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Real hypersurfaces in complex two-plane Grassmannians with ξ-invariant Ricci tensor[☆]

Young Jin Suh [*]

Kyungpook National University, Department of Mathematics, Taegu 702-701, Republic of Korea

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ABSTRACT

In this paper, first we introduce the full expression of the Ricci tensor of a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ from the equation of Gauss. Next, we give a new characterization of real hypersurfaces of type (A) in complex two-plane Grassamnnians with a vanishing Lie derivative of the Ricci tensor S in the direction of the Reeb vector field ξ, that is, an ξ-invariant Ricci tensor.

0. Introduction

In the geometry of real hypersurfaces in complex or quaternionic space forms it can be easily checked that there do not exist any real hypersurfaces with parallel shape operator A by virtue of Codazzi's equation.

However, in such kinds of space forms the proof of non-existence is not so easy if we consider a real hypersurface with a parallel Ricci tensor, that is, $\nabla S = 0$. In a class of Hopf hypersurfaces, Kimura [1] has asserted that there do not exist any real hypersurfaces in a complex projective space $\mathbb{C}P^m$ with a parallel Ricci tensor. Moreover, he has given a classification of Hopf hypersurfaces in $\mathbb{C}P^m$ with commuting Ricci tensor, that is $S\phi = \phi S$ (see [2]) and showed that M is locally congruent to one of real hypersurfaces of type A_1, A_2, B, C, D and E, that is, respectively, a tube of certain radius r over a totally geodesic $\mathbb{C}P^k$, a complex quadric $\mathbb{Q}^{m-1}$, $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{n-1}{2}}$, a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$ and an Hermitian symmetric space $SO(10)/U(5)$.

On the other hand, in a complex hyperbolic space $\mathbb{C}H^m$ Ki and the present author [3] have given a complete classification of Hopf hypersurfaces in $\mathbb{C}H^m$ with a commuting Ricci tensor and proved that M is locally congruent to a horosphere, a geodesic hypersphere, a tube over a tally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^m$.

In a quaternionic projective space $\mathbb{H}P^m$ Pérez [4] has considered the notion of $S\phi_i = \phi_i S, i = 1, 2, 3$, for real hypersurfaces in $\mathbb{H}P^m$ and classified that M is locally congruent to type A_1, or type A_2, that is, a tube over $\mathbb{H}P^k$ with radius $0 < r < \frac{\pi}{4}$. Moreover, in [4] he has also classified that real hypersurfaces in $\mathbb{H}P^m$ with a parallel Ricci tensor are locally congruent to an open subset of a geodesic hypersphere whose radius r satisfies $\cot^2 r = \frac{1}{2m}$.

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* Fax: +82 539506306.
E-mail address: yjsuh@knu.ac.kr.

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Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the formula concerned with the Ricci tensor mentioned above is not so simple if we consider a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ (see [5–10]).

In this paper we study an analogous question related to the Ricci tensor S of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. The ambient space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It becomes a compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see [11]).

In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. So, we have considered two natural geometric conditions that the 1-dimensional distribution $[\xi] = \text{Span}\,\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^{\perp} = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are invariant under the shape operator. By using such two conditions and the results in [12], Berndt and the present author [5] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^{\perp}$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

If the structure vector field ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator, that is, $A\xi = \alpha\xi$, $\alpha = g(A\xi, \xi)$, M is said to be a *Hopf real hypersurface*. In such a case, the integral curves of the structure vector field ξ are geodesics (see [6]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow*. Moreover, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ denotes the Lie derivative along the direction of the Reeb vector field ξ and g the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor ϕ commutes with the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.

When the Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, we say that M has a *commuting Ricci tensor*.

In the proof of Theorem A we have proved that the 1-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathfrak{D}^{\perp}$ or in the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$. The case (A) in Theorem A is just the case that the 1-dimensional distribution $[\xi]$ belongs to the distribution $\mathfrak{D}^{\perp}$. Of course, the Reeb vector field ξ of real hypersurfaces of type (A) is known to be Killing (see [6]). Then the commuting structure tensor implies the Ricci commuting. Moreover, we have given a characterization of type (A) in Theorem A in terms of the Lie derivative of the shape operator A along the direction of the Reeb vector field ξ, that is, $\mathcal{L}_\xi A = 0$ (see [9]). Then it can be easily checked that these kinds of hypersurfaces naturally satisfy $\mathcal{L}_\xi S = 0$ for the Ricci tensor of M in $G_2(\mathbb{C}^{m+2})$. When the Ricci tensor satisfies the formula $\mathcal{L}_\xi S = 0$, it is said to be a *ξ-invariant Ricci tensor*.

On the other hand, it is not difficult to check that the Ricci tensor S of real hypersurfaces of type (B) mentioned in Theorem A can not commute with the structure tensor ϕ and can not parallel. From such a view point it must be a natural problem to know ceratin hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with a *commuting Ricci tensor*. Along this direction we want to introduce a theorem due to the present author [13] as follows:

Theorem B. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with commuting Ricci tensor. Then M is congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Motivated by such a theorem, the main result of this paper is to give a characterization of real hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$ with a vanishing Lie derivative of the Ricci tensor S along the direction of the Reeb vector field ξ, $\mathcal{L}_\xi S = 0$, that is, the *ξ-invariant Ricci tensor*. Then we assert the following

Theorem. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with ξ-invariant Ricci tensor. Then M is congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

In Section 1 we recall Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. In Section 2 we will show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. In Section 3 the formulas for the Ricci tensor S and its Lie derivative $\mathcal{L}_\xi S$ along the direction of the Reeb vector field ξ will be shown explicitly. Also in this section we will give a complete proof of the main theorem.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [11,5,6]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o=eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m}\times\mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}),g)$ is eight.

When $m=1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m=2$, we note that the isomorphism $Spin(6)\simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m\geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k}=\mathfrak{su}(m)\oplus\mathfrak{su}(2)\oplus\mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu=J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2=I$ and $\mathrm{tr}(JJ_\nu)=0$. This fact will be used in the next sections.

A canonical local basis $\{J_1,J_2,J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1}=J_{\nu+2}=-J_{\nu+1}J_\nu$, where the index is taken as module three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar\nabla$ of $(G_2(\mathbb{C}^{m+2}),g)$, there exist a canonical local basis $\{J_1,J_2,J_3\}$ of $\mathfrak{J}$ and three local 1-forms q_1,q_2,q_3 such that

$$\bar\nabla_XJ_\nu=q_{\nu+2}(X)J_{\nu+1}-q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Moreover, in [11] it is known that the Riemannian curvature tensor $\bar R$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar R(X,Y)Z&=g(Y,Z)X-g(X,Z)Y+g(JY,Z)JX-g(JX,Z)JY-2g(JX,Y)JZ\\&\quad+\sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X-g(J_\nu X,Z)J_\nu Y-2g(J_\nu X,Y)J_\nu Z\}+\sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX-g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where $\{J_1,J_2,J_3\}$ is any canonical local basis of $\mathfrak{J}$, X, Y and Z, any vector fields on $G_2(\mathbb{C}^{m+2})$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a submanifold in $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g).

Now let us put

$$JX=\phi X+\eta(X)N,\qquad J_\nu X=\phi_\nu X+\eta_\nu(X)N \tag{2.1}$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$.

From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ,ξ,η,g) induced on M in such a way that

$$\phi^2X=-X+\eta(X)\xi,\quad \eta(\xi)=1,\qquad \phi\xi=0,\quad \eta(X)=g(X,\xi) \tag{2.2}$$

for any vector field X on M.

On the other hand, from the quaternionic Kähler structure $\{J_1,J_2,J_3\}$ of $\mathfrak{J}$ and the expression of (2.1) we have an almost contact metric 3-structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu=1,2,3$ on M. Moreover, from the commuting property of $J_\nu J=JJ_\nu$, $\nu=1,2,3$, in Section 1 and (2.1), the relation between these two contact metric structures (ϕ,ξ,η,g) and $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu=1,2,3$ can be given by

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\qquad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\&\phi\xi_\nu=\phi_\nu\xi,\qquad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\&\phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\&\phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}\end{aligned} \tag{2.3}$$

for any vector field X on M.

Using the above expressions (1.2) and (2.1) for the curvature tensor $\bar R$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned}R(X,Y)Z&=g(Y,Z)X-g(X,Z)Y+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y-2g(\phi X,Y)\phi Z\\&\quad+\sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X-g(\phi_\nu X,Z)\phi_\nu Y-2g(\phi_\nu X,Y)\phi_\nu Z\}\end{aligned}$$

$$+\sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X-g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}-\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X-\eta(X)\eta_\nu(Z)\phi_\nu\phi Y\}$$
$$-\sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z)-\eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu+g(AY,Z)AX-g(AX,Z)AY$$

and

$$(\nabla_X A)Y-(\nabla_Y A)X=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi+\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\}$$
$$+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu\,,$$

where R denotes the curvature tensor and A the shape operator of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

Then from the formulas (1.1) and (2.1), together with (2.2) and (2.3), the Kähler structure and the quaternionic Kähler structure of $G_2(\mathbb{C}^{m+2})$ give the following

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\qquad \nabla_X\xi=\phi AX, \tag{2.4}$$
$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX, \tag{2.5}$$
$$(\nabla_X\phi_\nu)Y=-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX-g(AX,Y)\xi_\nu. \tag{2.6}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi)&=\nabla_X(\phi\xi_\nu)\\&=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\&=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX-g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.\end{aligned}\tag{2.7}$$

Moreover, from $JJ_\nu=J_\nu J$, $\nu=1,2,3$, it follows that

$$\phi\phi_\nu X=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu. \tag{2.8}$$

3. Proof of main theorem

In this section, we consider a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with *ξ-invariant Ricci tensor*, that is, $\mathcal{L}_\xi S=0$, along the direction of the Reeb vector field ξ. Before giving the proof of our main theorem, let us check "what kind of hypersurfaces given in Theorem A satisfy the formula $\mathcal{L}_\xi S=0$".

In other words, it will be an interesting problem to know whether there exists a kind of hypersurface M in $G_2(\mathbb{C}^{m+2})$ with a Lie vanishing Ricci tensor. Then the *ξ-invariant Ricci tensor* $\mathcal{L}_\xi S=0$ gives

$$\begin{aligned}(\mathcal{L}_\xi S)X&=\mathcal{L}_\xi(SX)-S\mathcal{L}_\xi X\\&=\nabla_\xi(SX)-\nabla_{SX}\xi-S(\nabla_\xi X-\nabla_X\xi)\\&=(\nabla_\xi S)X-\nabla_{SX}\xi+S\nabla_X\xi\\&=(\nabla_\xi S)X-\phi ASX+S\phi AX\\&=0\end{aligned}$$

for any vector field X on M. Then the assumption $\mathcal{L}_\xi S=0$ holds if and only if $(\nabla_\xi S)X=\phi ASX-S\phi AX$. In this section, we will show that only a tube of certain radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ satisfies the formula $\mathcal{L}_\xi S=0$.

Now let us contract Y and Z in the equation of Gauss in Section 2. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}SX&=\sum_{i=1}^{4m-1}R(X,e_i)e_i\\&=(4m+10)X-3\eta(X)\xi-3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu+\sum_{\nu=1}^{3}\{(\mathrm{Tr}\,\phi_\nu\phi)\phi_\nu\phi X-(\phi_\nu\phi)^2X\}\\&\quad-\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X-\eta(X)\phi_\nu\phi\xi_\nu\}-\sum_{\nu=1}^{3}\{(\mathrm{Tr}\,\phi_\nu\phi)\eta(X)-\eta(\phi_\nu\phi X)\}\xi_\nu+hAX-A^2X,\end{aligned}\tag{3.1}$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, $\mathrm{Tr}\, JJ_\nu = 0$, $\nu = 1, 2, 3$, we calculate the following for any basis $\{e_1, \ldots, e_{4m-1}, N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$

$$\begin{aligned} 0 &= \mathrm{Tr}\, JJ_\nu \\ &= \sum_{k=1}^{4m-1} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N) \\ &= \mathrm{Tr}\, \phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN) \\ &= \mathrm{Tr}\, \phi\phi_\nu - 2\eta_\nu(\xi) \end{aligned} \tag{3.2}$$

and (2.8) gives that

$$\begin{aligned} (\phi_\nu\phi)^2 X &= \phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu) \\ &= \phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X){\phi_\nu}^2\xi \\ &= X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}. \end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$\begin{aligned} SX &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\} + hAX - A^2X \\ &= (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X. \end{aligned} \tag{3.4}$$

Now in order to compute the commuting Ricci tensor, we calculate the following

$$S\phi X = (4m+7)\phi X - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi^2 X - \eta(\phi_\nu\phi X)\phi_\nu\xi - \eta(\phi X)\eta_\nu(\xi)\xi_\nu\} + hA\phi X - A^2\phi X \tag{3.5}$$

and

$$\phi SX = -3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\phi\xi_\nu\} + h\phi AX - \phi A^2X. \tag{3.6}$$

Then from (3.5) and (3.6) it follows that

$$(\phi S - S\phi)X = -4\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu + h(\phi A - A\phi)X - (\phi A^2 - A^2\phi)X. \tag{3.7}$$

So we are able to calculate the following

$$\begin{aligned} \mathrm{Tr}\,(\phi S - S\phi)^2 &= h\mathrm{Tr}\,(\phi A - A\phi)(\phi S - S\phi) - \mathrm{Tr}\,(\phi A^2 - A^2\phi)(\phi S - S\phi) \\ &\quad - 4\sum_{\nu=1}^{3}\mathrm{Tr}\,(\eta_\nu \otimes \phi\xi_\nu)(\phi S - S\phi) + 4\sum_{\nu=1}^{3}\mathrm{Tr}\,(\eta \circ \phi \otimes \xi_\nu)(\phi S - S\phi). \end{aligned} \tag{3.8}$$

On the other hand, the terms in the right side of (3.8) are respectively given by

$$\begin{aligned} \mathrm{Tr}\,(\eta_\nu \otimes \phi\xi_\nu)(\phi S - S\phi) &= \sum_i g(\eta_\nu((\phi S - S\phi)e_i)\phi\xi_\nu, e_i) \\ &= \sum_i g((\phi S - S\phi)e_i, \xi_\nu)g(\phi\xi_\nu, e_i) = g((\phi S - S\phi)\phi\xi_\nu, \xi_\nu) \\ &= -g(\phi\xi_\nu, (\phi S - S\phi)\xi_\nu) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathrm{Tr}\,(\eta_\nu \circ \phi \otimes \xi_\nu)(\phi S - S\phi) &= \sum_i g(\eta_\nu((\phi^2 S - \phi S\phi)e_i)\xi_\nu, e_i) \\ &= \eta_\nu((\phi^2 S - \phi S\phi)\xi_\nu) = -g((\phi S - S\phi)\xi_\nu, \phi\xi_\nu). \end{aligned} \tag{3.10}$$

Then by (3.9) and (3.10), the formula (3.8) becomes

$$\begin{aligned} \mathrm{Tr}\,(\phi S - S\phi)^2 &= h\mathrm{Tr}\,(\phi A - A\phi)(\phi S - S\phi) - \mathrm{Tr}\,(\phi A^2 - A^2\phi)(\phi S - S\phi) \\ &= -\mathrm{Tr}\,(\phi A^2 - A^2\phi)(\phi S - S\phi). \end{aligned}$$

From this, the right side becomes

$$\begin{aligned}\mathrm{Tr}\,(\phi A^2 - A^2\phi)(\phi S - S\phi) &= \mathrm{Tr}\,,\phi A^2\phi S - \mathrm{Tr}\,,A^2\phi^2 S - \mathrm{Tr}\,,\phi A^2 S\phi + \mathrm{Tr}\,,A^2\phi S\phi \\ &= 2\mathrm{Tr}\,\phi A^2\phi S - \mathrm{Tr}\,,A^2\phi^2 S - \mathrm{Tr}\,,\phi A^2 S\phi. \end{aligned} \tag{3.11}$$

On the other hand, the symmetry of $\nabla_\xi S = \phi AS - S\phi A$, which is equivalent to $\mathcal{L}_\xi S = 0$, gives

$$(\phi A - A\phi)S = S(\phi A - A\phi).$$

This implies

$$\phi A(\phi AS - S\phi A + SA\phi - A\phi S) = 0,$$

so that we know

$$\mathrm{Tr}\,\phi ASA\phi = \mathrm{Tr}\,\phi A^2\phi S. \tag{3.12}$$

Then from (3.11) and (3.12) it follows that

$$\begin{aligned}\mathrm{Tr}\,(\phi S - S\phi)^2 &= -\mathrm{Tr}\,(\phi A^2 - A^2\phi)(\phi S - S\phi) \\ &= \mathrm{Tr}\,\phi^2 SA^2 + \mathrm{Tr}\,\phi^2 A^2 S - 2\mathrm{Tr}\,\phi^2 ASA. \end{aligned} \tag{3.13}$$

On the other hand, the right side of (3.13) can be calculated term by term as follows:

$$\mathrm{TR}\,\phi^2 ASA = \mathrm{Tr}\,(-ASA + \eta(ASA)\xi) = -\mathrm{Tr}\,ASA + \eta(ASA\xi),$$
$$\mathrm{TR}\,\phi^2 SA^2 = \mathrm{Tr}\,(-SA^2 + \eta(SA^2)\xi) = -\mathrm{Tr}\,SA^2 + \eta(SA^2\xi),$$

and

$$\mathrm{TR}\,\phi^2 A^2 A = \mathrm{Tr}\,(-A^2 S + \eta(A^2 S)\xi) = -\mathrm{Tr}\,A^2 S + \eta(A^2 S\xi).$$

Substituting these formulas into (3.13) gives the following

$$\begin{aligned}\mathrm{Tr}\,(\phi S - S\phi)^2 &= -3\mathrm{Tr}\,SA^2 + \eta(SA^2\xi) - \mathrm{Tr}\,A^2 S + \eta(A^2 S\xi) + 2\mathrm{Tr}\,ASA - 2\eta(ASA\xi) \\ &= 2\eta(SA^2\xi) - 2\eta(ASA\xi). \end{aligned} \tag{3.14}$$

Now from the expression of the Ricci tensor (3.4) for the Reeb vector field ξ, we have the following respectively

$$S\xi = 4(m+1)\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi,$$

and

$$\eta(SA^2\xi) = 4(m+1)\|A\xi\|^2 - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\xi_\nu, A^2\xi) + hg(A\xi, A^\xi) - g(A^2\xi, A^2\xi),$$

$$\begin{aligned}\eta(ASA\xi) &= g(SA\xi, A\xi) \\ &= (4m+7)g(A\xi, A\xi) - 3\eta(A\xi)^2 - 3\sum_{\nu=1}^{3}\eta_\nu(A\xi)^2 + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)g(\phi_\nu\phi A\xi, A\xi) - \eta(\phi_\nu A\xi)g(\phi_\nu\xi, A\xi) \\ &\quad - \eta(A\xi)\eta_\nu(\xi)\eta_\nu(A\xi)\} + hg(A^2\xi, A\xi) - g(A^3\xi, A\xi).\end{aligned}$$

Then the formula (3.14) for M in $G_2(\mathbb{C}^{m+2})$ becomes

$$\begin{aligned}\mathrm{Tr}\,(\phi S - S\phi)^2 &= 2\eta(SA^2\xi) - 2\eta(ASA\xi) \\ &= -6\|A\xi\|^2 + 6\eta(A\xi)^2 + 6\sum_{\nu=1}^{3}\eta_\nu(A\xi)^2 - 8\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(A^2\xi) - 2\sum_{\nu=1}^{3}\{\eta_\nu(\xi)g(\phi_\nu\phi A\xi, A\xi) \\ &\quad + \eta(\phi_\nu A\xi)^2 - \eta(A\xi)\eta_\nu(\xi)\eta_\nu(A\xi)\}. \end{aligned} \tag{3.15}$$

From this, together with (2.2)–(2.4) and the notion of Hopf, the right side of (3.15) should be vanishing for a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi S = 0$. This gives that the Ricci tensor S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. Then by Theorem B we can assert our main result. This gives a complete proof of our main theorem.

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