldots, x_n) \neq \lambda(x_1, \ldots, x_n)k_b$ then $\nabla_d C_{abc}^d = 0$ if and only if $\psi'_2(u) = 0$. Further if (6.6) holds, then (6.4) may be written as:

$$\nabla_c Z_{ab} = \varepsilon(x_1, \ldots, x_n, u)k_c k_a k_b,$$

being $\varepsilon(x_1, \ldots, x_n, u) = \lambda(x_1, \ldots, x_n)\psi_2(u) + \psi_1(x_1, \ldots, x_n)\psi'_2(u)$. In this way we can write:

$$\nabla_c Z_{ab} = \left\{ \frac{\varepsilon(x_1, \ldots, x_n, u)}{2\varepsilon(x_1, \ldots, x_n, u)} k_c \right\} Z_{ab} + \left\{ \frac{\varepsilon(x_1, \ldots, x_n, u)}{4\varepsilon(x_1, \ldots, x_n, u)} k_a \right\} Z_{bc}$$

$$+ \left\{ \frac{\varepsilon(x_1, \ldots, x_n, u)}{4\varepsilon(x_1, \ldots, x_n, u)} k_b \right\} Z_{ac}. \quad (6.8)$$

We have proved that an $n$-dimensional pp-wave space-time with $\psi(x_1, \ldots, x_n, u) = \psi_1(x_1, \ldots, x_n)\psi_2(u)$ and satisfying the condition $\nabla_b \psi_1(x_1, \ldots, x_n) = \lambda(x_1, \ldots, x_n)k_b$ results to be pseudo-Z symmetric with $\varphi = 0$ (and so pseudo-Ricci symmetric) and null associated covector $A_b = \frac{\varepsilon(x_1, \ldots, x_n, u)}{4\varepsilon(x_1, \ldots, x_n, u)} k_b$. It should be noted also that all the previous results hold under the stronger assumption for which $\psi_1(x_1, \ldots, x_n)$ is a real number, that is $\psi$ depends only on $u$, i.e. $\psi = \psi(u)$. In such a case it is $\nabla_b \psi_1(x_1, \ldots, x_n) = 0$ and thus immediately $\nabla_d C_{abc}^d = 0$. Further (6.8) is still valid with $\varepsilon(u) = \text{const}$ $\psi'_2(u)$.

We have thus proved the following.

**Theorem 6.1.** Let $M$ be an $n(\geq 4)$-dimensional pp-wave space-time endowed with the metric (6.1).

(a) The Ricci tensor is recurrent.

(b) If the function $\psi(x_1, \ldots, x_n, u) = \psi_1(x_1, \ldots, x_n)\psi_2(u)$ then $\nabla_d C_{abc}^d = 0$ is equivalent to each of the following conditions: (1) $\nabla_b \psi_1(x_1, \ldots, x_n) = \lambda(x_1, \ldots, x_n)k_b$; (2) $\psi'_2(u) = 0$.

(c) If $\psi(x_1, \ldots, x_n, u) = \psi_1(x_1, \ldots, x_n)\psi_2(u)$ and $\nabla_b \psi_1(x_1, \ldots, x_n) = \lambda(x_1, \ldots, x_n)k_b$, then the space is pseudo-Z symmetric with $\varphi = 0$. Moreover if $\psi = \psi(u)$ then the divergence of the Weyl tensor vanishes and the pp-wave is again a pseudo-Z symmetric space.

The previous result applies to a broad class of pp-wave metrics. It is well known that metrics (6.1) with different potential functions exhibit different curvature properties for the Weyl tensor. We summarize them here.

(1) The metrics (6.1) are conformally flat if and only if the potential function takes the form

$$H(x_1, \ldots, x_n, u) = a(u) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n b_i(u)x_i + c(u), \quad (6.9)$$
being \( a, b_1, \ldots, b_n, c \) arbitrary functions of \( u \) (see [56]); moreover as previously said they define a plane wave if \( H(x_1, \ldots, x_n, u) = \sum_{i=1}^{n} a_{ij}(u)x_ix_j \) where \( a_{ij}(u) \) are arbitrary matrix functions. In particular we have \( \psi(x_1, \ldots, x_n, u) = -\frac{1}{2} \nabla^2 H(x_1, \ldots, x_n, u) = -na(u) \). Thus conformally flat \( pp \)-waves are pseudo-Z symmetric spaces.

(2) Let us suppose a potential function of the form
\[
H(x_1, \ldots, x_n, u) = \sum_{i=1}^{n} a_{ij}(u)x_ix_j,
\]
being \( a_{ij}(u) = a(u)\alpha_{ij} + \beta \) for some non-constant function \( a(u) \), and for a non-vanishing matrix \( \beta_{ij} \) such that \( \sum_{i=1}^{n} \beta_{ii} = 0 \). Then metrics (6.1) become conformally symmetric (see [56, 37, 39]). In this case we have \( \psi(x_1, \ldots, x_n, u) = -\frac{1}{2} \nabla^2 H(x_1, \ldots, x_n, u) = -\sum_{i=1}^{n} a_{ii}(u) \). Again a \( pp \)-wave space-time with potential (6.10) is a pseudo-Z symmetric space.

(3) Metric (6.1) is two-symmetric, i.e. \( \nabla_{\alpha} \nabla_{\beta} R_{ijklm} = 0 \) if and only if the potential function takes the form (see [4, 13, 56])
\[
H(x_1, \ldots, x_n, u) = \sum_{i=1}^{n} a_{ij}(u)x_ix_j,
\]
being \( a_{ij}(u) = ua_{ij} + \beta_{ij}, \alpha_{ij} \) is a diagonal matrix with elements \( a_{11} \leq \cdots \leq a_{nn} \) non-null real numbers and \( \beta_{ij} \) an arbitrary symmetric matrix of real numbers. In this case we have \( \psi(x_1, \ldots, x_n, u) = -\frac{1}{2} \nabla^2 H(x_1, \ldots, x_n, u) = -\sum_{i=1}^{n} (ua_{ii} + \beta) \). Thus two-symmetric \( pp \)-waves are pseudo-Z symmetric spaces; moreover, the divergence of the conformal tensor vanishes.

(4) In [55] the author classified conformally recurrent Lorentzian manifolds. It was shown that either these manifolds are conformally flat, or \( \nabla_{\alpha} R_{ijklm} = 0 \), or they reduce to \( pp \)-waves. Further it was proved that the potential for which metrics (6.1) become conformally recurrent is given by
\[
H(x_1, \ldots, x_n, u) = a(u) \sum_{i=1}^{n} x_i^2 + F(u) \sum_{i=1}^{n} \lambda_i x_i^2,
\]
being \( a(u), F(u) \) functions and \( \lambda_i \in R, \sum_{i=1}^{n} \lambda_i = 0 \). In such a case we get \( \psi(x_1, \ldots, x_n, u) = -\frac{1}{2} \nabla^2 H(x_1, \ldots, x_n, u) = -na(u) \). Thus \( pp \)-waves with potential (6.12), i.e. conformally recurrent \( pp \)-waves result to be pseudo-Z symmetric spaces; moreover the divergence of the conformal tensor vanishes.

Examples (1)–(4) show \( pp \)-waves with different potentials that are pseudo-Z symmetric spaces. Nevertheless we are able to display a further simple differential curvature condition on the Weyl tensor that does not depend on the specific form of the potential function \( H(x_1, \ldots, x_n, u) \). In fact from \( \nabla_{\alpha} k_a = 0 \) and \( R_{ab} = \psi(x_1, \ldots, x_n, u)k_ak_b = T_{ab} \) (or directly from the Weyl type \( N \) condition) we have \( k_c C_{bde} = 0 \) and so \( T_{ac} C_{bde} = 0 \). In view of (2.11) we get immediately
\[
\nabla_{\alpha} \nabla_{\epsilon} C_{bde} + \nabla_{\beta} \nabla_{\epsilon} C_{cad} + \nabla_{\epsilon} \nabla_{\epsilon} C_{abd} = 0.
\]
Proposition 6.2. Let $M$ be an $(\geq 4)$-dimensional pp-wave space-time with the metric (6.1), then the curvature condition $D\Pi_{\alpha^\beta}(\gamma) = 0$ holds for the Weyl tensor.

Third from the recurrence of the Ricci tensor $\nabla_c R_{ab} = \beta_c R_{ab}$ being $\beta_c = \frac{\nabla \psi}{\psi}$ and again from the local components of the $(0, 4)$ conformal tensor we easily get:

$$\nabla_a C_{bcede} = \nabla_a R_{bcede} + \beta_a (C_{bcede} - R_{bcede}). \quad (6.14)$$

Write three versions of the previous equation with cyclically permuted indices $a, b, c$ and sum up the resulting equations using the second Bianchi identity for the Riemann tensor, we infer:

$$\nabla_a C_{bcede} + \nabla_b C_{cade} + \nabla_c C_{abde} = \beta_a C_{bcede} + \beta_b C_{cade} + \beta_c C_{abde} - (\beta_a R_{bcede} + \beta_b R_{cade} + \beta_c R_{abde}). \quad (6.15)$$

In [81, 82, 85] the notion of recurrent conformal 2-form was introduced by the present authors. The conformal 2-form $\Omega^2_{(C)\alpha\beta} = C_{abcd} dx^a \wedge dx^b$ is recurrent, i.e. satisfies the condition $D\Omega^2_{(C)\alpha\beta} = \beta \wedge \Omega^2_{(C)\alpha\beta}$ being $\beta = \beta \alpha dx^a$ an associated form if and only if (see [82, 85]):

$$\nabla_a C_{bcede} + \nabla_b C_{cade} + \nabla_c C_{abde} = \beta_a C_{bcede} + \beta_b C_{cade} + \beta_c C_{abde}. \quad (6.16)$$

In view of (6.16) and (6.15) the conformal 2-form is recurrent if and only if $\beta_a R_{bcede} + \beta_b R_{cade} + \beta_c R_{abde} = 0$, being $\beta_c = \frac{\nabla \psi}{\psi}$ a covector not necessarily coincident with $k_c$. Now we note that this condition is equivalent to have a curvature tensor of the form $R_{bcede} = \beta_b \beta_c D_{ac} - \beta_b \beta_d D_{ec} - \beta_a \beta_e D_{bd} + \beta_a \beta_d D_{be}$ and thus the following holds.

Proposition 6.3. Let $M$ be an $(\geq 4)$-dimensional pp-wave space-time endowed with the metric (6.1), then the conformal 2-form results to be recurrent if and only if $R_{bcede} = \beta_b \beta_c D_{ac} - \beta_b \beta_d D_{ec} - \beta_a \beta_e D_{bd} + \beta_a \beta_d D_{be}$, being $\beta_c = \frac{\nabla \psi}{\psi}$ and $D_{ab}$ a symmetric tensor.

Now if the function $\psi(x_1, \ldots, x_n, u)$ depends only on $u$ we have simply $\beta_c = \frac{\nabla \psi}{\psi} k_c$ so that from (6.15) we get $\nabla_a C_{bcede} + \nabla_b C_{cade} + \nabla_c C_{abde} = 0$ and consequently $\nabla_a C_{bcede} = 0$.

The second example we expose is essentially [84, Example 3.6] with some new refined results about curvature conditions. This is given in a simply connected domain of $\mathbb{R}^4$ with coordinates $(x^1, x^2, x^3, x^4)$ endowed with a pseudo-Riemannian metric as follows:

$$ds^2 = F(dx^1)^2 + 4dx^1 dx^2 + (dx^3)^2 + (kx^4)^2 dx^4, \quad (6.17)$$

being $F = \alpha_0 + \alpha_1 x^3 + \alpha_2 (x^3)^2, \alpha_0, \alpha_1, \alpha_2$ non-constant scalar functions of $x^1$ only and $k$ is a non-null arbitrary constant. It should be noted that metric (6.17) is a particular case one studied in [53, Lemma 3] and belongs to the Kundt class. The eigenvalues of the first fundamental form are $\lambda_{1,2} = (F \pm \sqrt{F^2 + 16})/2, \lambda_3 = 1, \lambda_4 = (kx^4)^2$. In this way it is easily seen that the metric signature is always Lorentzian in the open subset $x^1 \neq 0$. As shown in [84; 53, Lemma 3] the only
non-zero components of Christoffel symbols, the (1.3) and (0, 4) Riemann tensors
and the Ricci tensor are found to be:

\[ \Gamma_{11}^1 = \frac{1}{4} \nabla_1 F, \quad \Gamma_{13}^1 = \frac{1}{4} \nabla_3 F, \quad \Gamma_{11}^3 = -\frac{1}{2} \nabla_3 F, \]
\[ \Gamma_{14}^1 = \frac{1}{x_1}, \quad \Gamma_{14}^2 = -\frac{k^2}{2} x_1, \]
\[ R_{131}^3 = -\frac{\partial \Gamma_{11}^3}{\partial x^3} = \frac{1}{2} \nabla_3 \nabla_3 F, \]
\[ R_{133}^2 = -\frac{\partial \Gamma_{12}^2}{\partial x^3} = -\frac{1}{4} \nabla_3 \nabla_3 F, \]
\[ R_{1313} = g_{3p} R_{131}^p = g_{33} R_{131}^3 = \frac{1}{2} \nabla_3 \nabla_3 F = -R_{1331}, \]
\[ R_{11} = \frac{1}{2} \nabla_3 \nabla_3 F = \alpha_2(x_1) \neq 0. \]

From this we infer that the scalar curvature of the metric is null, i.e. \( R = g^{11} R_{11} = 0 \) and the only non-null covariant derivative of the Ricci tensor is \( \nabla_1 R_{11} = \frac{\partial R_{11}}{\partial x_1} = 2 R_{11}^m = \frac{\partial R_{11}}{\partial x_1} = \nabla_1 \psi \). Thus we have \( \psi = -\frac{R}{2} = 0 \) and from the definitions of pseudo-Z symmetric manifold (1.1), of the tensor \( \tilde{Z}_{kl} \), and considering the previous relations concerning the Ricci tensor and its covariant derivatives we have to satisfy \( \nabla_1 R_{11} = 4 A_{11} R_{11} \) for some 1-form \( A_{11} \). It is easily inferred that \( \nabla_1 \psi = 2 A_{10} \) which it follows that \( A_{11} = \frac{1}{4} \nabla_1 \ln |\psi_2| \). Thus the covector \( A_{1} = (A_{1}, 0, 0, 0) \) suitable for the definition of \( (PZS)_A \) manifold results to be locally a gradient: it may be rescaled to a constant null vector. We get also \( A_{1} = g^{11} A_{1} \) so that \( A_{1} = g^{11} A_{1} = 0, A_{2} = g^{22} A_{2} = \frac{1}{2} A_{2} - g_{33} A_{3} = 0, A_{3} = g^{33} A_{3} = 0, A_{4} = g^{44} A_{4} = 0 \) and thus \( A_{1} = (0, \frac{1}{4} A_{1}, 0, 0) \): we have also \( A_{1} = 0 \), i.e. \( A_{1} \) is a null covector.

The Ricci tensor may be written in the form \( R_{ij} = \psi A_{1} A_{j} \) with the choice \( \psi = \frac{16 a_{2}}{(1 + \ln |\psi_2|)^2} \). Moreover the only non-vanishing components of the Weyl tensor result to be

\[ C_{1331} = R_{1331} + g_{33} R_{11} = -\frac{1}{2} \alpha_2, \quad C_{1441} = g_{44} R_{11} = \frac{\alpha_2}{2} (kx_1)^2 = \beta_2 \]

and the metric is not conformally flat. It is easy to see that the equation

\[ \nabla^m C_{jklm} = \nabla^1 C_{jkl1} + \nabla^2 C_{jkl2} + \nabla^3 C_{jkl3} + \nabla^4 C_{jkl4} \]

has non-vanishing components only for \( j = 1, k = 3, l = 3; j = 1, k = 3, l = 1; j = 1, k = 4, l = 4 \) and \( j = 1, k = 4, l = 1 \): but in these cases the covariant derivatives are found to be zero: for example \( \nabla^1 \alpha_2 = g^{11} \nabla_1 \alpha_2 = 0, \nabla^3 \alpha_2 = g^{33} \nabla_1 \alpha_2 = 0 \) and analogous with \( \beta_2 \) so that finally we get \( \nabla^m C_{jklm} = 0 \) and the metric has harmonic curvature tensor. Also it is easily seen that \( A_{1} = 0 \) and thus the metric (6.17) is of Petrov type \( N \) [116]. Finally the components of the covariant derivative of the
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Weyl tensor are evaluated

\[ \nabla_1 C_{1331} = \frac{\partial C_{1331}}{\partial x^1} = -\frac{1}{2} \nabla_1 \alpha_2 = \frac{\nabla_1 \alpha_2}{\alpha_2} \left( -\frac{1}{2} \alpha_2 \right) = \frac{\nabla_1 \alpha_2}{\alpha_2} C_{1331}, \]

\[ \nabla_1 C_{1441} = \frac{\partial C_{1441}}{\partial x^1} - 2 \Gamma^4_{14} C_{1441} = \nabla_1 \beta_2 - \frac{2}{x^1} C_{1441} = \frac{\nabla_1 \beta_2}{\beta_2} \beta_2 - \frac{2}{x^1} C_{1441} \]

\[ = \left( \frac{\nabla_1 \beta_2}{\beta_2} - \frac{2}{x^1} \right) C_{1441}. \]

However it is \[ \frac{\nabla_1 \beta_2}{\beta_2} - \frac{2}{x^1} = \frac{\nabla_1 \alpha_2}{\alpha_2} \] and thus in this way if we choose the 1-form \( B_1 = (B_1, 0, 0, 0) \) with \( B_1 = \nabla_1 \ln|\alpha_2| \) the metric results to be conformally recurrent \([2, 105]\) with null recurrence parameter. We summarize the previous results (see also \([53, \text{Lemma 4}]\)). For a 4-dimensional space-time endowed with the metric \((6.17)\) the following properties hold:

(a) the metric is a non-conformally flat pseudo-Z symmetric space-time with null associated covector;
(b) the Ricci tensor is of rank 1;
(c) the Weyl tensor is divergence-free and of Petrov type \( N \);
(d) the Weyl tensor results to be recurrent with null recurrence parameter.

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References

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