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Horospheres and hyperbolicity of Hadamard manifolds

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ABSTRACT

From geometrical study of horospheres we obtain, among asymptotically harmonic Hadamard manifolds, a rigidity theorem of the complex hyperbolic space $\mathbb{C}H^m$ with respect to volume entropy. We also characterize $\mathbb{C}H^m$ horospherically in terms of holomorphic curvature boundedness. Corresponding quaternionic analogues are obtained.

1. Introduction

Geometry of horospheres and their defining function, the Busemann function, is one of interesting geometrical subjects for nonpositively curved manifolds. Let (X, g) be an n-dimensional Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of nonpositive curvature. In (X, g) a horosphere is defined as a level hypersurface $\mathcal{H} = \{x \in X \mid B_\theta(x) = \text{const.}\}$ of the Busemann function B_θ associated with a geodesic $\gamma = \gamma(t)$ which goes to an ideal point $\theta \in \partial X$ at infinity. The gradient field ∇B_θ and Hessian $\nabla\, dB_\theta$, respectively, stand for a unit normal field and the second fundamental form of the horosphere $\mathcal{H}$ whose hypersurface geometry can be described in terms of $\nabla\, dB_\theta$.

Horospheres of a typical Hadamard manifold have geometrically nice properties. In fact, a horosphere of the real hyperbolic space $\mathbb{R}H^n$ is flat, totally umbilic with constant principal curvature, and a horosphere of

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the complex hyperbolic space $\mathbb{C}H^m$ is characterized, from the results in [6], as one of Hopf real hypersurfaces with constant principal curvature among other tubular hypersurfaces. See also [7,8,14].

Taking an arbitrary real number t as level value of a Busemann function, we obtain, for a fixed $\theta \in \partial X$, a one parameter family of horospheres. In fact, for a given B_θ associated with a geodesic γ of $[\gamma] = \theta$ the horospheres $\{\mathcal{H}_{(\gamma(t),\theta)} = B_\theta^{-1}(-t) \mid t \in \mathbb{R}\}$, each of which passes through $\gamma(t)$ constitute a foliation of the ambient manifold X invariant by the geodesic flow. Geometric behavior of one parameter family of horospheres can be investigated by means of behavior of stable (or unstable) Jacobi tensor fields in time t along a geodesic γ.

An extremely important feature of stable (or unstable) Jacobi tensor fields is that along a geodesic γ tending to a $\theta \in \partial X$ they induce a one parameter family of shape operators $\mathcal{S}_t$, defined on the one parameter family of horospheres $\{\mathcal{H}_{(\gamma(t),\theta)} \mid t \in \mathbb{R}\}$. A one parameter family of shape operators $\{\mathcal{S}_t \mid t \in \mathbb{R}\}$ is a solution of the Riccati equation (for its precise definition see Section 3) so that $\{\mathcal{S}_t \mid t \in \mathbb{R}\}$ gives an appropriate tool for studying hypersurface geometry of horospheres. Here, we give an additional remark that the stable (or unstable) Jacobi tensor fields are also important in dynamical system of the geodesic flow on the unit sphere bundle of X. For this and behavior of the Anosov geodesic flow which is closely related to horospheres on a negatively curved closed manifolds, we refer to [5,13,18,22,35,37,49].

The purpose of this article is to present, from hypersurface geometry applied to one parameter families of horospheres, volume entropy rigidity theorems for the complex hyperbolic space $\mathbb{C}H^m$ and the quaternionic hyperbolic space $\mathbb{H}H^m$ (Theorems 1.5 and 1.7) and theorems which characterize $\mathbb{C}H^m$ and $\mathbb{H}H^m$ in terms of the value of second fundamental form $h(\cdot,\cdot)$ associated with structure vectors (Theorems 1.8 and 1.9).

In the volume entropy rigidity theorems an Hadamard manifold is assumed to be asymptotically harmonic. Here

Definition 1.1. (See [40].) An Hadamard manifold (X, g) is called *asymptotically harmonic* if $\Delta B_\theta(x)$ is a constant $-c$ for each $x \in X$ and $\theta \in \partial X$, where $\Delta = -\nabla^i\nabla_i$ is the Laplacian of the metric g.

The asymptotical harmonicity is equivalent to saying that the mean curvature of all horospheres in X is commonly constant $-c$. Here, by "mean" one means the sum of all principal curvatures. Furthermore (X, g) is asymptotically harmonic if and only if the positive function defined by $P(x,\theta) = \exp\{-cB_\theta(x)\}$ is harmonic on X for any $\theta \in \partial X$.

The motivation to our study is properly geometrical understanding of asymptotically harmonic Hadamard manifolds, since asymptotically harmonic manifolds appear in Fisher information geometry which plays a statistical role in the space $\mathcal{P}^+(\partial X, d\theta)$ of probability measures on the ideal boundary ∂X of an Hadamard manifold (X, g). As Theorem 1.3 in [31] illustrated, the constant $c > 0$ in Definition 1.1 appears as a homothety constant of the homothety map $\Phi : (X, g) \to (\mathcal{P}^+(\partial X, d\theta), G)$, where G is statistically defined metric over $\mathcal{P}^+(\partial X, d\theta)$, called Fisher information metric (see [1,23,31,34]).

Theorem 1.2. *(See [31].) Assume that an Hadamard manifold (X, g) admits a normalized Poisson kernel $P(x,\theta)$ (the fundamental solution to the Dirichlét problem at the ideal boundary; $\Delta u = 0$, $u_{|\partial X} = f$). Let Φ be a map from X to the space $\mathcal{P}^+(\partial X, d\theta)$, defined by $x \mapsto \mu(x) = P(x,\theta)\,d\theta$.*

Assume that the map Φ fulfills $\Phi^*G = \frac{c^2}{n} g$ ($c > 0$ is a constant) and further Φ is a harmonic map. Then, the Poisson kernel can be described as $P(x,\theta) = \exp\{-cB_\theta(x)\}$ in terms of B_θ (hence, $\Delta B_\theta = -c$, so (X, g) turns out to be asymptotically harmonic), and moreover (X, g) satisfies the axiom of visibility (see [20] for the notion of visibility).

The asymptotical harmonicity constant c appearing as the homothety constant in the above theorem has another geometrical meaning. The constant c coincides with the volume entropy $\rho(X)$ of X, as indicated in [33].

Theorem 1.3. *Let (X, g) be an Hadamard manifold. If (X, g) is asymptotically harmonic with $\Delta B_\theta \equiv -c$, then the volume entropy $\rho(X) = c$.*

Here, the *volume entropy* $\rho(X) = \lim_{r\to\infty} \frac{1}{r} \log \operatorname{Vol} B(x; r)$ is an invariant of a Riemannian manifold measuring the exponential growth rate of the volume of geodesic ball $B(x; r)$. Note that for a compact Riemannian manifold X of nonpositive curvature the volume entropy of its universal covering $\tilde{X}$ coincides with the topological entropy of the geodesic flow on X. See [43].

Let (X, g, J) be an almost Hermitian Hadamard manifold, i.e., an Hadamard manifold equipped with an almost complex structure J which satisfies $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. Then, we have the following volume entropy rigidity theorem for an almost Hermitian Hadamard manifold (X, g, J) which is nearly Kähler.

Definition 1.4. (See [25].) Let (X, g, J) be an almost Hermitian manifold. Then (X, g, J) is called *nearly Kähler*, if (X, g, J) satisfies

$$(\nabla_u J)u = 0$$

for any tangent vector u.

Being nearly Kähler is a notion in Hermitian geometry weaker than Kähler. Notice that for a horosphere $\mathcal{H}_{(x,\theta)}$ of (X, g, J) the vector field $\xi = J\nabla B_\theta$ defined at any point $y \in \mathcal{H}_{(x,\theta)}$ is tangent to $\mathcal{H}_{(x,\theta)}$ and parallel along a geodesic, an integral curve of Busemann function. We call ξ a structure vector field.

Theorem 1.5. *Let (X^n, g, J) be a nearly Kähler Hadamard manifold of real dimension $n = 2m$ $(\geqslant 4)$ of Ricci curvature $\mathrm{Ric}_g \geqslant -2(m + 1)$. Assume that* (i) *$(X, g, J)$ is asymptotically harmonic and* (ii) *$h(\xi, \xi) \leqslant -2$ for any $(x, \theta) \in X \times \partial X$, where $h(\xi, \xi)$ is the value of the second fundamental form of the field $\xi = J\nabla B_\theta$. Then $\rho(X) \leqslant 2m$ and equality $\rho(X) = 2m$ holds if and only if (X, g, J) is biholomorphically isometric to $\mathbb{C}H^m$ of constant holomorphic curvature -4.*

A similar rigidity theorem for compact manifolds is appeared in [41]. Let M be a compact Kähler manifold M with biholomorphic curvature $\geqslant -2$. Then, $\rho(\tilde{M}) \leqslant 2m$ for the universal covering $\tilde{M}$, and equality holds if and only if the universal covering $\tilde{M}$ is isometric to $\mathbb{C}H^m$ of constant holomorphic curvature -4. Munteanu obtained in [44] a similar rigidity theorem in terms of the bottom spectrum λ_1 of the Laplacian Δ.

Next we consider the case of quaternionic Kähler Hadamard manifolds.

Definition 1.6. (See [11,29,39].) A Riemannian manifold X is called *quaternionic Kähler*, if it admits a rank three vector subbundle $V \subset \mathrm{End}(TX)$ satisfying the following:

(i) in any local coordinate of X there exists a local basis $\{J_1, J_2, J_3\}$ of V such that $\{J_1, J_2, J_3\}$ gives a quaternionic structure on X and

$$\langle J_i u, J_i v\rangle = \langle u, v\rangle, \quad i = 1, 2, 3$$

for any $u, v \in TX$, and

(ii) for any section $\varphi \in \Gamma(V)$ and any vector field $\boldsymbol{u}$ on X the covariant derivative $\nabla_{\boldsymbol{u}}\varphi$ belongs to $\Gamma(V)$.

Theorem 1.7. *Let (X, g, V) be a quaternionic Kähler Hadamard manifold of real dimension $4m$ $(\geqslant 8)$ which is asymptotically harmonic, and with scalar curvature $\mathrm{scal}_g \geqslant -16m(m + 2)$. If $\rho(X) = 2(2m + 1)$, then (X, g, V) is isometric to the quaternionic hyperbolic space $\mathbb{H}H^m$ of constant holomorphic curvature -4.*

The scalar curvature condition can be replaced with $\mathrm{Ric}_g \geqslant -4(m+2)$, since a quaternionic Kähler manifold of dimension $4m \geqslant 8$ is Einstein.

Based on the quaternionic curvature identities stated in Section 7, Theorem 1.7 does not require any condition on the second fundamental form of horospheres, compared with Theorem 1.5. Moreover in general, it holds $\rho(X) \leqslant 2(2m+1)$ for a complete quaternionic Kähler manifold (X, g, V) of $\mathrm{scal}_g \geqslant -16m(m+2)$. See [39] for this.

Replacing the Ricci curvature assumption from Theorem 1.5 with a holomorphic curvature boundedness from below, we obtain the following characterization of the complex hyperbolic space, without the assumption of asymptotical harmonicity.

Theorem 1.8. *Let (X, g, J) be a nearly Kähler Hadamard manifold of real dimension $n = 2m$ ($\geqslant 4$) and of holomorphic curvature $\geqslant -4$. Assume that the structure vector field $\xi = J\nabla B_\theta$ on any horosphere $\mathcal{H}_{(x,\theta)}$ satisfies $h(\xi, \xi) \leqslant -2$ for any $(x, \theta) \in X \times \partial X$. Then (X, g, J) is biholomorphically isometric to $\mathbb{C}H^m$ of constant holomorphic curvature -4.*

A characterization of a quaternionic Kähler Hadamard manifold is similarly stated as follows.

Theorem 1.9. *Let (X, g, V) be a quaternionic Kähler Hadamard manifold of real dimension $n = 4m$ ($\geqslant 8$) and with scalar curvature $\mathrm{scal}_g \geqslant -16m(m+2)$. If $\sum_{i=1}^{3} h(\xi_i, \xi_i) \leqslant -6$ ($\xi_i = J_i \nabla B_\theta$, $i = 1, 2, 3$) for any $(x, \theta) \in X \times \partial X$, then (X, g, V) is isometric to $\mathbb{H}H^m$ of constant holomorphic curvature -4.*

We give finally the volume entropy rigidity of a real hyperbolic space as follows.

Theorem 1.10. *Let (X, g) be an n-dimensional Hadamard manifold of Ricci curvature $\mathrm{Ric}_g \geqslant -(n-1)$. If (X, g) is asymptotically harmonic, then*

$$\rho(X) \leqslant n - 1$$

and equality holds if and only if (X, g) is isometric to the real hyperbolic space $\mathbb{R}H^n$ of constant curvature -1.

The inequality $\rho(X) \leqslant n-1$ is derived from the standard volume comparison argument for an Hadamard manifold of $\mathrm{Ric}_g \geqslant -(n-1)$ (see for this [24,42]). For the corresponding rigidity theorem in a compact case refer to [41,42].

This paper is organized as follows. In Section 2 we give notations and basic properties of Hadamard manifolds and geometry of horospheres needed in the subsequent sections. In Section 3 we introduce (un)stable Jacobi tensor fields and the Riccati equation with respect to shape operators and in Section 4 we treat geometry of asymptotically harmonic manifolds and give a sketch of a proof to Theorem 1.3. In Section 5 we deals with proving Theorem 1.7. Strum's argument applied to the scalar Riccati differential inequality together with a proof of Theorem 1.8 are given in Section 6. In final section we deal with quaternionic Kähler geometry with basic curvature identities and give a proof to Theorems 1.7 and 1.9.

2. Preliminaries

Let (X, g) be an n-dimensional Hadamard manifold. For convenience sake, we write sometimes a Riemannian metric g as $\langle \cdot, \cdot \rangle$. From the Cartan–Hadamard theorem X is diffeomorphic to the Euclidean space or an open n-ball.

Let γ_1 and γ_2 be two geodesic rays on (X, g); $\gamma_i : [0, \infty) \to X$, $i = 1, 2$. Then, γ_1 and γ_2 are said to be *asymptotically equivalent*, if there exists a constant $C > 0$ such that $d(\gamma_1(t), \gamma_2(t)) < C$ for all $t \geqslant 0$. Throughout this paper geodesics are assumed to have unit speed. The set of asymptotical equivalence classes

$[\gamma]$ of all geodesic rays on (X, g) is called the *ideal boundary* of (X, g), denoted by ∂X. We say a geodesic ray γ converges to an ideal point $\theta \in \partial X$, when γ is a representative of θ.

For any point $x \in X$ and any $\theta \in \partial X$ there exists a unique geodesic γ such that $\gamma(0) = x$ and $[\gamma] = \theta$ so that we identify ∂X with the unit tangent sphere $S_o X \cong S^{n-1}$ at a certain base point $o \in X$ by identifying $v \in S_o X$ with $[\gamma_v] \in \partial X$, where γ_v is the geodesic ray such that $\gamma_v(0) = o$, $\gamma_v'(0) = v$. The space $X \cup \partial X$ is equipped with the cone topology which gives a compactification of X homeomorphic to a closed n-disk in $\mathbb{R}^n$ (see [4] for the details).

Fix a point $o \in X$ and $\theta \in \partial X$ and let $\gamma = \gamma(t)$ be a geodesic such that $\gamma(0) = o$ and $[\gamma] = \theta$. Then, the *Busemann function* on an Hadamard manifold (X, g), normalized at the point o, is defined by

$$B_\theta(x) = \lim_{t\to\infty} \bigl(d\bigl(x, \gamma(t)\bigr) - t\bigr). \tag{1}$$

We call occasionally B_θ the Busemann function associated with the geodesic γ, when we strengthen the geodesic defining the Busemann function.

The Busemann function on an Hadamard manifold is convex, at least C^2 (see [19,28]), and is characterized as a convex C^1-function b on X satisfying $|\nabla b| \equiv 1$ (see [4, Lemma 3.4] and [46]). So, if $\overline{B}_\theta$ is the Busemann function normalized at another point but associated with a geodesic tending to the same θ, the difference $\overline{B}_\theta - B_\theta$ must be a constant function on X.

A level hypersurface of the Busemann function B_θ that contains a point $x \in X$ is called a *horosphere* centered at θ and passing through x, denoted by $\mathcal{H}_{(x,\theta)}$, i.e.,

$$\mathcal{H}_{(x,\theta)} = \bigl\{y \in X \bigm| B_\theta(y) = B_\theta(x)\bigr\}. \tag{2}$$

Note that the horosphere $\mathcal{H}_{(x,\theta)}$ is derived also from taking limit of the geodesic spheres $\mathcal{G}(\gamma(t), t)$ centered at $\gamma(t)$ with radius t as $t \to \infty$, where γ is the geodesic such that $\gamma(0) \in \mathcal{H}_{(x,\theta)}$ and $[\gamma] = \theta$.

The gradient field ∇B_θ is globally defined on X and of unit norm. The integral curves of ∇B_θ are exactly geodesics converging to θ of the reversed parameter t. The restriction of ∇B_θ to a horosphere $\mathcal{H}_{(x,\theta)}$ gives a C^1 unit vector field ν outward normal to $\mathcal{H}_{(x,\theta)}$. The second fundamental form h of $\mathcal{H}_{(x,\theta)}$ defined by

$$h(v, w) = \langle \nabla_v \tilde{w}, \nu\rangle,$$

where $\tilde{w}$ is a locally defined, smooth extension of $w \in T_y\mathcal{H}_{(x,\theta)}$, is a symmetric bilinear form of $T_y\mathcal{H}_{(x,\theta)}$, whereas the shape operator $\mathcal{S}$ of $\mathcal{H}_{(x,\theta)}$ with respect to the unit normal ν is a self-adjoint endomorphism of $T_y\mathcal{H}_{(x,\theta)}$, defined by

$$\mathcal{S}v = -\nabla_v \nu$$

(see [38] and [19, 1.10.8]). Then,

$$h(v, w) = \langle \mathcal{S}v, w\rangle = -\nabla\, dB_\theta(v, w), \quad v, w \in T_y\mathcal{H}_{(x,\theta)}.$$

The Hessian $\nabla\, dB_\theta$ is positive semi-definite at any point, because B_θ is convex. Hence, the second fundamental form h is negative semi-definite and then the principal curvatures of any horosphere are nonpositive. Note that $\nabla\, dB_\theta(\nabla B_\theta, w) = 0$ for any vector w.

Example 2.1. Let (X, g) be the n-dimensional real hyperbolic space $\mathbb{R}H^n$ of constant curvature -1. Take the Poincaré unit ball model for it. Then the Busemann function on it normalized at the origin o has the form

$$B_\theta(x) = \log \frac{|x - \theta|^2}{1 - |x|^2}, \quad x \in X,\ \theta \in \partial X \cong S^{n-1}(1).$$

3. Stable Jacobi tensor fields

Let $\gamma : \mathbb{R} \to X$ be a geodesic on an Hadamard manifold (X, g). Let $\gamma(t)^\perp = \{v \in T_{\gamma(t)}X \mid v \perp \gamma'(t)\}$ be the space of tangent vectors orthogonal to γ at t.

Definition 3.1. Let $Y(t)$, $t \in (a, b)$ be a smooth bundle endomorphism of $\gamma(t)^\perp$. We call $Y(t)$ a *Jacobi tensor field* along γ, if it satisfies

$$Y''(t) + R(t)Y(t) = O, \tag{3}$$

where $R(t) = R_t : v \mapsto R(v, \gamma'(t))\gamma'(t)$ is the Jacobi operator, the self-adjoint endomorphism of $\gamma(t)^\perp$ defined by the Riemannian curvature tensor R;

$$R(u, v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w.$$

For Jacobi tensors refer to [15,18,21,26,27,37]. The existence and uniqueness of solution to (3) for a given initial condition is guaranteed in $t \in \mathbb{R}$.

Let $Y(t)$ be a Jacobi tensor field along γ, $t \in (a, b)$. Then, for any perpendicular parallel vector field $v = v(t)$ along γ, $y(t) = Y(t)v(t)$ is a perpendicular Jacobi vector field along γ. When $Y(t)$ is invertible on (a, b), we set $\mathcal{Y}(t) = Y'(t)Y(t)^{-1}$ as an endomorphism of $\gamma(t)^\perp$ and find that $\mathcal{Y}(t)$ satisfies the *Riccati equation* along γ, $t \in (a, b)$;

$$\mathcal{Y}'(t) + \mathcal{Y}(t)^2 + R(t) = O. \tag{4}$$

The *Wronskian tensor field* $\mathcal{W}(Y, Z)(t)$ is defined by

$$\mathcal{W}(Y, Z)(t) = Y'(t)^*Z(t) - Y(t)^*Z'(t)$$

for endomorphisms $Y(t)$ and $Z(t)$ along γ. Here, the asterisk means the adjoint endomorphism. If, $Y(t)$ and $Z(t)$ are Jacobi tensor fields along γ, then, $\mathcal{W}(Y, Z)(t)$ is seen to be parallel along γ. If $Y(t)$ is invertible on some interval, then $\mathcal{W}(Y, Y)(t) = O$ implies the self-adjointness of the tensor field $\mathcal{Y}(t) = Y'(t)Y(t)^{-1}$.

We say that a Jacobi tensor field $Y(t)$ is *stable* when (i) $Y(t)$ vanishes nowhere, and (ii) $\lim_{t\to\infty} \mathrm{tr}(Y(t)^*Y(t)) = 0$, or equivalently, there exists a constant $C > 0$ such that $\mathrm{tr}(Y(t)^*Y(t)) < C$ for any $t \geqslant 0$. Note that, for any covariantly constant invertible endomorphism $D = D(t)$, $Y_1(t) = Y(t)D(t)$ is also a stable Jacobi tensor field. A Jacobi tensor field $J(t)$, $t \in \mathbb{R}$ is called unstable, if $J^-(t) = J(-t)$ is stable. The existence of stable (or unstable) Jacobi tensor field along any geodesic is guaranteed (see [18,37] for example). We notice that any stable (or unstable) Jacobi tensor field is invertible. Moreover, it holds $\mathcal{W}(Y, Y)(t) = \mathcal{W}(U, U)(t) = O$ for a stable Jacobi tensor field Y and an unstable U.

Lemma 3.2. *(See [18,21,37].) Let $\gamma(t)$ be a geodesic converging to a $\theta \in \partial X$ and $Y(t)$ a stable Jacobi tensor field along $\gamma(t)$, normalized as $Y(0) = \mathrm{id}_{\gamma(0)^\perp}$. For each t, let $\mathcal{H}_{(\gamma(t),\theta)}$ be a horosphere centered at θ, passing through a point $\gamma(t)$. Then, the shape operator $\mathcal{S}(t)$ of the $\mathcal{H}_{(\gamma(t),\theta)}$ at $\gamma(t)$ with respect to the unit normal $\nu = \nabla B_\theta$ $(= -\gamma'(t))$ is represented by $\mathcal{S}(t) = Y'(t)Y^{-1}(t)$, which fulfills*

$$\mathcal{S}'(t) + \mathcal{S}^2(t) + R(t) = O.$$

Note 3.3. In terms of an unstable Jacobi tensor field $U(t)$ along γ $U'(t)U^{-1}(t)$ stands for the shape operator of the horosphere $\mathcal{H}_{(\gamma(t),-\theta)}$ with respect to the unit normal vector field $\nu_1 = \nabla B_{(-\theta)} = \gamma'(t)$. Here $-\theta \in \partial X$ is an ideal point at infinity given by $\gamma(t) \to -\theta$ as $t \to -\infty$.

Now, let $\mathcal{G}(x,r)$ be a geodesic sphere centered at x and of radius $r>0$ and $\gamma(t)=\exp_x tu$, $u\in S_xX$ be a geodesic. So, we have $\gamma(t)\in\mathcal{G}(x,t)$ at any $t>0$. As the situation is similar to the horospheres, there exists a unique Jacobi tensor field $A(t)$, $t\in\mathbb{R}$ such that $A(0)=O$ and $A'(0)=\mathrm{id}_{\gamma(0)^\perp}$. From the nonpositive curvature assumption $A(t)$ is invertible for $t\neq 0$. Therefore, $\mathcal{S}_G(t)=A'(t)A(t)^{-1}$, $t>0$ gives the shape operator of $\mathcal{G}(x,t)$ and satisfies the Riccati equation. See [16,48] for this. $\mathcal{S}_G(t)$ is positive semi-definite with respect to the outward normal field ν_G which is given by ∇r, since $r(\gamma(t))=d(x,\gamma(t))=t$ for the geodesic γ.

We will close this section by presenting the following example.

Example 3.4. Let (X,g) be an n-dimensional real hyperbolic space $\mathbb{R}H^n$ of constant curvature $-k^2$, $k>0$. Along any geodesic $\gamma=\gamma(t)$ $Y(t)=\exp(-kt)\,\mathrm{id}\,\gamma(t)^\perp$ gives a stable Jacobi tensor field and $U(t)=\exp(kt)\,\mathrm{id}_{\gamma(t)^\perp}$ an unstable tensor field, which are both normalized at $t=0$. The shape operator of a horosphere $\mathcal{H}_{(\gamma(t),\theta)}$ centered at $\theta=[\gamma]$ is given by $\mathcal{S}_H(t)=Y'(t)Y^{-1}(t)=-k\,\mathrm{id}_{\gamma(t)^\perp}$ so $\operatorname{tr}\mathcal{S}_H(t)=-(n-1)k$, whereas $U'(t)U^{-1}(t)=k\,\mathrm{id}_{\gamma(t)^\perp}$ is an endomorphism of $\gamma(t)^\perp$ giving the shape operator of a horosphere $\mathcal{H}_{(\gamma(t),-\theta)}$, which plays a crucial role in proving Theorem 1.3 in Section 4.

On the other hand the Jacobi tensor field $A(t)=\sinh kt\,\mathrm{id}_{\gamma(t)^\perp}$ satisfies $A(0)=O$, $A'(0)=\mathrm{id}_{\gamma(0)^\perp}$. So the shape operator of the geodesic sphere $\mathcal{G}(\gamma(0),t)$ is $\mathcal{S}_G(t)=A'(t)A^{-1}(t)=k\frac{\cosh kt}{\sinh kt}\,\mathrm{id}_{\gamma(t)^\perp}$.

4. Asymptotically harmonic manifolds

Let (X,g) be an n-dimensional Hadamard manifold. Assume that (X,g) is asymptotically harmonic with $\Delta B_\theta(x)\equiv -c$ for a constant $c\geqslant 0$.

An aim of this section is to verify Theorem 1.3. We will give an outline of its proof. The proof is a direct consequence of an asymptotic formula of the mean curvature of a geodesic sphere. The detailed proof is given in [33].

Let $x\in X$ and $\theta\in\partial X$. Take a geodesic $\gamma=\gamma(t)=\exp_x tu$, $u\in S_xX$ such that $\gamma(0)=x$ and $[\gamma]=\theta$.

The volume $\mathcal{V}(r)$ of the geodesic ball $B(x;r)=\{y\in X\mid d(y,x)\leqslant r\}$ is represented as $\mathcal{V}(r)=\int_{B(x;r)}dv_g=\int_0^r\mathcal{A}(t)\,dt$ in terms of the area $\mathcal{A}(r)$ of the geodesic sphere $\mathcal{G}(x,r)$.

If $\frac{\mathcal{A}'(r)}{\mathcal{A}(r)}$ converges to c as $r\to\infty$, the theorem is verified by applying l'Hospital's rule to $\rho(X)=\lim_{r\to\infty}\frac{1}{r}\log\mathcal{V}(r)$ as

$$\rho(X)=\lim_{r\to\infty}\frac{\mathcal{V}'(r)}{\mathcal{V}(r)}=\lim_{r\to\infty}\frac{\mathcal{A}(r)}{\int_0^r\mathcal{A}(t)\,dt}=\lim_{r\to\infty}\frac{\mathcal{A}'(r)}{\mathcal{A}(r)}=c.$$

So, we will show $\lim_{r\to\infty}\frac{\mathcal{A}'(r)}{\mathcal{A}(r)}=c$.

Recall that the density function $\sqrt{\det(g_{ij})}$ at the point $\gamma(t)$ of dv_g is represented by

$$J(u,t)\,t^{n-1}=\sqrt{\det\langle y_i(t),y_j(t)\rangle},$$

where $y_i(t)$ is a Jacobi vector field along γ such that $y_i(0)=0$ and $y_i'(0)=e_i(0)$, $i=1,\dots,n-1$ for the parallel orthonormal frame field $\{e_i(t)\}$ so that

$$\mathcal{V}(r)=\int_0^r\left(\int_{u\in S_xX}J(u,t)t^{n-1}\,du\right)dt$$

and

$$\mathcal{A}(r) = \int\limits_{u\in S_xX} J(u,r)r^{n-1}\,du.$$

Since each Jacobi field y_i is expressed as $y_i(t) = A(t)e_i(t)$ in terms of the Jacobi tensor field $A(t) = A_u(t)$ along γ, defined in Section 3, we have $\sqrt{\det\langle y_i(t), y_j(t)\rangle} = \det A_u(t)$ so $\mathcal{A}(r) = \int_{u\in S_xX} \det A_u(r)\,du$ and hence $\mathcal{A}'(r) = \int_{u\in S_xX} \frac{\partial}{\partial r}\det A_u(r)\,du$ is expressed as

$$\mathcal{A}'(r) = (n-1)\int\limits_{S_xX} \mu_G(u,r)\det A_u(r)\,du,$$

where $\mu_G(u,r)$ is the mean curvature of $\mathcal{G}(x,r)$ at $\gamma(r) = \exp_x ru$. This is immediate from the well known formula, seen in [24, p. 143],

$$\big(\det A_u(t)\big)' = \big(\det A_u(t)\big)\times \operatorname{tr}\big(A_u'(t)A_u^{-1}(t)\big).$$

From Lemma 4.2 below, we have $\mu_G(u,r) = c + \varepsilon_u(r)$, $|\varepsilon_u(r)| < \frac{n-1}{r}$ so $\frac{\mathcal{A}'(r)}{\mathcal{A}(r)}$ has a limit as $r\to\infty$ and its value is equal to c, as we observe

$$\left|\frac{\mathcal{A}'(r)}{\mathcal{A}(r)} - c\right| = \left|\frac{\int_{S_xX}(c+\varepsilon_u(r))\det A_u(r)\,du}{\int_{S_xX}\det A_u(r)\,du} - c\right| < \frac{n-1}{r}$$

which shows Theorem 1.3.

We will investigate a relation between the shape operator of a geodesic sphere and the shape operator of horospheres. Let $\mathcal{S}_G(t)$ be the shape operator of the geodesic sphere $\mathcal{G}_t = \mathcal{G}(x,t)$ at point $\gamma(t)$, $t > 0$, where $\gamma(t) = \exp_x tu$ is a geodesic, $u \in S_xX$. Then, we have $\mathcal{S}_G(t) = A'(t)A^{-1}(t)$ in terms of the Jacobi tensor field $A(t)$ as stated in Section 3.

On the other hand, let $\mathcal{H}_t = \mathcal{H}_{(\gamma(t),-\theta)}$ be a horosphere centered at $-\theta$, passing through $\gamma(t)$, $t > 0$, where $-\theta$ is the asymptotical equivalence class $[\gamma_-]$ represented by the reversely oriented geodesic $\gamma_-(t) = \exp_x(-tu)$. Since $\mathcal{H}_t \perp \gamma'(t)$, we have $\gamma(t)^\perp = T_{\gamma(t)}\mathcal{H} = T_{\gamma(t)}\mathcal{G}$, that is, $\mathcal{G}_t$ contacts $\mathcal{H}_t$ at point $\gamma(t)$, $t > 0$. The shape operator of $\mathcal{H}_t$ with respect to the normal vector $\gamma'(t)$ is $\mathcal{S}_H(t) = U'(t)U^{-1}(t)$, where $U = U(t)$, $t\in\mathbb{R}$, is the unstable Jacobi tensor field along γ with $U(0) = \mathrm{id}_{\gamma(0)^\perp}$.

Lemma 4.1. *For any $t > 0$*

$$\mathcal{S}_G(t) - \mathcal{S}_H(t) = \big(A^*\big)^{-1}(t)U^{-1}(t). \tag{5}$$

Proof. Since the shape operators are self-adjoint, we have by using Wronskian

$$\begin{aligned}\mathcal{S}_G(t) - \mathcal{S}_H(t) &= \big(A'(t)A^{-1}(t)\big)^* - U'(t)U^{-1}(t)\\ &= \big(A^*\big)^{-1}(t)\big\{\big(A'\big)^*(t)U(t) - A^*(t)U'(t)\big\}U^{-1}(t)\\ &= \big(A^*\big)^{-1}(t)\mathcal{W}(A,U)(t)U^{-1}(t).\end{aligned}$$

Here the Wronskian $\mathcal{W}(A,U)(t)$ is equal to $\mathrm{id}_{\gamma(t)^\perp}$, since $\mathcal{W}(A,U)(t)$ is a covariant constant tensor field and $\mathcal{W}(A,U)(0) = (A')^*(0)U(0) - A^*(0)U'(0) = \mathrm{id}_{\gamma(0)^\perp}$. $\square$

Lemma 4.2. *For any $t > 0$*

$$\big|\mu_G(t) - \mu_H(t)\big| < \frac{n-1}{t}.$$

Here $\mu_H(t)$ is equal to the constant c.

Proof. From (5) we have

$$\mu_G(t) - \mu_H(t) = \operatorname{tr}\bigl(\bigl(A^*\bigr)^{-1}(t)U^{-1}(t)\bigr). \tag{6}$$

We set the RHS of (6) as $L(t)$. From Schwarz inequality on the trace inner product $\operatorname{tr} CD^*$ for endomorphisms C, D of $\gamma(t)^\perp$, we have

$$L(t)^2 \leqslant \operatorname{tr}\bigl(\bigl(A^*\bigr)^{-1}(t)A^{-1}(t)\bigr) \cdot \operatorname{tr}\bigl(\bigl(U^*\bigr)^{-1}(t)U^{-1}(t)\bigr).$$

We claim that for any $t > 0$ and any unit vector $v \in \gamma(t)^\perp$

$$\bigl\langle A(t)v, A(t)v\bigr\rangle \geqslant t^2\langle v, v\rangle.$$

In fact, we compare the Hadamard manifold (X, g) with the Euclidean space $(\overline{X}, \bar{g})$ of the same dimension where any perpendicular Jacobi field $\overline{J}(t)$ with $\overline{J}(0) = 0$, $|\overline{J}'(0)| = 1$ along a straight line is just $t\bar{v}$ for a unit perpendicular vector $\bar{v}$. From Rauch comparison theorem (see [17]), we have

$$\operatorname{tr}\bigl(\bigl(A^*\bigr)^{-1}(t)A^{-1}(t)\bigr) = \sum_{i=1}^{n-1}\bigl\langle A^{-1}(t)e_i, A^{-1}(t)e_i\bigr\rangle \leqslant \sum_{i=1}^{n-1}\frac{1}{t^2} = \frac{n-1}{t^2}, \tag{7}$$

where $\{e_i\}$ is an orthonormal frame field along γ such that $e_n = \gamma'(t)$.

We claim next that for $t > 0$

$$\bigl\langle U(t)v, U(t)v\bigr\rangle \geqslant \langle v, v\rangle$$

for any perpendicular vector v. This is a direct consequence of the convexity of Jacobi field over an Hadamard manifold together with that $U(t)$ is unstable. So,

$$\operatorname{tr}\bigl(\bigl(U^*\bigr)^{-1}(t)U^{-1}(t)\bigr) = \sum_{i=1}^{n-1}\bigl\langle U^{-1}(t)e_i, U^{-1}(t)e_i\bigr\rangle \leqslant \sum_i\langle e_i, e_i\rangle = n - 1. \tag{8}$$

From (7), (8) we obtain

$$L(t)^2 \leqslant \frac{(n-1)^2}{t^2}$$

and Lemma 4.2. $\square$

Remark 4.3. Let (X, g) be an asymptotically harmonic Hadamard manifold.

(i) If (X, g) admits a smooth compact quotient, the volume entropy $\rho(X)$ of X coincides with the constant c. This is a direct consequence of the formula $\rho(X) = -\frac{1}{C}\int_M(\int_{\partial X}\Delta B_\theta(x)\,d\nu_x)\,dv_g$, $C = \int_M \nu_x(\partial X)\,dv_g$, if (X, g) has a smooth compact quotient M (see [36,41,49]). Here ν_x is the Patterson–Sullivan measure. However, our theorem (Theorem 1.3) can be applied to an Hadamard manifold admitting no smooth quotient of finite volume.

(ii) If (X, g) is of negative curvature and admits a compact smooth quotient, then it turns out from [12, 9.18 Corollaire] to be a rank one symmetric space of noncompact type.

(iii) Since $\Delta B_\theta \equiv -c$ is called a Poisson equation with constant c, regularity of B_θ can be recovered so that every Busemann function is smooth (refer for the regularity problem to [11, p. 466] and [3, 3.54, p. 85]). So, all horospheres of (X, g) are smooth hypersurfaces and moreover the stable and unstable foliations in the unit tangent bundle SX are smooth, since their leaves W^s, W^u are written by means of the gradient field ∇B_θ, so we obtain directly the conclusion of Proposition in [22, p. 98]. When (X, g) is further harmonic, every Busemann function on (X, g) is analytic (see [45]).

(iv) An Hadamard manifold which is harmonic is automatically asymptotically harmonic. Refer to [10,47]. A Damek–Ricci space is a typical example of harmonic Hadamard manifold, as remarked in [9]. See also [2,30].

(v) The square norm of shape operator $|\mathcal{S}|^2(y)$, $y \in \mathcal{H}_{(x,\theta)}$ is given by

$$|\mathcal{S}|^2(y) = -\mathrm{Ric}_g(\nabla B_\theta, \nabla B_\theta). \tag{9}$$

So, (X^n, g), $n \geqslant 3$ is Einstein if and only if $|\mathcal{S}|^2$ is constant and this constant is common for all horospheres.

(vi) When (X^n, g) is further Einstein, $|\mathcal{S}_G|^2(t)$ of the shape operator of a geodesic sphere of radius t has an asymptotical representation

$$|\mathcal{S}_G|^2(t) = -\frac{1}{n}\mathrm{scal}_g - \frac{2(n-1)}{nt}\mathrm{scal}_g + O\left(\frac{1}{t^2}\right).$$

See [33] for the detail.

5. Rigidity theorems for asymptotically harmonic manifolds

We will prove in this section the rigidity theorems for an asymptotically harmonic Hadamard manifold. We begin with a proof of Theorem 1.10.

Proof of Theorem 1.10. Since (X, g) is asymptotically harmonic, from Theorem 1.3 we have $\Delta B_\theta = -\mathrm{tr}(\nabla\, dB_\theta) = -\rho$ for any $\theta \in \partial X$, where $\rho = \rho(X)$ is the volume entropy of (X, g).

The inequality $\rho \leqslant n-1$ follows from $(\mathrm{tr}\, T)^2 \leqslant (n-1)\,\mathrm{tr}\, T^*T$ for an endomorphism T of an $(n-1)$-dimensional real vector space. In fact, let $\gamma = \gamma(t)$ be a geodesic and $\mathcal{S}(t)$ be the shape operator of the horosphere $\mathcal{H}_{(\gamma(t),\theta)}$ along γ, $[\gamma] = \theta$. Then, $\mathrm{tr}\,\mathcal{S}(t) = \Delta B_\theta(\gamma(t)) = -\rho$ is constant. So, we have $(\mathrm{tr}\,\mathcal{S}')(t) = 0$ and from the Riccati equation (4) $\mathrm{tr}\,\mathcal{S}^2(t) + \mathrm{tr}\, R(t) = 0$, in other words, $\mathrm{tr}\,\mathcal{S}^2(t) = -\mathrm{Ric}_g(\gamma'(t), \gamma'(t))$. Since from the assumption the Ricci curvature is equal to or less than $n-1$, we have

$$\rho^2 = \big(\mathrm{tr}\,\mathcal{S}(t)\big)^2 \leqslant (n-1)\,\mathrm{tr}\,\mathcal{S}^2(t) \leqslant (n-1)^2.$$

So $\rho \leqslant n-1$.

When equality $\rho = n-1$ holds, $(\mathrm{tr}\,\mathcal{S}(t))^2 = (n-1)\mathrm{tr}\,\mathcal{S}^2(t) = (n-1)^2$. So, $\mathcal{S}(t) = -\mathrm{id}_{\gamma'(t)^\perp}$ for any t. This means that the horosphere is totally umbilic of principal curvature -1 for any t. Since the geodesic γ can be taken arbitrarily, the theorem follows from the following.

Theorem 5.1. *(See [32, Theorem 1.1].) Let (X, g) be an n-dimensional Hadamard manifold. Then, (X, g) is isometric to $\mathbb{R}H^n$ if and only if there exists a constant k such that all horospheres of X are totally umbilic with constant principal curvature k.* $\square$

Now we will give a proof of Theorem 1.5, the rigidity theorem of complex hyperbolic space. So, let (X, g, J) be an asymtotically harmonic, nearly Kähler Hadamard manifold of real dimension n, and of Ricci curvature $\mathrm{Ric}_g \geqslant -(n+2)$. Let $\mathcal{H}_{(x,\theta)}$ be a horosphere centered at θ and passing through x. Let $\mathcal{S}$ be the shape operator of $\mathcal{H}_{(x,\theta)}$.

We compare the shape operator $\mathcal{S}$ with the shape operator of a horosphere in a complex hyperbolic space. So, consider the self-adjoint endomorphism of $T_y\mathcal{H}_{(x,\theta)}$, $y \in \mathcal{H}_{(x,\theta)}$;

$$\mathcal{S} - \frac{\rho}{n}\mathcal{S}_0,$$

where $\mathcal{S}_0$ is a self-adjoint endomorphism defined by

$$\mathcal{S}_0\, v = -\big(v + \langle v, \xi\rangle \xi\big), \quad v \in T_y\mathcal{H}_{(x,\theta)}.$$

Taking trace of $(\mathcal{S} - \frac{\rho}{n}\mathcal{S}_0)^2$, we have

$$\begin{aligned}\operatorname{tr}\left(\mathcal{S} - \frac{\rho}{n}\mathcal{S}_0\right)^2 &= \operatorname{tr}\mathcal{S}^2 - \frac{2\rho}{n}\operatorname{tr}\mathcal{S}\mathcal{S}_0 + \frac{\rho^2}{n^2}\operatorname{tr}\mathcal{S}_0^2 \\ &= \operatorname{tr}\mathcal{S}^2 + \frac{2\rho}{n}\big(\operatorname{tr}\mathcal{S} + \langle \mathcal{S}\xi, \xi\rangle\big) + \frac{\rho^2}{n^2}(n+2).\end{aligned}$$

Since $\operatorname{tr}(\mathcal{S} - \frac{\rho}{n}\mathcal{S}_0)^2 \geqslant 0$ and $\operatorname{tr}\mathcal{S} = -\rho$, we have

$$\operatorname{tr}\mathcal{S}^2 \geqslant -\frac{2\rho}{n}\big(-\rho + \langle \mathcal{S}\xi, \xi\rangle\big) - \frac{\rho^2}{n^2}(n+2) = \frac{\rho^2(n-2)}{n^2} - \frac{2\rho}{n}a,$$

where $a = h(\xi, \xi) = \langle \mathcal{S}\xi, \xi\rangle$. We apply (9) and the Ricci curvature assumption to have

$$\rho^2 - \frac{2an}{n-2}\rho - \frac{n^2(n+2)}{n-2} \leqslant 0.$$

The assumption $a \leqslant -2$ implies

$$0 \geqslant \rho^2 + \frac{4n}{n-2}\rho - \frac{n^2(n+2)}{n-2} = (\rho - n)\left(\rho + \frac{n(n+2)}{n-2}\right).$$

It is concluded that $\rho \leqslant n$, since ρ is nonpositive.

Now, we consider the equality case; $\rho = n$. Equality holds if and only if $\mathcal{S} = \frac{\rho}{n}\mathcal{S}_0 = \mathcal{S}_0$, which implies that the vector ξ is a principal direction of principal curvature -2. Therefore, applying the following, we find that (X, g, J) is biholomorphically isometric to $\mathbb{C}H^m$, $n = 2m$, of constant holomorphic curvature -4.

Theorem 5.2. *(See [32, Theorem 1.3].) Let (X, g, J) be a $2m$-dimensional nearly Kähler Hadamard manifold. Assume that for every horosphere $\mathcal{H}_{(x,\theta)}$ of X the tangent vector $\xi = J\nabla B_\theta$ at $y \in \mathcal{H}_{(x,\theta)}$ is a principal direction whose principal curvature $k = k_{(x,\theta)}(y)$ $(\neq 0)$ is constant in y and is independent of the choice of $x \in X$ and $\theta \in \partial X$. Then, (X, g, J) must be a complex hyperbolic space $\mathbb{C}H^m$ of constant holomorphic curvature $-k^2$.*

Remark 5.3. Theorem 1.10, the rigidity theorem of real hyperbolic space, can be similarly obtained by setting $\mathcal{S}_0 = -\frac{\rho}{n-1}\mathrm{id}$.

6. Sturm's argument and a proof of Theorem 1.8

The goal of this section is to prove Theorem 1.8, a characterization of $\mathbb{C}H^m$ in terms of holomorphic curvature boundedness.

For this we will show the following lemma.

Lemma 6.1. *Let $s(t)$, $-\infty < t < \infty$ be a C^1-solution of*

$$s'(t) + s^2(t) - k^2 \leqslant 0, \tag{10}$$

where k is a positive constant. Then, $s(t) \geqslant -k$ for all t.

Proof. Assume that there exists a t_0 such that $s(t_0) < -k$ and show that this leads to a contradiction.

Set $s(t_0) = -(k+\varepsilon)$, $\varepsilon > 0$ and let $k_1 = k + \frac{\varepsilon}{2}$ so $0 < k_1 < k + \varepsilon$. Define a function

$$\sigma(t) = k_1 \coth(k_1 t - \ell),$$

where ℓ is an arbitrary real number. Then, $\sigma(t)$ satisfies

$$\sigma'(t) + \sigma^2(t) - k_1^2 = 0. \tag{11}$$

Fix an ℓ such that $\sigma(t_0) = s(t_0)$. In fact, if we choose an ℓ as

$$\ell = \left(k + \frac{\varepsilon}{2}\right) t_0 - \frac{1}{2} \log \frac{\varepsilon}{4k + 3\varepsilon},$$

then $\sigma(t_0) = s(t_0)$. Remark that if we let $\varepsilon > 0$ be sufficiently small, then the ℓ must be large. Subtract (11) from (10). Then we have

$$s'(t) - \sigma'(t) \leqslant \sigma^2(t) - s^2(t) + k^2 - k_1^2$$

from which it follows at $t = t_0$

$$s'(t_0) - \sigma'(t_0) \leqslant k^2 - k_1^2 < 0,$$

since $\sigma(t_0) = s(t_0)$. Thus, the function $s(t) - \sigma(t)$ is decreasing in t at t_0, and consequently we obtain $s(t) \leqslant \sigma(t)$ for $t_0 \leqslant t \leqslant t_0 + \delta$ for a positive number δ. Therefore, by applying Sturm's argument given in [26,27], we conclude that $s(t) \leqslant \sigma(t)$ for all $t \geqslant t_0$. However, $\lim_{t \to 1/k_1 - 0} \sigma(t) = -\infty$, while $s(t)$ is well defined at $t = \frac{1}{k_1}$. This is a contradiction. So, $s(t) \geqslant -k$ for all t. $\square$

The following is a generalization of the above lemma.

Lemma 6.2. *If $s(t)$, $-\infty < t < \infty$ is a C^1 function which satisfies*

$$s'(t) + \frac{1}{K} s^2(t) - Kk^2 \leqslant 0$$

for positive constants k, K, then $s(t) \geqslant -kK$ for all t.

This is derived by comparing $s(t)$ with the function $\sigma(t) = kK\coth(kt - \ell)$.

We apply Lemma 6.1 to the proof of Theorem 1.8. Let (X^{2m}, g, J) be a nearly Kähler Hadamard manifold of $\dim X = 2m$ ($\geqslant 4$), whose holomorphic curvature is equal or greater than -4.

Let $\gamma(t)$ be a geodesic such that $[\gamma] = \theta$ and $\mathcal{S}(t)$ be the shape operator of $\mathcal{H}_{(\gamma(t),\theta)}$ at a point $\gamma(t)$ with respect to the unit normal vector $\nu = \nabla B_\theta = -\gamma'(t)$.

Set $s(t) = \langle \mathcal{S}(t)\,\xi(t), \xi(t)\rangle$. Here $\xi(t) = J\nu = -J\gamma'(t)$ is a unit vector field along γ which is parallel, since (X, g, J) is nearly Kähler. So, $s'(t) = \langle \mathcal{S}'(t)\xi(t), \xi(t)\rangle$. Since $\mathcal{S}(t)$ satisfies the Riccati equation (4),

$$s'(t) + \bigl\langle \mathcal{S}(t)\xi(t), \mathcal{S}(t)\xi(t)\bigr\rangle + \bigl\langle R_t\xi(t), \xi(t)\bigr\rangle = 0,$$

from which

$$s'(t) + s^2(t) - 4 \leqslant 0,$$

since $\langle R_t\xi(t), \xi(t)\rangle \geqslant -4$ from the holomorphic curvature assumption, and also $\langle \mathcal{S}(t)\xi(t), \mathcal{S}(t)\xi(t)\rangle \geqslant s^2(t)$, which is from the decomposition of $\mathcal{S}(t)\xi(t)$ with respect to $\xi(t)$ as $\mathcal{S}(t)\xi(t) = s(t)\xi(t) + \xi^\perp(t)$.

We are then able to apply Lemma 6.1 with $k = 2$. Since $s(t) \leqslant -2$ from the assumption, we conclude that $s(t) \equiv -2$ and hence $s'(t) = 0$. Therefore $\langle \mathcal{S}(t)\xi(t), \mathcal{S}(t)\xi(t)\rangle = -\langle R_t\xi(t), \xi(t)\rangle \leqslant 4$. On the other hand, from the decomposition $\langle \mathcal{S}(t)\,\xi(t), \mathcal{S}(t)\xi(t)\rangle = s^2(t) + |\xi^\perp(t)|^2 = 4 + |\xi^\perp(t)|^2$. Thus, $\xi^\perp(t)$ must be zero for any t which means $\mathcal{S}(t)\xi(t) = -2\xi(t)$ for any t. Theorem 1.8 follows, then, from Theorem 5.2.

7. Quaternionic Kähler Hadamard manifolds and the quaternionic curvature identities

In final section we will verify Theorems 1.7 and 1.9. So, let (X, g, V) be a quaternionic Kähler Hadamard manifold.

From (ii) of Definition 1.7 there exist on (X, g, V) local one-forms α_1, α_2 and α_3 such that

$$\begin{aligned}
\nabla_u J_1 &= \alpha_3(u)J_2 - \alpha_2(u)J_3,\\
\nabla_u J_2 &= -\alpha_3(u)J_1 + \alpha_1(u)J_3,\\
\nabla_u J_3 &= \alpha_2(u)J_1 - \alpha_1(u)J_2,
\end{aligned}\tag{12}$$

so that the Riemannian curvature tensor R satisfies

$$\begin{aligned}
\bigl[R(u,v), J_1\bigr] &= \beta_3(u,v)J_2 - \beta_2(u,v)J_3,\\
\bigl[R(u,v), J_2\bigr] &= -\beta_3(u,v)J_1 + \beta_1(u,v)J_3,\\
\bigl[R(u,v), J_3\bigr] &= \beta_2(u,v)J_1 - \beta_1(u,v)J_2,
\end{aligned}\tag{13}$$

where local two-forms β_i, $i = 1, 2, 3$ are given by $\beta_i = d\alpha_i + \alpha_j \wedge \alpha_k$ with respect to a cyclic permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

Proposition 7.1. *(See [39].) A quaternionic Kähler manifold (X^{4m}, g, V) satisfies*

$$\begin{aligned}
\bigl\langle R(u,v)w, J_1w\bigr\rangle + \bigl\langle R(u,v)J_2w, J_3w\bigr\rangle &= \beta_1(u,v)|w|^2,\\
\bigl\langle R(u,v)w, J_2w\bigr\rangle + \bigl\langle R(u,v)J_3w, J_1w\bigr\rangle &= \beta_2(u,v)|w|^2,\\
\bigl\langle R(u,v)w, J_3w\bigr\rangle + \bigl\langle R(u,v)J_1w, J_2w\bigr\rangle &= \beta_3(u,v)|w|^2.
\end{aligned}\tag{14}$$

Furthermore, (X^{4m}, g, V), $m \geqslant 2$, is Einstein. So, there exists a constant δ such that

$$\mathrm{Ric}_g = 4(m+2)\delta g$$

with constant scalar curvature $\mathrm{scal}_g = 16m(m+2)\delta$. Here, the 2-forms β_i satisfy

$$\beta_i(u, J_i v) = -\frac{1}{4m+2}\mathrm{Ric}_g(u,v) = -4\delta g(u,v).$$

Proposition 7.2. *(See [39].) For a quaternionic Kähler manifold (X^{4m}, g, V), $m \geqslant 2$, one has the following curvature identities*;

(i) *For any $u \in TX$,*

$$\langle R(J_1u,u)u, J_1u\rangle + \langle R(J_2u,u)u, J_2u\rangle + \langle R(J_3u,u)u, J_3u\rangle = 12\delta|u|^4.$$

(ii) *For any tangent vectors u, v satisfying $\langle v,u\rangle = \langle v, J_1u\rangle = \langle v, J_2u\rangle = \langle v, J_3u\rangle = 0$,*

$$\langle R(v,u)u,v\rangle + \langle R(J_1v,u)u, J_1v\rangle + \langle R(J_2v,u)u, J_2v\rangle + \langle R(J_3v,u)u, J_3v\rangle = 4\delta|u|^2|v|^2.$$

Proof. For (i) we substitute $v = J_iu$ in the i-th equality in (14), $i = 1,2,3$. Then we have

$$\begin{aligned}
\langle R(u,J_1u)u, J_1u\rangle + \langle R(u,J_1u)J_2u, J_3u\rangle &= \beta_1(u,J_1u)|u|^2,\\
\langle R(u,J_2u)u, J_2u\rangle + \langle R(u,J_2u)J_3u, J_1u\rangle &= \beta_2(u,J_2u)|u|^2,\\
\langle R(u,J_3u)u, J_3u\rangle + \langle R(u,J_3u)J_1u, J_2u\rangle &= \beta_3(u,J_3u)|u|^2.
\end{aligned}$$

By summing up the above and applying the first Bianchi identity to the summation of the second terms, we obtain the required equality in view of $\beta_i(u, J_iu) = -4\delta|u|^2$.

For (ii) we observe from the assumption and the first formula of (13) that $[R(\cdot,\cdot), J_1]v$ is perpendicular to the vectors u, J_1u and then see

$$\langle R(u,J_1v)J_1v, u\rangle = -\langle R(u,J_1v)v, J_1u\rangle, \tag{15}$$

$$\langle R(v,u)J_1v, J_1u\rangle = \langle R(v,u)v,u\rangle. \tag{16}$$

The RHS of (15) is from the first Bianchi identity reduced to

$$-\left(-\langle R(J_1v,v)u, J_1u\rangle - \langle R(v,u)J_1v, J_1u\rangle\right)$$

and then, from (16), to

$$\langle R(J_1v,v)u, J_1u\rangle + \langle R(v,u)v,u\rangle.$$

Therefore we have

$$\langle R(u,J_1v)J_1v, u\rangle + \langle R(u,v)v,u\rangle = -\langle R(v,J_1v)u, J_1u\rangle.$$

Substitute J_2v into v in the above to have

$$\langle R(u,J_3v)J_3v, u\rangle + \langle R(u,J_2v)J_2v, u\rangle = -\langle R(J_2v,J_3v)u, J_1u\rangle.$$

So, summing up these equalities we get (ii), since

$$\begin{aligned}\bigl\langle R(u,v)v,u\bigr\rangle+\sum_{i=1}^{3}\bigl\langle R(u,J_iv)J_iv,u\bigr\rangle&=-\bigl\langle R(v,J_1v)u,J_1u\bigr\rangle-\bigl\langle R(J_2v,J_3v)u,J_1u\bigr\rangle\\&=-\beta_1(u,J_1u)|v|^2=4\delta|u|^2|v|^2.\qquad\square\end{aligned}$$

Let $\gamma=\gamma(t)$ be a geodesic in a quaternionic Kähler manifold (X,g,V). Then we have a notion of *adapted* orthonormal frame along γ, namely an orthonormal frame $\{\boldsymbol{e}_i=\boldsymbol{e}_i(t),\ i=1,\dots,4m\}$ satisfying the following; $\boldsymbol{e}_1(t)=\gamma'(t)$ with $\boldsymbol{e}_2(t)=J_1\boldsymbol{e}_1(t)$, $\boldsymbol{e}_3(t)=J_2\boldsymbol{e}_1(t)$, $\boldsymbol{e}_4(t)=J_3\boldsymbol{e}_1(t)$ and further $\boldsymbol{e}_{4j+1}(t)$, $j=1,\dots,m-1$ are parallel such that $\boldsymbol{e}_{4j+2}$, $\boldsymbol{e}_{4j+3}$, $\boldsymbol{e}_{4j+4}$ are given by $\boldsymbol{e}_{4j+2}=J_1\boldsymbol{e}_{4j+1}$, $\boldsymbol{e}_{4j+3}=J_2\boldsymbol{e}_{4j+1}$, $\boldsymbol{e}_{4j+4}=J_3\boldsymbol{e}_{4j+1}$. We set linear subspaces $E_0=E_0(t), E_j=E_j(t)\subset\gamma^{\perp}(t)$ by

$$\begin{aligned}E_0(t)&=\mathrm{span}\{\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4\},\\E_j(t)&=\mathrm{span}\{\boldsymbol{e}_{4j+1},\boldsymbol{e}_{4j+2},\boldsymbol{e}_{4j+3},\boldsymbol{e}_{4j+4}\}.\end{aligned}$$

Then, $E_0\oplus E_1\oplus\cdots\oplus E_{m-1}=\gamma^{\perp}(t)$ is the orthogonal direct sum of $\gamma^{\perp}(t)$.

We notice that each E_j, $j=1,\dots,m-1$, is invariant by the quaternionic structure and also each E_j, $j=0,\dots,m-1$, by the covariant derivative along γ. An adapted orthonormal frame can always be constructive by an inductive argument. Notice also a choice of an adapted orthonormal frame depends on a local choice of quaternionic structure J_i, $i=1,2,3$, whereas each E_k, $k=1,\dots,m-1$, is independent of local quaternionic structure.

Let $\mathcal{H}_{(\gamma(t),\theta)}$ be a horosphere centered at $\theta=[\gamma]$ and passing through the point $\gamma(t)$ so we have a foliation $\{\mathcal{H}_{(\gamma(t),\theta)}\mid-\infty<t<\infty\}$ of X. The unit vector field $\nu=\nabla B_\theta$ is a normal to $\mathcal{H}_{(\gamma(t),\theta)}$ at the point $\gamma(t)$.

Associated with the shape operator $\mathcal{S}=\mathcal{S}(t)$ of $\mathcal{H}_{(\gamma(t),\theta)}$ we define functions $s(t)$ and $s_j=s_j(t)$, $j=1,\dots,m-1$, respectively along γ by

$$s(t)=\sum_{k=2}^{4}\langle\mathcal{S}\boldsymbol{e}_k,\boldsymbol{e}_k\rangle=\sum_{i=1}^{3}\langle\mathcal{S}J_i\boldsymbol{e}_1,J_i\boldsymbol{e}_1\rangle$$

and

$$\begin{aligned}s_j(t)&=\sum_{k=1}^{4}\langle\mathcal{S}\boldsymbol{e}_{4j+k},\boldsymbol{e}_{4j+k}\rangle\\&=\langle\mathcal{S}\boldsymbol{e}_{4j+1},\boldsymbol{e}_{4j+1}\rangle+\sum_{i=1}^{3}\langle\mathcal{S}J_i\boldsymbol{e}_{4j+1},J_i\boldsymbol{e}_{4j+1}\rangle\end{aligned}$$

for $j=1,\dots,m-1$.

Notice 7.3. We have the following inequality

$$\frac{1}{3}s^2(t)\leqslant\sum_{j=2}^{4}\langle\mathcal{S}\boldsymbol{e}_j,\mathcal{S}\boldsymbol{e}_j\rangle=\sum_{i=1}^{3}\langle\mathcal{S}J_i\boldsymbol{e}_1,\mathcal{S}J_i\boldsymbol{e}_1\rangle\tag{17}$$

and equality holds if and only if each of $\boldsymbol{e}_j$, $j=2,3,4$ is an eigenvector of $\mathcal{S}$ with the same eigenvalue.

Furthermore

$$\frac{1}{4}s_j^2(t) \leqslant \sum_{k=1}^{4}\langle \mathcal{S}\boldsymbol{e}_{4j+k}, \mathcal{S}\boldsymbol{e}_{4j+k}\rangle$$
$$= \langle \mathcal{S}\boldsymbol{e}_{4j+1}, \mathcal{S}\boldsymbol{e}_{4j+1}\rangle + \sum_{i=1}^{3}\langle \mathcal{S}J_i\boldsymbol{e}_{4j+1}, \mathcal{S}J_i\boldsymbol{e}_{4j+1}\rangle$$

and equality holds if and only if each of $\boldsymbol{e}_{4j+k}$, $k = 1, 2, 3, 4$ is an eigenvector of $\mathcal{S}$ with the same eigenvalue.

Lemma 7.4. *$s(t)$ and $s_j(t)$, $j = 1, \dots, m-1$ satisfy the following for all t;*

$$s'(t) + \frac{1}{3}s^2(t) + 12\delta \leqslant 0, \tag{18}$$

$$s_j'(t) + \frac{1}{4}s_j^2(t) + 4\delta \leqslant 0. \tag{19}$$

Proof. We differentiate, with respect to t, the function

$$s(t) = \sum_{i=1}^{3}\langle \mathcal{S}J_i\boldsymbol{e}_1, J_i\boldsymbol{e}_1\rangle.$$

Then, from (12) and the fact that the shape operator is self-adjoint we obtain

$$s'(t) = \sum_{i=1}^{3}\bigl\langle \mathcal{S}'J_i\boldsymbol{e}_1, J_i\boldsymbol{e}_1\bigr\rangle. \tag{20}$$

In fact, we can write $\langle \mathcal{S}J_i\boldsymbol{e}_1, J_i\boldsymbol{e}_1\rangle'$ as $\langle \mathcal{S}'J_i\boldsymbol{e}_1, J_i\boldsymbol{e}_1\rangle + 2\langle \mathcal{S}J_i'\boldsymbol{e}_1, J_i\boldsymbol{e}_1\rangle$ to have

$$s'(t) = \sum_{i=1}^{3}\bigl\langle \mathcal{S}'J_i\boldsymbol{e}_1, J_i\boldsymbol{e}_1\bigr\rangle + 2\sum_{i=1}^{3}\bigl\langle \mathcal{S}J_i'\boldsymbol{e}_1, J_i\boldsymbol{e}_1\bigr\rangle.$$

From (12) the second term reduces to zero so we have (20). Then, from the Riccati equation (4) we have

$$0 = s'(t) + \sum_{i=1}^{3}\langle \mathcal{S}J_i\boldsymbol{e}_1, \mathcal{S}J_i\boldsymbol{e}_1\rangle + \sum_{i=1}^{3}\langle R_tJ_i\boldsymbol{e}_1, J_i\boldsymbol{e}_1\rangle$$
$$= s'(t) + \sum_{i=1}^{3}\langle \mathcal{S}J_i\boldsymbol{e}_1, \mathcal{S}J_i\boldsymbol{e}_1\rangle + 12\delta. \tag{21}$$

With the aid of the inequality (17) in Notice 7.3 we have the first inequality of Lemma 7.4.

Similarly, we have

$$s_j'(t) = \bigl\langle \mathcal{S}'\boldsymbol{e}_{4j+1}, \boldsymbol{e}_{4j+1}\bigr\rangle + \bigl\langle \mathcal{S}'J_1\boldsymbol{e}_{4j+1}, J_1\boldsymbol{e}_{4j+1}\bigr\rangle$$
$$+ \bigl\langle \mathcal{S}'J_2\boldsymbol{e}_{4j+1}, J_2\boldsymbol{e}_{4j+1}\bigr\rangle + \bigl\langle \mathcal{S}'J_3\boldsymbol{e}_{4j+1}, J_3\boldsymbol{e}_{4j+1}\bigr\rangle$$

to get the second inequality of Lemma 7.4.

We can then apply Lemma 6.2 to Lemma 7.4 to conclude for all t

$$s(t) \geqslant -6\sqrt{-\delta}, \tag{22}$$

and

$$s_j(t) \geqslant -4\sqrt{-\delta}. \tag{23}$$

In fact, we can set for $s(t)$ of Lemma 6.2 $K = 3$ and $k > 0$ such that $12\delta = -Kk^2$, i.e., $k = 2\sqrt{-\delta}$ so that $-kK = -6\sqrt{-\delta}$, and for $s_j(t)$ of Lemma 6.2 set $K = 4$ and $k > 0$ such that $4\delta = -Kk^2$ so $-kK = -4\sqrt{-\delta}$. $\square$

Now, we will give a proof to Theorem 1.7.

Since (X, g, V) is asymptotically harmonic, the trace of $\mathcal{S}$ is equal to $-\rho(X)$ so that it is equal to $-2(2m+1)$. On the other hand, from (22) and (23), we find that

$$\begin{aligned}\operatorname{tr}\mathcal{S} &= s(t) + \sum_{j=1}^{m-1} s_j(t) \\ &\geq \{-6 + (m-1)(-4)\}\sqrt{-\delta} = -2(2m+1)\sqrt{-\delta}.\end{aligned} \tag{24}$$

So, we have $1 \leqslant \sqrt{-\delta}$ and hence $\delta \leqslant -1$.

However, $\delta \geqslant -1$ from the scalar curvature assumption $\mathrm{scal}_g = 12m(m+2)\delta \geqslant -12m(m+2)$. So δ must be -1. Therefore equality holds in the inequality (24) so $s(t)$ and $s_j(t)$ must satisfy $s(t) \equiv -6$ and $s_j(t) \equiv -4$, $j = 1, \ldots, m-1$. Then, from (21) we have

$$\sum_{i=1}^{3} \langle \mathcal{S}J_i\boldsymbol{e}_1, \mathcal{S}J_i\boldsymbol{e}_1\rangle = 12 = \frac{1}{3}s^2(t) \tag{25}$$

on any horosphere $\mathcal{H}_{(\gamma(t),\theta)}$. Hence, from Notice 7.3, we find that $J_1\boldsymbol{e}_1$, $J_2\boldsymbol{e}_1$ and $J_3\boldsymbol{e}_1$ are eigenvectors of $\mathcal{S}$ with eigenvalue -2, that is, each is a principal direction of principal curvature -2. As this statement holds for an arbitrary $\theta \in \partial X$, the proof is immediate from the following.

Theorem 7.5. *(See [32, Theorem 1.3].) Let (X, g, V) be a $4m$ $(\geqslant 8)$-dimensional quaternionic Kähler Hadamard manifold. Assume that for every horosphere $\mathcal{H}_{(x,\theta)}$ of X the tangent vector $\xi_i = J_i\nabla B_\theta$, $i = 1, 2, 3$ is a principal direction whose principal curvature $k = k(y)$ is negative constant in $y \in \mathcal{H}_{(x,\theta)}$ and further is independent of the choice of x and θ. Then, (X, g, V) must be a quaternionic hyperbolic space $\mathbb{H}H^m$ of constant holomorphic curvature $-k^2$.*

As a direct consequence of Theorem 1.7 we have the following corollary.

Corollary 7.6. *Let (X^{4m}, g, V), $m \geqslant 2$ be a quaternionic Kähler Hadamard manifold with $\mathrm{scal}_g \geqslant -16m \times (m+2)$. If the bottom $\lambda_1(X)$ of spectrum of the Laplacian satisfies $\lambda_1(X) \geqslant (2m+1)^2$, then (X^{4m}, g, V) is isometric to $\mathbb{H}H^m$ of constant holomorphic curvature -4.*

Here, for a complete Riemannian manifold X the bottom $\lambda_1(X)$ of the spectrum of the Laplacian is defined

$$\lambda_1(X) = \inf_{f \in C_o^\infty(X)} \frac{\int_X |df|^2\, dv_g}{\int_X f^2\, dv_g}.$$

See [40] for this.

In fact, Corollary 7.6 follows from the above theorem, since $\lambda_1(X) \leqslant \frac{\rho^2(X)}{4}$ holds for any Hadamard manifold X by the result in [40] and then from [39] $\rho = \rho(X) \leqslant 4m + 2$ so that from the assumption $\lambda_1(X) \geqslant (2m + 1)^2$ one has equality $\rho(X) = 4m + 2$.

We will now give a proof to Theorem 1.9 as follows. From the argument in the proof to Theorem 1.7, it is enough to show Eq. (25).

From the assumption of Theorem 1.9 we have $\mathrm{scal}_g = 16m(m + 2)\delta \geqslant -16m(m + 2)$ and then $\delta \geqslant -1$. Hence, from (18) we obtain

$$s'(t) + \frac{1}{3}s^2(t) - 12 \leqslant 0. \tag{26}$$

Now we apply Lemma 6.2 with $K = 3$, $k = 2$ to the above to conclude that $s(t) \geqslant -6$ so that $s(t) \equiv -6$ holds in all t from the assumption $s(t) \leqslant -6$.

Thus, from (21) we have

$$\sum_{i=1}^{3} \langle \mathcal{S}J_i \boldsymbol{e}_1, \mathcal{S}J_i \boldsymbol{e}_1 \rangle = -12\delta \leqslant 12 = \frac{1}{3}s^2(t)$$

from which, together with (17), we obtain (25).

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