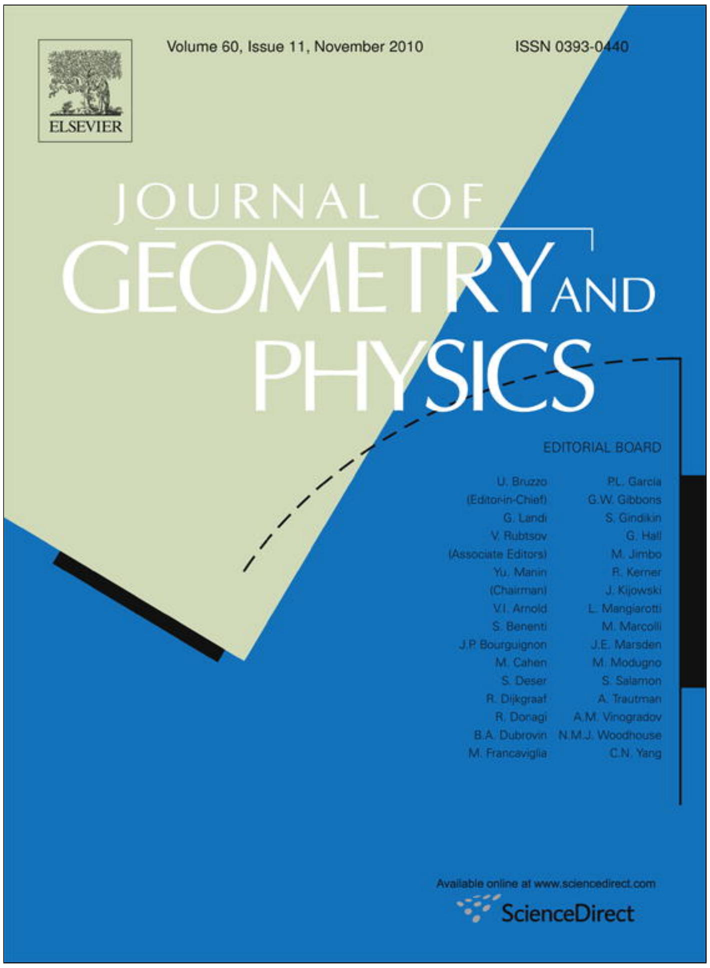
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Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor ☆

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ABSTRACT

In this paper, first we introduce the full expression for the Ricci tensor of a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ from the equation of Gauss. Next we prove that a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$. Finally it can be verified that there do not exist any Hopf Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms $Q_m(c)$ Kimura [1,2] (resp. Pérez and the author [3]) considered real hypersurfaces in $M_n(c)$ (resp. in $Q_m(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$ (resp. $S\phi_i = \phi_i S,\ i = 1, 2, 3$) where S and ϕ (resp. S and $\phi_i,\ i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of real hypersurfaces in $M_m(c)$ (resp. in $Q_m(c)$).

In [1,2], Kimura has classified that a Hopf hypersurface M in complex projective space $P_m(\mathbb{C})$ with commuting Ricci tensor is locally congruent of type (A), to a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), to a tube over a complex quadric Q_{m-1}, $\cot^2 2r = m - 2$, of type (C), to a tube over $P_1(\mathbb{C}) \times P_{(m-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{m-2}$ where m is odd, of type (D), to a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ with $m = 9$, of type (E), to a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ with $m = 15$.

The notion of Hopf hypersurfaces means that the structure vector ξ defined by $\xi = -JN$ satisfies $A\xi = \alpha\xi$, where J denotes a Kähler structure of $P_m(\mathbb{C})$, N and A a unit normal and the shape operator of M in $P_m(\mathbb{C})$ (see [4]).

On the other hand, for in a quaternionic projective space $\mathbb{Q}P^m$ Pérez and the author [3] have classified real hypersurfaces in QP^m with commuting Ricci tensor $S\phi_i = \phi_i S,\ i = 1, 2, 3$, where S (resp. ϕ_i) denotes that the Ricci tensor (resp. the structure tensor) of M in $\mathbb{Q}P^m$ is locally congruent of type A_1, A_2, that is, to a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k \in \{0, \dots, m-1\}$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_iN,\ i = 1, 2, 3$, where $J_i,\ i = 1, 2, 3$, denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$. Moreover, Pérez and the present author [5]

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have considered the notion of $\nabla_{\xi_i} R = 0,\ i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{Q}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbb{Q}P^k$.

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor is not so simple and will be quite different from the cases mentioned above.

So in this paper we consider a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $S\phi = \phi S$, where S and ϕ denote the Ricci tensor and the structure tensor of M in $G_2(\mathbb{C}^{m+2})$, respectively. The curvature tensor $R(X, Y)Z$ of M in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $\bar{R}(X, Y)Z$ of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for any vector fields X, Y and Z on M. Then by contraction and using the geometric structure $JJ_i = J_iJ,\ i = 1, 2, 3$, connecting the Kähler structure J and the quaternionic Kähler structure $J_i,\ i = 1, 2, 3$, we can derive the Ricci tensor S given by (see Section 3)

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_{4m-1}\}$ denotes a basis of the tangent space T_xM of M, $x \in M$, in $G_2(\mathbb{C}^{m+2})$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see [6,7]). So, for in $G_2(\mathbb{C}^{m+2})$ we have two natural geometrical conditions for real hypersurfaces: that $[\xi] = \text{Span}\,\{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of geometric conditions Berndt and the present author [6] have proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the structure vector field ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator A, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the structure vector field ξ are geodesics (see [7]). The flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow*.

On the other hand, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field ξ, which gives a characterization of real hypersurfaces of type (A) in Theorem A. Moreover, it was verified in [8] that $\mathcal{L}_\xi g = 0$ is equivalent to $\mathcal{L}_\xi A = 0$ for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.

When the Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, we say that M has a *commuting Ricci tensor*. In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ belongs to either the three-dimensional distribution $\mathfrak{D}^\perp$ or to the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. The case (A) in Theorem A is just the case where the one-dimensional distribution $[\xi]$ belongs to the distribution $\mathfrak{D}^\perp$. Of course they satisfy that the Reeb vector ξ is Killing, that is, the structure tensor ϕ commutes with the shape operator A. But it is not difficult to check that the Ricci tensor S of real hypersurfaces of type (B) mentioned in Theorem A cannot commute with the structure tensor ϕ. Moreover, in Section 5 we can check that any real hypersurface of type (A) in Theorem A has a commuting Ricci tensor.

In this paper we consider such a converse problem and want to give a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $S\phi = \phi S$ as follows:

Theorem. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, it is known that the Ricci tensor S of an Einstein hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by $S = ag$ for a constant a and a Riemannian metric g defined on M. Naturally the Ricci tensor S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. So by virtue of our theorem mentioned above it becomes a hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$. But by Proposition C in Section 5 it can be easily checked that any tubes of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ cannot be Einstein (see [9]). Then, as an application of our theorem in the direction of mathematical physics, we assert the following:

Corollary. *There do not exist any Hopf Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.*

In Section 2 we recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and in Section 3 we will show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. The formula for the Ricci tensor S and its covariant derivative ∇S will be shown explicitly in this section. In Sections 4 and 5 we will give a complete proof of the main theorem according to the geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$ or the geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^\perp$.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$; for details we refer the reader to [10,6,7,11]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is 8.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$ part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\operatorname{tr}(JJ_1) = 0$. This fact will be used in later sections.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulus 3. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local 1-forms q_1, q_2, q_3 such that

$$\bar\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar R$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ + \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y \\ &\quad - 2g(J_\nu X, Y)J_\nu Z\} + \sum_{\nu=1}^{3}\{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a submanifold in $G_2(\mathbb{C}^{m+2})$ with real codimension 1. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for the curvature tensor $\bar R$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned}R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + \sum_{\nu=1}^{3}\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y, Z)\phi_\nu\phi X - g(\phi_\nu\phi X, Z)\phi_\nu\phi Y\} - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ &\quad - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\}\,\xi_\nu + g(AY, Z)AX - g(AX, Z)AY\end{aligned}$$

and

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\,\xi_\nu,\end{aligned}$$

where R denotes the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward manner and will be used frequently in subsequent calculations (see [12,9,8,11]):

$$\begin{aligned}
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \qquad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi\xi_\nu = \phi_\nu\xi, \qquad \eta_\nu(\phi X) = \eta(\phi_\nu X),\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.
\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \qquad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \qquad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{2.4}$$

Summing these formulas, we find the following:

$$\begin{aligned}
\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\
&= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\
&= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.
\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

3. Proof of main theorem

In this section let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, that is, $S\phi = \phi S$.

Now let us contract Y and Z in the equation of Gauss in Section 2. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}
SX &= \sum_{i=1}^{4m-1} R(X,e_i)e_i\\
&= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{(\mathrm{Tr}\,\phi_\nu\phi)\phi_\nu\phi X - (\phi_\nu\phi)^2X\}\\
&\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(X)\phi_\nu\phi\xi_\nu\} - \sum_{\nu=1}^{3}\{(\mathrm{Tr}\,\phi_\nu\phi)\eta(X) - \eta(\phi_\nu\phi X)\}\xi_\nu + hAX - A^2X,
\end{aligned} \tag{3.1}$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, $\mathrm{Tr}\,JJ_\nu = 0$, $\nu = 1, 2, 3$, we calculate the following for any basis $\{e_1, \ldots, e_{4m-1}, N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$:

$$\begin{aligned}
0 &= \mathrm{Tr}\,JJ_\nu\\
&= \sum_{k=1}^{4m-1} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N)\\
&= \mathrm{Tr}\,\phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN)\\
&= \mathrm{Tr}\,\phi\phi_\nu - 2\eta_\nu(\xi)
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
(\phi_\nu\phi)^2X &= \phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu)\\
&= \phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X)\phi_\nu{}^2\xi\\
&= X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}.
\end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$\begin{aligned}
SX &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X\\
&= (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X.
\end{aligned} \tag{3.4}$$

Now let us take a covariant derivative of $S\phi = \phi S$. This gives that

$$(\nabla_Y S)\phi X + S(\nabla_Y \phi)X = (\nabla_Y \phi)SX + \phi(\nabla_Y S)X. \tag{3.5}$$

Then the first term of (3.5) becomes

$$\begin{aligned}(\nabla_Y S)\phi X = & -3g(\phi AY, \phi X)\xi - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\phi X) - q_{\nu+1}(Y)\eta_{\nu+2}(\phi X) + g(\phi_\nu AY, \phi X)\}\xi_\nu \\ & - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu A\phi X\} \\ & + \sum_{\nu=1}^{3}\Big[Y(\eta_\nu(\xi))\phi_\nu\phi^2 X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi^2 X + q_{\nu+2}(Y)\phi_{\nu+1}\phi^2 X \\ & + \eta_\nu(\phi^2 X)AY - g(AY, \phi^2 X)\xi_\nu\} - \eta_\nu(\xi)g(AY, \phi X)\phi_\nu\xi - g(\phi AY, \phi_\nu\phi X)\phi_\nu\xi \\ & + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}\phi X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}\phi X) - \eta_\nu(\phi X)\eta(AY) + \eta(\xi_\nu)g(AY, \phi X)\}\phi_\nu\xi \\ & - \eta(\phi_\nu\phi X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY - \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} - g(\phi AY, \phi X)\eta_\nu(\xi)\xi_\nu\Big] \\ & + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X.\end{aligned}$$

The second term of (3.5) becomes

$$\begin{aligned}S(\nabla_Y \phi)X = & \,\eta(X)\Big[(4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu AY)\phi_\nu\xi - \eta(AY)\eta_\nu(\xi)\xi_\nu\} \\ & + hA^2Y - A^3Y\Big] - g(AY, X)\Big[(4m+7)\xi - 3\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi\Big].\end{aligned}$$

The first term of the right side in (3.5) becomes

$$(\nabla_Y \phi)SX = \eta(SX)AY - g(AY, SX)\xi,$$

and the second term of the right side in (3.5) is given by

$$\begin{aligned}\phi(\nabla_Y S)X = & -3\eta(X)\phi^2 AY - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, \phi X)\}\phi\xi_\nu \\ & - 3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\} \\ & + \sum_{\nu=1}^{3}\Big[Y(\eta_\nu(\xi))\phi\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi\phi_{\nu+2}\phi X + q_{\nu+2}(Y)\phi\phi_{\nu+1}\phi X \\ & + \eta_\nu(\phi X)\phi AY - g(AY, \phi X)\phi\xi_\nu\} + \eta_\nu(\xi)\{\eta(X)\phi\phi_\nu AY - g(AY, X)\phi\phi_\nu\xi\} \\ & - g(\phi AY, \phi_\nu X)\phi\phi_\nu\xi + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) - \eta_\nu(X)\eta(AY) \\ & + \eta(\xi_\nu)g(AY, X)\}\phi\phi_\nu\xi - \eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi\phi_{\nu+2}\xi + \phi\phi_\nu\phi AY \\ & - \eta(AY)\phi\xi_\nu + \eta(\xi_\nu)\phi AY\} - g(\phi AY, X)\eta_\nu(\xi)\phi\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\phi\xi_\nu - \eta(X)\eta_\nu(\xi)\phi\nabla_Y\xi_\nu\Big] \\ & + (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X.\end{aligned}$$

Putting $X = \xi$ into (3.5) and using that the structure vector ξ is principal, that is, $A\xi = \alpha\xi$, then we have

$$\begin{aligned}S(\nabla_Y \phi)\xi = & \Big[(4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu\phi AY)\phi_\nu\xi - \alpha\eta(Y)\eta_\nu(\xi)\xi_\nu\} + hA^2Y - A^3Y\Big] \\ & - \alpha\eta(Y)\Big[4(m+1)\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + (\alpha h - \alpha^2)\xi\Big].\end{aligned}$$

Moreover, the right side of (3.5) becomes

$$\begin{aligned}(\nabla_Y\phi)S\xi+\phi(\nabla_YS)\xi &= \eta(S\xi)AY-g(AY,S\xi)\xi+\phi(\nabla_YS)\xi\\ &=\left[\{4(m+1)+h\alpha-\alpha^2\}-4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2\right]AY-3\eta(X)\phi^2AY\\ &\quad-\left\{\{4(m+1)\alpha+h\alpha^2-\alpha^3\}\eta(Y)-4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(AY)\right\}\xi\\ &\quad-3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi)-q_{\nu+1}(Y)\eta_{\nu+2}(\xi)+\eta_\nu(\phi AY)\}\phi\xi_\nu\\ &\quad-3\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(Y)\phi\xi_{\nu+1}-q_{\nu+1}(Y)\phi\xi_{\nu+2}+\phi\phi_\nu AY\}\\ &\quad+\sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)\{\phi\phi_\nu AY-\alpha\eta(Y)\phi^2\xi_\nu\}-g(\phi AY,\phi\xi_\nu)\phi^2\xi_\nu\\ &\quad-Y(\eta_\nu(\xi))\phi\xi_\nu-\eta_\nu(\xi)\phi\nabla_Y\xi_\nu\Big]+h\phi(\nabla_YA)\xi-\phi(\nabla_YA^2)\xi.\end{aligned}$$

From this, putting $Y=\xi$ into $L=R$, then it follows that

$$0=\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi)-q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi\xi_\nu+\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi\xi_{\nu+1}-q_{\nu+1}(\xi)\phi\xi_{\nu+2}+\alpha\phi^2\xi_\nu\}.$$

Now in order to show that ξ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$, let us assume that $\xi=X_1+X_2$ for some $X_1\in\mathfrak{D}$ and $X_2\in\mathfrak{D}^\perp$. Then it follows that

$$\begin{aligned}0&=\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi)-q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}(\phi_\nu X_1+\phi_\nu X_2)\\ &\quad+\sum_{\nu=1}^{3}\eta_\nu(\xi)\left\{q_{\nu+2}(\xi)(\phi_{\nu+1}X_1+\phi_{\nu+1}X_2)-q_{\nu+1}(\xi)(\phi_{\nu+2}X_1+\phi_{\nu+2}X_2)-\alpha\xi_\nu+\alpha\eta(\xi_\nu)(X_1+X_2)\right\}.\end{aligned}\tag{3.6}$$

Then by comparing the $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of (3.6), we have respectively

$$\begin{aligned}0&=\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi)-q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi_\nu X_1+\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)^2X_1\\ &\quad+\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi_{\nu+1}X_1-q_{\nu+1}(\xi)\phi_{\nu+2}X_1\},\end{aligned}\tag{3.7}$$

$$\begin{aligned}0&=\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi)-q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi_\nu X_2\\ &\quad+\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi_{\nu+1}X_2-q_{\nu+1}(\xi)\phi_{\nu+2}X_2-\alpha\xi_\nu+\alpha\eta(\xi_\nu)X_2\}.\end{aligned}\tag{3.8}$$

Taking an inner product (3.7) with X_1, we have

$$\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)^2=0.\tag{3.9}$$

Then $\alpha=0$ or $\eta_\nu(\xi)=0$ for $\nu=1,2,3$. So for a non-vanishing geodesic Reeb flow we have $\eta_\nu(\xi)=0$, $\nu=1,2,3$. This means that $\xi\in\mathfrak{D}$, which gives a contradiction to our assumption $\xi=X_1+X_2$. Including this, we are able to assert the following:

Lemma 3.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor. Then the Reeb vector ξ belongs either to the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.*

Proof. When the geodesic Reeb flow is non-vanishing, that is $\alpha \neq 0$, (3.9) gives $\xi \in \mathfrak{D}$. When the geodesic Reeb flow is vanishing, we differentiate $A\xi = 0$. Then by Berndt and Suh [7] we know that

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y) = 0.$$

From this, on replacing Y by ϕY, it follows that

$$\sum_{\nu=1}^{3} \eta_\nu^2(\xi)\eta(Y) = 0.$$

So if there are some $Y \in \mathfrak{D}$ such that $\eta(Y) \neq 0$, then $\eta_\nu(\xi) = 0$ for $\nu = 1, 2, 3$. This means that $\xi \in \mathfrak{D}$. If $\eta(Y) = 0$ for any $Y \in \mathfrak{D}$, then we know that $\xi \in \mathfrak{D}^\perp$. □

4. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$

Let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, that is, $S\phi = \phi S$ and $\xi \in \mathfrak{D}$. From this, differentiating, we have

$$(\nabla_Y S)\phi X + S(\nabla_Y \phi)X = (\nabla_Y \phi)SX + \phi(\nabla_Y S)X. \tag{4.1}$$

In this section let us show that the distribution $\mathfrak{D}$ of M in $G_2(\mathbb{C}^{m+2})$ satisfies $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$ for the case $\xi \in \mathfrak{D}$.

Now using $\xi \in \mathfrak{D}$ in (4.1), the first term becomes

$$\begin{aligned}
(\nabla_Y S)\phi X &= -3g(\phi AY, \phi X)\xi - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\phi X) - q_{\nu+1}(Y)\eta_{\nu+2}(\phi X) + g(\phi_\nu AY, \phi X)\}\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu A\phi X\} \\
&\quad + \sum_{\nu=1}^{3}\Big[\{\alpha\eta(Y)\eta(\phi_\nu X) - \eta(X)\eta_\nu(\phi AY) - g(AY, \phi_\nu X)\}\phi_\nu\xi \\
&\quad - \{q_{\nu+1}(Y)\eta_{\nu+2}(X) - q_{\nu+2}(Y)\eta_{\nu+1}(X) + \alpha\eta_\nu(\phi X)\eta(Y)\}\phi_\nu\xi \\
&\quad + \eta_\nu(X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY - \alpha\eta(Y)\xi_\nu\}\Big] \\
&\quad + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X.
\end{aligned}$$

The second term of (4.1) becomes

$$\begin{aligned}
S(\nabla_Y \phi)X &= \eta(X)S(AY) - g(AY, X)S\xi \\
&= \eta(X)\left[(4m+7)AY - 3\alpha\eta(Y)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu - \sum_{\nu=1}^{3}\eta(\phi_\nu AY)\phi_\nu\xi + hA^2Y - A^3Y\right] \\
&\quad - g(AY, X)\left\{4(m+1)\xi + (h\alpha - \alpha^2)\xi\right\}.
\end{aligned}$$

The first term of the right side in (4.1) becomes

$$\begin{aligned}
(\nabla_Y \phi)SX &= \eta(SX)AY - g(AY, SX)\xi \\
&= 4(m+1)\eta(X)AY + (h\alpha - \alpha^2)\eta(X)AY - g(AY, SX)\xi,
\end{aligned}$$

and the second term of the right side in (4.1) is given by

$$\begin{aligned}
\phi(\nabla_Y S)X &= -3\eta(X)\phi^2 AY - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, X)\}\phi\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\} + \sum_{\nu=1}^{3} g(\phi AY, \phi_\nu X)\xi_\nu \\
&\quad - \sum_{\nu=1}^{3}\{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) - \alpha\eta_\nu(X)\eta(Y)\}\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\eta(\phi_\nu X)\left[\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2}\} - \phi\phi_\nu\phi AY + \alpha\eta(Y)\phi\xi_\nu\right] \\
&\quad + (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X.
\end{aligned}$$

Substituting these formulas into (4.1) and putting $X = \xi_\mu$ into the equation obtained, and next using that the structure vector ξ is in $\mathfrak{D}$ and (2.1), we have

$$\begin{aligned}
&-3g(AY,\xi_\mu)\xi + (Yh)A\phi\xi_\mu + h(\nabla_Y A)\phi\xi_\mu - (\nabla_Y A^2)\phi\xi_\mu + \{4(m+1)+(h-\alpha)\alpha\}g(AY,\xi_\mu)\xi \\
&= -g(AY,S\xi_\mu)\xi - 4\sum_{\nu=1}^{3}\left\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi_\mu) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi_\mu) + g(\phi_\nu AY,\xi_\mu)\right\}\phi\xi_\mu \\
&\quad - 4\left\{q_{\mu+2}(Y)\phi\xi_{\mu+1} - q_{\mu+1}(Y)\phi\xi_{\mu+2} + \phi\phi_\mu AY\right\} + 4\sum_{\nu=1}^{3} g(\phi AY,\phi_\nu\xi_\mu)\xi_\nu \\
&\quad + \alpha\eta(Y)\xi_\mu + (Yh)\phi A\xi_\mu + h\phi(\nabla_Y A)\xi_\mu - \phi(\nabla_Y A^2)\xi_\mu.
\end{aligned} \tag{4.2}$$

Putting $X = \xi_\mu$ into (3.4) and using $\xi \in \mathfrak{D}$, we have

$$S\xi_\mu = (4m+7)\xi_\mu - 3\xi_\mu + hA\xi_\mu - A^2\xi_\mu.$$

So the first term of the right side of (4.2) becomes

$$-g(AY,S\xi_\mu)\xi = -4(m+1)g(AY,\xi_\mu)\xi - hg(A\xi_\mu,AY)\xi + g(A^2\xi_\mu,AY)\xi.$$

Then substituting this into (4.2), we have

$$\begin{aligned}
&4\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi_\mu) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi_\mu) + g(\phi_\nu AY,\xi_\mu)\}\phi\xi_\nu \\
&\quad + 4\{q_{\mu+2}(Y)\phi\xi_{\mu+1} - q_{\mu+1}(Y)\phi\xi_{\mu+2} + \phi\phi_\mu AY\} \\
&\quad + \{(8m+5)+(h-\alpha)\alpha\}g(AY,\xi_\mu)\xi + hg(A\xi_\mu,AY)\xi - g(A^2\xi_\mu,AY)\xi \\
&\quad - 4\sum_{\nu=1}^{3} g(\phi AY,\phi_\nu\xi_\mu)\xi_\nu - \alpha\eta(Y)\xi_\mu + (Yh)(A\phi-\phi A)\xi_\mu \\
&\quad + h\{(\nabla_Y A)\phi - \phi(\nabla_Y A)\}\xi_\mu - \{(\nabla_Y A^2)\phi - \phi(\nabla_Y A^2)\}\xi_\mu \\
&= 0.
\end{aligned} \tag{4.3}$$

From this, taking the inner product with ξ, we have

$$\begin{aligned}
&\{(8m+5)+(h-\alpha)\alpha\}g(AY,\xi_\mu) + hg(A\xi_\mu,AY) - g(A^2\xi_\mu,AY) \\
&\quad + hg((\nabla_Y A)\phi\xi_\mu,\xi) - g((\nabla_Y A^2)\phi\xi_\mu,\xi) = 0.
\end{aligned} \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}
&g((\nabla_Y A)\phi\xi_\mu,\xi) = \alpha g(\phi AY,\phi\xi_\mu) - g(A\phi AY,\phi\xi_\mu), \\
&g((\nabla_Y A^2)\phi\xi_\mu,\xi) = \alpha^2 g(\phi AY,\phi\xi_\mu) - g(A^2\phi AY,\phi\xi_\mu).
\end{aligned}$$

From this, together with (4.4), we have

$$\{(8m+5)+2(h-\alpha)\alpha\}A\xi_\mu + hA^2\xi_\mu - A^3\xi_\mu + hA\phi A\phi\xi_\mu - A\phi A^2\phi\xi_\mu = 0. \tag{4.5}$$

Now putting $X = \xi$ in (4.1) and using $\xi \in \mathfrak{D}$, then we have

$$\begin{aligned}
&\left[(4m+7)AY - 3\alpha\eta(Y)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi AY)\phi_\nu\xi + hA^2Y - A^3Y\right] \\
&\quad - \alpha\eta(Y)\{4(m+1)\xi + \alpha(h-\alpha)\xi\} \\
&= \left[\{4(m+1)+(h-\alpha)\alpha\}AY - \{4(m+1)\alpha + (h-\alpha)\alpha^2\}\eta(Y)\xi\right] + (3-\alpha h+\alpha^2)AY \\
&\quad - (3\alpha - \alpha^2 h + \alpha^3)\eta(Y)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\phi\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu - h\phi A\phi AY + \phi A^2\phi AY.
\end{aligned} \tag{4.6}$$

From this, putting $Y = \xi_\mu$ and also using $\xi \in \mathfrak{D}$, we have

$$2(4m+7)A\xi_\mu - 2\sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\xi_\nu - 4\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\phi_\nu\xi + hA^2\xi_\mu - A^3\xi_\mu - h\phi A\phi A\xi_\mu + \phi A^2\phi A\xi_\mu = 0. \tag{4.7}$$

On the other hand, we have assumed that M has a commuting Ricci tensor, that is $S\phi = \phi S$. From this, together with $\xi \in \mathfrak{D}$, we have

$$hA\phi X - A^2\phi X = h\phi AX - \phi A^2 X - 4\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu. \tag{4.8}$$

Then by putting $X = A\xi_\mu$ into (4.8) we have

$$hA\phi A\xi_\mu - A^2\phi A\xi_\mu = h\phi A^2\xi_\mu - \phi A^3\xi_\mu - 4\sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\phi\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\xi_\nu.$$

From this, on applying ϕ to the left side we know that

$$h\phi A\phi A\xi_\mu - \phi A^2\phi A\xi_\mu = -hA^2\xi_\mu + A^3\xi_\mu + 4\sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\phi\xi_\nu. \tag{4.9}$$

Also, by putting $X = \xi_\mu$ into (4.8) we have

$$hA\phi\xi_\mu - A^2\phi\xi_\mu = h\phi A\xi_\mu - \phi A^2\xi_\mu - 4\phi\xi_\mu - 4\xi_\mu. \tag{4.10}$$

From this, on applying $A\phi$ to the left side and using that ξ is principal we have

$$hA\phi A\phi\xi_\mu - A\phi A^2\phi\xi_\mu = -hA^2\xi_\mu + A^3\xi_\mu + 4A\xi_\mu - 4A\phi\xi_\mu. \tag{4.11}$$

Then substituting (4.11) into (4.5), we have

$$A\phi\xi_\mu = \beta A\xi_\mu, \tag{4.12}$$

where we have put $\beta = \frac{1}{4}\{(8m+9) + 2(h-\alpha)\alpha\}$.

On the other hand, substituting (4.9) into (4.7), we have

$$hA^2\xi_\mu - A^3\xi_\mu = 3\sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\phi_\nu\xi - (4m+7)A\xi_\mu. \tag{4.13}$$

Now substituting (4.12) into (4.10), we have

$$\beta(hA\xi_\mu - A^2\xi_\mu) = h\phi A\xi_\mu - \phi A^2\xi_\mu - 4\phi\xi_\mu - 4\xi_\mu. \tag{4.14}$$

Then by applying the structure tensor ϕ to the left side of (4.14), we have

$$\beta(h\phi A\xi_\mu - \phi A^2\xi_\mu) = -(hA\xi_\mu - A^2\xi_\mu) + 4\xi_\mu - 4\phi\xi_\mu.$$

From this, together with (4.14), on applying the function β to both sides, we have

$$\begin{aligned}\beta^2(hA\xi_\mu - A^2\xi_\mu) &= \beta(h\phi A\xi_\mu - \phi A^2\xi_\mu) - 4\beta\phi\xi_\mu - 4\beta\xi_\mu \\ &= -(hA\xi_\mu - A^2\xi_\mu) + 4\xi_\mu - 4\phi\xi_\mu - 4\beta\phi\xi_\mu - 4\beta\xi_\mu.\end{aligned}$$

Then we put this as follows:

$$hA\xi_\mu - A^2\xi_\mu = \lambda\xi_\mu + \mu\phi\xi_\mu, \tag{4.15}$$

where λ (resp. μ) denotes $\frac{-4(\beta-1)}{\beta^2+1}$ $\left(\text{resp. } \mu = \frac{-4(\beta+1)}{\beta^2+1}\right)$. From this, together with (4.12), we have

$$hA^2\xi_\mu - A^3\xi_\mu = \lambda A\xi_\mu + \mu A\phi\xi_\mu = (\lambda + \mu\beta)A\xi_\mu. \tag{4.16}$$

On the other hand, by (4.13) the left side of (4.16) becomes

$$(\lambda + \mu\beta + 4m + 7)A\xi_\mu = 3\sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\phi_\nu\xi. \tag{4.17}$$

Then (4.17) gives the following for $\xi \in \mathfrak{D}$:

$$\begin{aligned}(\lambda + \mu\beta + 4m + 7)g(A\xi_\mu, \phi_\delta\xi) &= 4\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)g(\phi_\nu\xi, \phi_\delta\xi) \\ &= 4\eta_\delta(\phi A\xi_\mu) = -4g(A\xi_\mu, \phi\xi_\delta),\end{aligned}$$

which means that $g(A\phi_\delta\xi, \xi_\mu) = 0$, because $\lambda + \mu\beta + 4m + 11 > 0$. Then (4.17), together with $\lambda + \mu\beta + 4m + 7 > o$, gives $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. Then by Theorem A we know that M is locally congruent of type (B), that is, to a tube over a totally real and totally geodesic $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$. Concerned with such a tube, we are able to recall a proposition given by Berndt and the present author [6] as follows:

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures:*

$$\alpha = -2\tan(2r), \qquad \beta = 2\cot(2r), \qquad \gamma = 0, \qquad \lambda = \cot(r), \qquad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \qquad m(\beta) = 3 = m(\gamma), \qquad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \qquad T_\beta = \mathfrak{J}J\xi, \qquad T_\gamma = \mathfrak{J}\xi, \qquad T_\lambda, \qquad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp, \qquad \mathfrak{J}T_\lambda = T_\lambda, \qquad \mathfrak{J}T_\mu = T_\mu, \qquad JT_\lambda = T_\mu.$$

Now it remains to check whether the Ricci tensor for real hypersurfaces of type (B) in Theorem A is commuting or not. So let us suppose that the Ricci tensor S of type (B) is commuting, that is $S\phi = \phi S$. Then this gives (4.8). So if we consider an eigenvector $X \in T_\lambda$, by Proposition B we know that $\phi X \in T_\mu$. Then applying such a situation to (4.8), we have

$$(\lambda - \mu)(h - \lambda - \mu) = 0,$$

where the function h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.

Since $\lambda - \mu \neq 0$, we know that

$$h = \lambda + \mu = \cot r - \tan r = 2\cot 2r.$$

By Proposition B, we also know that

$$h = -2\tan 2r + 6\cot 2r + (4n - 4)(\cot r - \tan r).$$

Then by comparing two formulas for the function h we know that

$$\cot^2 2r = \frac{1}{2(2n-1)}. \tag{4.18}$$

On the other hand, by putting $X = \xi_\mu$, $\mu = 1, 2, 3$, into (4.8) we have

$$\phi_\mu\xi + hA\phi\xi_\mu - A^2\phi\xi_\mu = -3\phi\xi_\mu + h\phi A\xi_\mu - \phi A^2\xi_\mu.$$

In this formula, if we consider an eigenvector $\xi_\mu \in T_\beta$, then $\phi\xi_\mu \in T_\gamma$, $A\phi\xi_\mu = 0$, $\phi A\xi_\mu = 2\cot 2r\phi\xi_\mu$, and $\phi A^2\xi_\mu = (2\cot 2r)^2\phi\xi_\mu$. So it follows that

$$(4\cot^2 2r - 2h\cot 2r + 4)\phi_\mu\xi = 0,$$

where the trace h is given by $h = -2\tan 2r + 2(4n-1)\cot 2r$. Then substituting this, we have another formula:

$$\cot^2 2r = \frac{1}{2n-1}. \tag{4.19}$$

Then from (4.18) and (4.19) we have a contradiction. So we have shown that there do not exist any real hypersurfaces of type (B) satisfying $S\phi = \phi S$. Accordingly, we have proved that no real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor can exist for the case $\xi \in \mathfrak{D}$.

5. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^\perp$

Now let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and $\xi \in \mathfrak{D}^\perp$. Now differentiating $S\phi = \phi S$ gives

$$(\nabla_Y S)\phi X + S(\nabla_Y\phi)X = (\nabla_Y\phi)SX + \phi(\nabla_Y S)X.$$

In this section by Lemma 3.1 we only discuss the geodesic Reeb flow ξ belonging to the distribution $\mathfrak{D}^\perp$. Since we have assumed that $\xi \in \mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, there exists a Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1N$, that is, $\xi = \xi_1$. Then it follows that

$$\phi\xi_2 = \phi_2\xi = \phi_2\xi_1 = -\xi_3, \qquad \phi\xi_3 = \phi_3\xi_1 = -\xi_2. \tag{5.1}$$

From this, together with the expression for (3.4) and $\xi \in \mathfrak{D}^{\perp}$, we have

$$\begin{aligned}
&(4m+1)g(AX,Y)\xi - 3[\{q_3(Y)\eta_3(X)+q_2(Y)\eta_2(X)\}\xi_1 - q_1(Y)\eta_2(X)\xi_2 - q_1(Y)\eta_3(X)\xi_3] + 2\eta(X)\eta_2(AY)\xi_2\\
&\quad + 2\eta(X)\eta_3(AY)\xi_3 + \sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\phi AY + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X + \eta(X)\{hA^2Y - A^3Y\}\\
&= \{-g(AY,SX) - \eta_3(X)\eta_3(AY) - \eta_2(X)\eta_2(AY)\}\xi + 4[g(\phi_2 AY,X)\xi_3 - g(\phi_3 AY,X)\xi_2] - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\phi_\nu AY\\
&\quad + 4\sum_{\nu=1}^{3} g(\phi AY,\phi_\nu X)\xi_\nu + \eta_3(X)\phi_2 AY - \eta_2(X)\phi_3 AY + (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X. \qquad (5.2)
\end{aligned}$$

Now putting $X = \xi$ in (5.2), we have

$$\begin{aligned}
&(4m+1)g(A\xi,Y)\xi + 2\eta_2(AY)\xi_2 + 2\eta_3(AY)\xi_3 + \phi_1\phi AY + hA^2Y - A^3Y\\
&\quad = -g(AY,S\xi)\xi + 4\{g(\phi_2 AY,\xi)\xi_3 - g(\phi_3 AY,\xi)\xi_2\} - 3\phi\phi_1 AY + 4g(\phi AY,\phi_2\xi)\xi_2\\
&\qquad + 4g(\phi AY,\phi_3\xi)\xi_3 + h\phi(\nabla_Y A)\xi - \phi(\nabla_Y A^2)\xi.
\end{aligned}$$

From this, if we use the following formulas:

$$S\xi = 4(m+1)\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi = (4m + h\alpha - \alpha^2)\xi$$

and

$$g(AY,S\xi) = \alpha(4m + h\alpha - \alpha^2)\eta(Y),$$

then it follows that

$$\phi_1\phi AY + hA^2Y - A^3Y = 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - h\phi A\phi AY + \phi A^2\phi AY - 3\phi\phi_1 AY. \qquad (5.3)$$

On the other hand, by the equation of Codazzi in [6] (see page 6), we have

$$\begin{aligned}
A\phi AY &= \phi Y + \sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y - 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} + \alpha(A\phi + \phi A)Y\\
&= \phi Y + \phi_1 Y + \eta_2(Y)\phi\xi_2 + \eta_3(Y)\phi\xi_3 + \eta_2(\phi Y)\xi_2 + \eta_3(\phi Y)\xi_3 + \alpha(A\phi + \phi A)Y. \qquad (5.4)
\end{aligned}$$

So for any $Y \in \mathfrak{D}$, (5.4) gives that $A\phi AY = \phi Y + \phi_1 Y + \alpha(A\phi + \phi A)Y$. This implies

$$\begin{aligned}
\phi A^2\phi AY &= \phi A(A\phi AY) = \phi A(\phi Y + \phi_1 Y)\\
&= \phi A\phi Y + \phi A\phi_1 Y + \alpha\phi A(A\phi + \phi A)Y.
\end{aligned}$$

From this, together with (5.3), it follows that

$$\begin{aligned}
\phi_1\phi AY + hA^2Y - A^3Y &= 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - h(-Y + \phi\phi_1 Y) - h\alpha\phi(A\phi + \phi A)Y + \phi A\phi Y\\
&\quad + \phi A\phi_1 Y - 3\phi\phi_1 AY + \alpha\phi A(A\phi + \phi A)Y. \qquad (5.5)
\end{aligned}$$

On the other hand, we calculate the following:

$$S\phi Y = (4m+7)\phi Y - 3\eta_2(\phi Y)\xi_2 - 3\eta_3(\phi Y)\xi_3 + \phi_1\phi^2 Y - \eta(\phi_2\phi Y)\phi_2\xi - \eta(\phi_3\phi Y)\phi_3\xi + hA\phi Y - A^2\phi Y,$$

$$\phi SY = (4m+7)\phi Y - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi\xi_\nu + \phi\phi_1\phi Y - \eta(\phi_2 Y)\phi_2\xi - \eta(\phi_3 Y)\phi_3\xi + h\phi AY - \phi A^2 Y.$$

So for any $X \in \mathfrak{D}$ the condition $S\phi = \phi S$ implies that

$$-\phi_1 Y + hA\phi Y - A^2\phi Y = \phi\phi_1\phi Y + h\phi AY - \phi A^2 Y.$$

Then by replacing Y by ϕY for $Y \in \mathfrak{D}$ we have

$$hA^2Y - A^3Y = -A\phi_1\phi Y + A\phi\phi_1 Y - hA\phi A\phi Y + A\phi A^2\phi Y. \qquad (5.6)$$

Now by using (5.4) for $Y \in \mathfrak{D}$, the terms in the right side become respectively

$$\begin{aligned}
A\phi A\phi Y &= \phi^2 Y + \phi_1\phi Y + \alpha(A\phi + \phi A)Y\\
&= -Y + \phi 1\phi Y + \alpha(A\phi + \phi A)\phi Y
\end{aligned}$$

and

$$A\phi A^2\phi Y = \phi A\phi Y + \phi_1 A\phi Y + \eta_2(A\phi Y)\phi\xi_2 + \eta_3(A\phi Y)\phi\xi_3 + \eta_2(\phi A\phi Y)\xi_2 + \eta_3(\phi A\phi Y)\xi_3 + \alpha(A\phi + \phi A)\phi Y.$$

From these, together with (5.5) and (5.6), we have

$$\begin{aligned}
&\phi_1\phi AY - A\phi_1\phi Y + A\phi\phi_1 Y + hY - h\phi_1\phi Y - \alpha h(A\phi + \phi A)\phi Y + \phi A\phi Y + \alpha(A\phi + \phi A)A\phi Y\\
&\quad + \{\phi_1 A\phi Y + \eta_2(A\phi Y)\phi\xi_2 + \eta_3(A\phi Y)\phi\xi_3 + \eta_2(\phi A\phi Y)\xi_2 + \eta_3(\phi A\phi Y)\xi_3\}\\
&= 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - h\phi\phi_1 Y + \phi A\phi_1 Y - 3\phi\phi_1 AY + hY + \phi A\phi Y\\
&\quad - h\alpha\phi(A\phi + \phi A)Y + \alpha\phi A(A\phi + \phi A)Y.
\end{aligned}$$

Then this can be rearranged as follows:

$$\begin{aligned}
&\phi_1\phi AY - A\phi_1\phi Y + A\phi\phi_1 Y - h\phi_1\phi Y + \alpha\phi_1\phi Y\\
&\quad + \{\phi_1 A\phi Y + \eta_2(A\phi Y)\phi\xi_2 + \eta_3(A\phi Y)\phi\xi_3 + \eta_2(\phi A\phi Y)\xi_2 + \eta_3(\phi A\phi Y)\xi_3\}\\
&= 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - h\phi\phi_1 Y + \phi A\phi_1 Y - 3\phi\phi_1 AY + \alpha\phi\phi_1 Y,
\end{aligned} \tag{5.7}$$

where we have used the following formulas obtained from (5.4):

$$\alpha A\phi A\phi Y = -\alpha Y + \alpha\phi_1\phi Y + \alpha^2(A\phi + \phi A)\phi Y$$

and

$$\alpha\phi A\phi AY = -\alpha Y + \alpha\phi\phi_1 Y + \alpha^2\phi(A\phi + \phi A)Y.$$

Now let us take the inner product (5.7) with ξ_2. Then for any $Y \in \mathfrak{D}$ we have

$$\begin{aligned}
&g(\phi_1\phi AY, \xi_2) - g(\phi_1\phi Y, A\xi_2) + g(\phi\phi_1 Y, A\xi_2) - (h-\alpha)g(\phi_1\phi Y, \xi_2) - g(A\phi X, \phi_1\xi_2) + \eta_3(A\phi X) + \eta_2(\phi A\phi Y)\\
&= 6\eta_2(AY) + g(\phi A\phi_1 Y, \xi_2) - 3g(\phi\phi_1 AY, \xi_2) - (h-\alpha)g(\phi\phi_1 Y, \xi_2).
\end{aligned} \tag{5.8}$$

Then by a direct calculation in (5.8) for any $Y \in \mathfrak{D}$, we have

$$\eta_3(A\phi Y) = 2\eta_2(AY) + \eta_3(A\phi_1 Y). \tag{5.9}$$

Similarly, if we take the inner product (5.7) with ξ_3, then it follows that

$$-\eta_2(A\phi Y) = 2\eta_3(AY) - \eta_2(A\phi_1 Y) \tag{5.10}$$

for any vector field $Y \in \mathfrak{D}$. Then in this section we know that the distribution $\mathfrak{D}$ can be decomposed into two distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ defined in such a way that

$$\mathfrak{D}_1 = \{Y \in \mathfrak{D} | \phi Y = \phi_1 Y\}$$

and

$$\mathfrak{D}_2 = \{Y \in \mathfrak{D} | \phi Y = -\phi_1 Y\}.$$

So first let us consider the distribution $\mathfrak{D}_1$. The formulas (5.9) and (5.10) imply that $\eta_\nu(AY) = 0$ for any $Y \in \mathfrak{D}_1$ and $\nu = 1, 2, 3$. Then we get our assertions on the distribution $\mathfrak{D}_1$.

Next we consider the distribution $\mathfrak{D}_2$. Then by (5.9) and (5.10) on such a distribution $\mathfrak{D}_2$ we have

$$\eta_2(A\phi Y) = -\eta_3(AY) \quad \text{and} \quad \eta_3(A\phi Y) = \eta_2(AY). \tag{5.11}$$

Substituting these formulas into (5.7), we have for any $Y \in \mathfrak{D}_2$,

$$\phi_1\phi AY + \phi_1 A\phi Y = 4\eta_2(AY)\xi_2 + 4\eta_3(AY)\xi_3 + \phi A\phi_1 Y - 3\phi\phi_1 AY. \tag{5.12}$$

From (5.4) and using $\phi Y = -\phi_1 Y,\ Y \in \mathfrak{D}_2$, we have

$$A\phi AY = 0 \quad \text{and} \quad A\phi A\phi Y = 0.$$

So from this, together with (5.3) and (5.6), it follows that

$$\begin{aligned}
4\phi_1\phi AY + hA^2 Y - A^3 Y &= 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3\\
&= 4\phi_1\phi AY + A\phi A^2\phi Y\\
&= 4\phi_1\phi AY + \phi A\phi X + \phi_1 A\phi Y + \eta_2(A\phi Y)\phi\xi_2\\
&\quad + \eta_3(A\phi Y)\phi\xi_3 + \eta_2(\phi A\phi Y)\xi_2 + \eta_3(\phi A\phi Y)\xi_3
\end{aligned} \tag{5.13}$$

where in the first equality we have used $A\phi A\phi Y = 0$ and the fact that $\phi_1\phi AY = \phi\phi_1 AY + \eta(AY)\xi_1 = \phi\phi_1 AY$ for any $Y \in \mathfrak{D}_2$.

Now let us consider eigenvectors $Y, \phi Y \in \mathfrak{D}_2$ such that $\phi Y = -\phi_1 Y$. Then we can put

$$AY = \lambda Y + \sum_{\nu=1}^{3} \eta_\nu(AY)\xi_\nu$$

and

$$A\phi Y = \bar{\lambda}\phi Y + \sum_{\nu=1}^{3} \eta_\nu(A\phi Y)\xi_\nu.$$

Then this also implies that

$$\phi AY = \lambda\phi Y + \sum_{\nu=1}^{3} \eta_\nu(AY)\phi\xi_\nu.$$

From these formulas and (5.13) it follows that

$$4\left\{\lambda Y + \sum_{\nu=1}^{3} \eta_\nu(AY)\phi_1\phi\xi_\nu\right\} + \bar{\lambda}\left\{Y + \sum_{\nu=1}^{3} \eta_\nu(A\phi Y)\phi_1\xi_\nu\right\}$$
$$= 4\eta_2(AY)\xi_2 + 4\eta_3(AY)\xi_3 + \bar{\lambda}\left\{Y - \sum_{\nu=1}^{3} \eta_\nu(A\phi Y)\phi\xi_\nu\right\}. \tag{5.14}$$

Then we have $\lambda = 0$ and similarly, $\bar{\lambda} = 0$. So it follows that

$$AY = \sum_{\nu=1}^{3} \eta_\nu(AY)\xi_\nu = g(A\xi_2, Y)\xi_2 + g(A\xi_3, Y)\xi_3.$$

Then for $\phi Y \in \mathfrak{D}_2$ we know that

$$A\phi Y = g(A\xi_2, \phi Y)\xi_2 + g(A\xi_3, \phi Y)\xi_3. \tag{5.15}$$

From this, applying ϕ and ϕ_1 respectively, we have

$$\phi A\phi Y = -g(A\xi_2, \phi Y)\xi_3 + g(A\xi_2, \phi Y)\xi_2$$

and

$$\phi_1 A\phi Y = g(A\xi_2, \phi Y)\xi_3 - g(A\xi_3, \phi Y)\xi_2.$$

From these formulas, together with (5.13), we have

$$\phi_1\phi AY = \eta_2(AY)\xi_2 + \eta_3(AY)\xi_3.$$

Then by applying ϕ_1 we have

$$\phi AY = \eta_3(AY)\xi_2 - \eta_2(AY)\xi_3. \tag{5.16}$$

By comparing (5.15) and (5.16), and using (5.11), we know that

$$A\phi = -\phi A \tag{5.17}$$

on the distribution $\mathfrak{D}_2$. From this and (5.4) it follows that for any $Y \in \mathfrak{D}_2$,

$$0 = \phi Y + \phi_1 Y = A\phi AY = -A^2\phi Y.$$

So $\phi Y \in \mathfrak{D}_2$ gives $A^2 Y = 0$. Then from this and (5.5), and using $\phi Y = -\phi_1 Y$ we have

$$6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - 4\phi_1\phi AY = 0,$$

where we have used that $\phi\phi_1 AY = \phi_1\phi AY + \eta_1(AY)\xi = \phi_1\phi AY$. From this, taking an inner product with ξ_2 and using the formulas in Section 2, we have

$$\eta_2(AY) = 0.$$

Similarly, we can assert that $\eta_3(AY) = 0$ for any $Y \in \mathfrak{D}_2$. So combining this with the fact that $\eta_\nu(AY) = 0$ for any $\nu = 1, 2, 3$, and any $Y \in \mathfrak{D}_1$, we have proved that $\eta_\nu(AY) = 0$ for any $Y \in \mathfrak{D}$, $\nu = 1, 2, 3$. Accordingly, we have $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$ for Hopf hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and its Reeb vector $\xi \in \mathfrak{D}^\perp$. Then, by virtue of Theorem A we know that M is locally congruent to real hypersurfaces of type (A), that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. □

We introduce in Theorem A, relating to this kind of hypersurface, another proposition due to Berndt and the present author [6] as follows:

Proposition C. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \alpha_i = \sqrt{8}\cot\left(\sqrt{8}r\right), \qquad \alpha_j = \alpha_k = \sqrt{2}\cot\left(\sqrt{2}r\right), \qquad \lambda = -\sqrt{2}\tan\left(\sqrt{2}r\right), \qquad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. *The corresponding multiplicities are*

$$m(\alpha_i) = 1, \qquad m(\alpha_j) = 2, \qquad m(\lambda) = 2m - 2 = m(\mu),$$

and for the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1X\}. \end{aligned}$$

In the paper [7] due to Berndt and the present author we have given a characterization of real hypersurfaces of type (A) in Theorem A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, which is equivalent to the condition that *the Reeb flow on M is isometric*, that is $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$ (resp. g) denotes the Lie derivative (resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ. Namely, Berndt and the present author [7] proved the following:

Theorem D. *Let M be a connected orientable real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. *Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

Now let us check for the real hypersurfaces of type (A) mentioned in Proposition C and Theorem D whether they satisfy a commuting Ricci tensor, that is, $S\phi = \phi S$. Then by Theorem D for the commuting shape operator, that is, $A\phi = \phi A$, the commuting Ricci tensor $S\phi = \phi S$ implies

$$\begin{aligned} &-3\eta_2(\phi Y)\xi_2 - 3\eta_3(\phi Y)\xi_3 + \phi_1\phi^2 Y - \eta(\phi_2\phi Y)\phi_2\xi - \eta(\phi_3\phi Y)\phi_3\xi \\ &\quad = -3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi\xi_\nu + \phi\phi_1\phi Y - \eta(\phi_2 Y)\phi_2\xi - \eta(\phi_3 Y)\phi_3\xi. \end{aligned} \tag{5.18}$$

Now let us check case by case whether the two sides in (5.18) are equal to each other as follows:

Case 1. $Y = \xi = \xi_1$.

In this case it can be easily checked that the two sides are equal to each other.

Case 2. $Y = \xi_2, \xi_3$.

Then by putting $X = \xi_2$ in (5.18) we have

$$-3\eta_2(\phi\xi_2)\xi_3 - \phi_1\xi_2 + \eta_2(\xi_2)\phi_2\xi + \eta_3(\xi_2)\phi_3\xi = -3\phi\xi_2 + \phi\phi_1\phi\xi_2 - \eta(\phi_3\xi_2)\phi_3\xi,$$

which implies that both sides are equal to ξ_3.

Case 3. $Y \in T_\lambda \oplus T_\mu$.

In such a case we have immediately $S\phi Y = \phi SY$.

Remark 5.1. In the paper due to Pérez and the author [13] we have proved that there do not exist any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Such a geometric condition is stronger than our commuting Ricci tensor in this paper. In the paper [12] we also have proved the non-existence property for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, $A\phi_i = \phi_i A,\ i = 1, 2, 3$.

References

[1] M. Kimura, Some real hypersurfaces of a complex projective space, Saitama Math. J. 5 (1987) 1–5.
[2] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, Trans. Amer. Math. Soc. 296 (1986) 137–149.
[3] J.D. Pérez, Y.J. Suh, Certain conditions on the Ricci tensor of real hypersurfaces in quaternionic projective space, Acta Math. Hungar. 91 (2001) 343–356.
[4] T.E. Cecil, P.J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. 269 (1982) 481–499.
[5] J.D. Pérez, Y.J. Suh, Real hypersurfaces of quaternionic projective space satisfying $\nabla_{\xi_i}R = 0$, Differential Geom. Appl. 7 (1997) 211–217.
[6] J. Berndt, Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians, Monatsh. Math. 127 (1999) 1–14.
[7] J. Berndt, Y.J. Suh, Isometric flows on real hypersurfaces in complex two-plane Grassmannians, Monatsh. Math. 137 (2002) 87–98.
[8] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives, Canad. Math. Bull. 49 (2006) 134–143.
[9] Y.J. Suh, Pseudo-Einstein real hypersurfaces in complex two-plane Grassmannians, Bull. Aust. Math. Soc. 73 (2006) 183–200.
[10] D.V. Alekseevskii, Compact quaternion spaces, Funct. Anal. Appl. 2 (1966) 106–114.
[11] Y.J. Suh, Real hypersurfaces of type *B* in complex two-plane Grassmannians, Monatsh. Math. 147 (2006) 337–355.
[12] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator, Bull. Aust. Math. Soc. 68 (2003) 379–393.
[13] J.D. Pérez, Y.J. Suh, The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians, J. Korean Math. Soc. 44 (2007) 211–235.