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Conformally symmetric semi-Riemannian manifolds ☆

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Abstract

In this paper we introduce the concept of conformal curvature-like tensor on a semi-Riemannian manifold, which is weaker than the notion of conformal curvature tensor defined on a Riemannian manifold. By such kind of conformal curvature-like tensor we give a complete classification of conformally symmetric semi-Riemannian manifolds with generalized non-null stress energy tensor.

MSC: Primary 53C40; Secondary 53C15

Keywords: Weyl curvature tensor; Conformally symmetric; Conformal-like curvature tensor; Semi-Riemannian manifold

1. Introduction

Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold with a metric tensor g and a Riemannian connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M. Any two self-adjoint (1,1) tensor fields A, B

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on a semi-Riemannian manifold (M, g) we define Kulkrani–Nomizu tensor product $A \otimes B$ of End $\Lambda^2 TM$ in such a way that

$$(A \otimes B)(X, Y) = \frac{1}{2}(AX \wedge BY + BY \wedge AY).$$

Then the conformal curvature tensor C of a semi-Riemannian manifold (M, g) acting on two-forms is given by

$$C = R - 2(n-1)^{-1}\,\mathrm{id} \otimes S + (n-1)^{-1}(n-2)^{-1} r\,\mathrm{id} \otimes \mathrm{id},$$

where id denotes the identity tensor of type (1,1) on (M, g). The conformal curvature tensor C should be the trace free part of the Riemannian curvature tensor R in above orthogonal decomposition, that is, $\mathrm{Ric}(C) = 0$, and is *conformally invariant*. Moreover, the conformal curvature tensor C, if n is at least 4, vanishes if and only if the metric is *conformally flat*.

Such a *conformal flatness* is equivalent to the vanishing of the Weyl conformal curvature tensor in dimension not less than 4. This should be an interesting subject, because there are many other examples of *conformally flat* manifolds which are not spaces of constant curvature, and because of its important applications to physics (see [6,8,9]).

Now the components C_{ijkl} of the conformal curvature tensor C can be written by

$$\begin{aligned} C_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(S_{il}g_{jk} - S_{ik}g_{jl} + S_{jk}g_{il} - S_{jl}g_{ik}) \\ &+ \frac{r}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}), \end{aligned} \tag{1.1}$$

where R_{ijkl} (resp. S_{ij}) denotes the components of the curvature tensor R(resp. the Ricci tensor S) on M.

We say that M is *conformally symmetric* if the conformal curvature tensor C is parallel, that is $\nabla C = 0$. Such kind of conformally flat or conformally symmetric semi-Riemannian manifolds have been studied by Besse [1], Bourguignon [2], Derdziński and Shen [4], Ryan [10], Simon [15], Weyl [17,18], Yano [19], Yano and Bochner [20]. More generally, conformally symmetric semi-Riemannian manifolds with indices 0 and 1 are investigated by Derdziński and Roter [3]. In particular, in semi-Riemannian manifolds with index 0, which are also said to be Riemannian manifolds, Derdziński and Roter [3] and Miyazawa [7] proved the following

Theorem A. *An $n(\geq 4)$-dimensional conformally symmetric manifold is conformally flat or locally symmetric.*

In particular, Derdziński and Roter [3] investigated the structure of analytic conformally symmetric indefinite Riemannian manifold of index 1 which is neither *conformally flat* nor *locally symmetric*.

The symmetric tensor K of type (0,2) with components K_{ij} is called the *Weyl tensor*, if it satisfies

$$K_{ijl} - K_{ilj} = \frac{1}{2(n-1)}(k_l g_{ij} - k_j g_{il}), \tag{1.2}$$

where $k = TrK$ and K_{ijl} (resp. k_j) are components of the covariant derivative ∇K (resp. ∇k).

On the other hand, in Weyl [17,18] it can be easily seen that the Ricci tensor is equal to the Weyl tensor when we only consider an $n(\geq 4)$-dimensional conformally flat Riemannian manifold.

Now as a generalization of conformal curvature tensor we introduce a new notion of conformal curvature-like tensor $B(T, U)$, which is defined in [12–14] due to the present authors. It was given as follows:

Let T be any curvature-like tensor (see its define in Section 2, in detail) and let U be any symmetric tensor of type (0,2) satisfying $2C_{12}(\nabla U) = C_{23}(\nabla U)$, where C_{12} and C_{23} denote the metric contraction defined by $2\sum_i \epsilon_i \nabla U(E_i, E_i, X) = \sum_i \epsilon_i \nabla U(X, E_i, E_i)$ for any vector X at x and for any orthonormal basis $\{E_j\}$ for the tangent space $T_x M$ to M at x. Then let us define the tensor $B = B(T, U)$ with components B_{ijkl} such that

$$\begin{aligned} B_{ijkl} = T_{ijkl} &- \frac{1}{n-2}(U_{il}g_{jk} - U_{ik}g_{jl} + U_{jk}g_{il} - U_{jl}g_{ik}) \\ &+ \frac{u}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}) \end{aligned} \tag{1.3}$$

where $u = \mathrm{Tr}U$. Then such a tensor $B = B(T, U)$ is said to be the *conformal curvature-like tensor*, which is a general extension of the usual conformal curvature tensor C in (1.1). For this kind of conformal curvature-like tensor $B(T, U)$ we also define the notion of *conformally flat* or *conformally symmetric* according as $B = 0$ or $\nabla B = 0$ respectively.

The Ricci-like tensor $\mathrm{Ric}(B)$ of B is defined by $\mathrm{tr}\{Z \to B(Z, X)Y\}$. Then the components B_{ij} of $\mathrm{Ric}(B)$ is defined by $B_{ij} = \sum_k \epsilon_k B_{kijk}$. Thus by (1.3), its components can be written as $B_{ij} = T_{ij} - U_{ij}$, where $T_{ij} = \sum_k \epsilon_k T_{kijk}$.

Now the tensor $\mathrm{Ric}(B) - \frac{b}{n} g$ defined on a semi-Riemannian manifold is said to be the *generalized stress energy* tensor for the conformal curvature-like tensor $B = B(R, U)$, where $b = \mathrm{Tr}(\mathrm{Ric}(B)) = C_{12}(\mathrm{Ric}(B))$. The physical meaning of such kind of stress energy tensor can be explained in more detail as follows:

In general relativity there can be no universal a priori geometry, since for any spacetime the Einstein equation already determines the stress energy tensor T, which is given by

$$T = \frac{1}{8\pi}\left(\mathrm{Ric} - \frac{1}{2}Sg\right),$$

where $S = C_{12}(\mathrm{Ric})$ denotes the scalar curvature. This is an Einstein equation between the stress energy tensor in physics and the Ricci curvature in differential geometry of spacetime. Thus a given spacetime can be used to model matter only in the unlikely case that T happens to be a physically realistic stress energy tensor (See [6,8,9]).

When the curvature-like tensor T(resp. the symmetric tensor U) mentioned above is equal to the curvature tensor R(resp. the Ricci tensor S), then $B(R, S)$ can be identified with the conformal curvature tensor C. Moreover, a semi-Riemannian manifold M is said to be *locally symmetric* if its derivative of the curvature tensor R vanishes, that is, $\nabla R = 0$.

Now in this paper we want to make a generalization of Theorem A in the direction of semi-Riemannian manifolds with symmetric conformal curvature-like tensor. In order to do this we need a geometric physical condition, that is, the *generalized non-null stress energy tensor* which is weaker than the notion of *stress energy tensor* given in B.O'Neill [8,9]. That is, we show the following

Theorem. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold with generalized non-null stress energy tensor. Let U be the symmetric tensor in D^2M and its trace ∇u is non-null and let $B = B(R, U)$ be the conformal curvature-like tensor. If it is conformally symmetric for the conformal curvature-like tensor $B(R, U)$, then it is locally symmetric, conformally flat or $\langle \nabla U, \nabla U \rangle = 0$.*

In the proof of our Theorem, we have used a useful Corollary 5.1 given in Section 5 and Lemma 6.3 and Theorem 6.1 given in Section 6. Now we will give its brief outline of the proof as follows:

In Corollary 5.1, under the assumption that ∇u is non-null we have proved the scalar-like curvature u is constant. Moreover, in Lemma 6.3, if the conformal curvature-like tensor $B = B(R, U)$ is symmetric on a semi-Riemannian manifold M, that is $\nabla B = 0$, we have proved that the generalized non-null stress energy tensor vanishes when M is not locally symmetric. By using such results, in Theorem 6.4 we are able to show that M is conformally flat when M is not locally symmetric.

If M is a Riemannian manifold, the result $\langle \nabla U, \nabla U \rangle = 0$ in our Theorem implies that the symmetric tensor U is parallel on M. From this together with the assumption of conformal symmetry $\nabla B = 0$ we can assert that $\nabla R = 0$, that is M is locally symmetric.

2. Preliminaries

Let M be an $n(\geq 2)$-dimensional semi-Riemannian manifold of index s $(0 \leq s \leq n)$ equipped with semi-Riemannian metric tensor ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now we can choose a local field $\{E_j\} = \{E_1, \ldots, E_n\}$ of orthonormal frames on a neighborhood of M. Here and in the sequel, the Latin small indices $i, j, k, \ldots,$ run from 1 to n. With respect to the semi-Riemannian metric we have $g(E_j, E_k) = \epsilon_j \delta_{jk}$, where

$$\epsilon_j = -1 \text{ or} 1 \text{ according as } 0 \leq j \leq s \text{ or } s + 1 \leq j \leq n.$$

Let $\{\theta_j\}$, $\{\theta_{ij}\}$ and $\{\Theta_{ij}\}$ be the canonical form, the connection form and the curvature form on M, respectively, with respect to the field $\{E_j\}$ of orthonormal frames. Then we have the structure equations

$$\mathrm{d}\theta_i + \sum_j \epsilon_j \theta_{ij} \wedge \theta_j = 0, \qquad \theta_{ij} + \theta_{ji} = 0, \qquad \mathrm{d}\theta_{ij} + \sum_k \epsilon_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij},$$

$$\Theta_{ij} = -\frac{1}{2} \sum_{k,l} \epsilon_{kl} R_{ijkl} \theta_k \wedge \theta_l,$$

where $\epsilon_{ij\ldots k} = \epsilon_i \epsilon_j \ldots \epsilon_k$ and R_{ijkl} denotes the components of the Riemannian curvature tensor R of M(See [9,11,12,14]).

Now, let C be the conformal curvature tensor with components C_{ijkl} on M, which is given by

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(\epsilon_j S_{il}\delta_{jk} - \epsilon_j S_{ik}\delta_{jl} + \epsilon_i S_{jk}\delta_{il} - \epsilon_i S_{jl}\delta_{ik}) + \frac{r}{(n-1)(n-2)}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \tag{2.1}$$

where S_{ij} denotes the components of the Ricci tensor S with respect to the orthonormal frame field $\{E_j\}$.

Remark 2.1. If M is Einstein, the conformal curvature tensor C satisfies

$$C_{ijkl} = R_{ijkl} - \frac{r}{n(n-1)}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

This yields that the conformal curvature tensor of an Einstein Riemannian manifold is the concircular curvature one [20]. In particular, if M is a space of constant curvature, the conformal curvature tensor vanishes identically.

Let $D^r M$ be the vector bundle consisting of differentiable r-forms and $DM = \sum_{r=0}^n D^r M$, where $D^0 M$ is the algebra of differentiable functions on M. For any tensor field K in $D^r M$ the components K_{ijklh} of the covariant derivative ∇K of K are defined by (for simplicity, we consider the case $r = 4$)

$$\sum_h \epsilon_h K_{ijklh}\theta_h = \mathrm{d}K_{ijkl} - \sum_h \epsilon_h (K_{hjkl}\theta_{hi} + K_{ihkl}\theta_{hj} + K_{ijhl}\theta_{hk} + K_{ijkh}\theta_{hl}).$$

Now we denote by TM the tangent bundle of M. Let T be a quadrilinear mapping of $TM \times TM \times TM \times TM$ into $\mathbb{R}$ satisfying the curvature-like properties:

(a) $T(X, Y, Z, U) = -T(Y, X, Z, U) = -T(X, Y, U, Z)$,
(b) $T(X, Y, Z, U) = T(Z, U, X, Y)$,
(c) $T(X, Y, Z, U) + T(Y, Z, X, U) + T(Z, X, Y, U) = 0$.

Then T is called the *curvature-like tensor* on M (see B.O. N'eill [8], for example). For an orthonormal frame $\{E_j\}$, let T_{ijkl} be the components of T associated with the orthonormal frame. Accordingly, the components T_{ijkl} are given by $T_{ijkl} = T(E_i, E_j, E_k, E_l)$. The components of T corresponding to the conditions (a), (b) and (c) are given by respectively

$$T_{ijkl} = -T_{jikl} = -T_{ijlk}, \tag{2.2}$$

$$T_{ijkl} = T_{klij} = T_{lkji}, \tag{2.3}$$

$$T_{ijkl} + T_{jkil} + T_{kijl} = 0. \tag{2.4}$$

If the components T_{ijkl} of a tensor T in $D^4M = \otimes^4 T^*M$ satisfy (2.2), (2.3) and (2.4), then it becomes a curvature-like tensor. Let FM be the ring consisting of all smooth functions on M and let $T_r^s M$ be the module over FM consisting of all tensor fields of type (r, s) defined on M. Let $H = H(M, g)$ be the vector subbundle in D^3M which, at any point x in M, consists of all trilinear mapping ξ of T_xM into $\mathbb{R}$ such that $\xi(X, Y, Z) = \xi(X, Z, Y)$ for any vectors at x and

$$2\sum_r \epsilon_r \xi(E_r, E_r, X) = \sum_r \epsilon_r \xi(X, E_r, E_r)$$

for any vector X at x and for any orthonormal basis $\{E_j\}$ for the tangent space T_xM to M at x.

For any integers a and b such that $1 \leq a < b \leq s$ the metric contraction reduced by a and b is denoted by $\mathcal{C}_{ab}$: $T_s^r M \to T_{s-2}^r M$ with respect to the orthonormal frame $\{E_j\}$. In terms of the metric contraction, the section ξ in $C^\infty(H)$ satisfies that $\xi(X, Y, Z)$ is symmetric with respect to Y and Z and $2\mathcal{C}_{12}(\xi) = \mathcal{C}_{23}(\xi)$.

Let U be a symmetric tensor of type (0,2) in D^2M with components $U_{ij}(= U_{ji}) = U(E_i, E_j)$. The symmetric tensor U in D^2M is called the *Weyl tensor* if its components of the covariant derivative ∇U of U satisfy

$$U_{ijk} - \frac{1}{2(n-1)} u_k \epsilon_i \delta_{ij} = U_{ikj} - \frac{1}{2(n-1)} u_j \epsilon_i \delta_{ik} \tag{2.5}$$

where $u = \mathcal{C}_{12}U$. In particular, if u is constant, then U is called the *Codazzi tensor*. Now we define the covariant derivative ∇U of the symmetric tensor U in such a way that $\nabla U(X, Y, Z) = \nabla_X U(Y, Z)$. Since U is symmetric, so is ∇U with respect to Y and Z. Moreover, we know that

$$\nabla_{E_k} U(E_i, E_j) = \nabla U(E_k, E_i, E_j) = U_{ijk}(= U_{jik}).$$

Then by (2.5) and the expression of ∇U it can be easily seen that

$$\sum_k \epsilon_k U_{kjk} = \frac{1}{2} u_j,$$

where $u_j = \sum_l \epsilon_l U_{llj} = \sum_l \epsilon_l \nabla U(E_j, E_l, E_l)$. This means $C_{12}(\nabla U) = C_{23}(\nabla U)/2$. Accordingly, we know that ∇U is the section of the bundle H. For such a pair (T, U), we define the tensor $B = B(T, U)$ with components B_{ijkl} by

$$B_{ijkl} = T_{ijkl} - \frac{1}{n-2}(\epsilon_i U_{jk}\delta_{il} - \epsilon_i U_{jl}\delta_{ik} + \epsilon_j U_{il}\delta_{jk} - \epsilon_j U_{ik}\delta_{jl}) + \frac{u}{(n-1)(n-2)}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \tag{2.6}$$

which is said to be the *conformal curvature-like tensor* for T and U. The Ricci-like tensor $\mathrm{Ric}(B)$ of B is defined by $\mathrm{tr}\{Z \to B(Z, X)Y\}$. Then the components B_{ij} of $\mathrm{Ric}(B)$ is given by $B_{ij} = \sum_k \epsilon_k B_{kijk}$. By (2.6), we get

$$B_{ij} = T_{ij} - U_{ij}. \tag{2.7}$$

Remark 2.2. For a given semi-Riemannian manifold M with Riemannian connection ∇, there exist so many kind of pairs (T, U) for the curvature-like tensor T and the symmetric tensor of D^2M such that ∇U is contained in $C^\infty(H)$. Among them the most popular pair is (R, S). In particular, let U be the Weyl tensor and let K be the parallel symmetric tensor in D^2M. Then $U + K$ is also the Weyl tensor.

For an orthonormal frame $\{E_j\}$, let $\theta_0 = \{\theta_j\}$, $\theta = \{\theta_{ij}\}$ and $\Theta = \{\Theta_{ij}\}$ be the canonical form, the connection form and the curvature form on M. In the same way as we associate a curvature form Θ to the Riemannian curvature tensor R, we associate a two-form $\Phi_T = \{\Phi_{ij}\}$ to the curvature-like tensor T in the following

$$\Phi_{ij} = \sum_{k,l} \epsilon_{kl} T_{ijkl}\theta_k \wedge \theta_l, \tag{2.8}$$

which is the analogue to the curvature form Θ except for the coefficient. So it is called the *curvature-like form* for the curvature-like tensor T. The canonical form $\theta_0 = \{\theta_j\}$ can be regarded as a vector in $\mathbb{R}^n$ and the connection form $\theta = \{\theta_{ij}\}$ and the curvature-like form $\Phi_T = \{\Phi_{ij}\}$ can be regarded as $n \times n$ skew-symmetric matrices. We call the equation

$$\mathrm{d}\Phi_T = \Phi_T \wedge \theta - \theta \wedge \Phi_T \tag{2.9}$$

the *second Bianchi equation* for the curvature-like form Φ_T. Then it is not difficult to assert the following proposition:

Proposition 2.1. *On the semi-Riemannian manifold M, let T be the curvature-like tensor in D^4M on M. Then the following assertions are equivalent*:

(a) *The curvature-like form Φ_T satisfies the second Bianchi* Eq. (2.9).
(b) *Its components satisfy*

$$T_{ijklh} + T_{ijlhk} + T_{ijhkl} = 0. \tag{2.10}$$

Now let T be the curvature-like tensor in D^4M and let Ψ_T be the associated curvature-like form with T. The mapping $\delta : \wedge^4 T_*M \to \wedge^3 T_*M$ defined by the divergence: $\delta(\Psi_T) = -(\mathcal{C}_{15}\nabla T)$, where $\mathcal{C}_{ab}$ is the metric contraction defined by C_{ab}: $T_s^r M \to T_{s-2}^r M$. This is a generalization of the well-known differential operators on $\mathbb{R}^3$. For the orthonormal frame $\{E_j\}$, in terms of coordinates, the components of $\delta(\Psi_T)$ is given by

$$\delta(\Psi_T)_{ijk} = -\sum_l \epsilon_l T_{lijkl}.$$

If $\delta(\Psi_T) = 0$, then the form Ψ_T is said to be *coclosed*.

Remark 2.3. On the Riemannian manifold (M, g) with Riemannian connection ∇, it has a formal adjoint $\nabla^* : T^*M \times T^r_s M \to T^r_s M$ defined as follows: for any vector fields $X_1, \ldots, X_r$ and any α in $T^*M \times T^r_s M$, $\nabla^*\alpha$ is given by

$$\nabla^*\alpha(X_1, \ldots, X_r) = -\sum_k \epsilon_k \nabla_{E_k}\alpha(E_k, X_1, \ldots, X_r),$$

where $\{E_j\}$ is the orthonormal frame. Namely, $\nabla^*\alpha(X_1, \ldots, X_r)$ is the opposite of the trace with respect to g of the $D^s M$ valued two-form

$$(X, Y) \longrightarrow \nabla_X\alpha(Y, X_1, \ldots, X_r).$$

For the exterior differential d: $D^r M \to D^{r+1} M$ let us denote by δ: $D^r M \to D^{r-1} M$ its formal adjoint. For the orthonormal base $\{E_j\}$ of $T_x M$ at any point x, the components of $\delta(\Psi_T)$ are given by

$$\delta(\Psi_T(T))(X, Y, Z) = -\sum_k \epsilon_k \nabla_{E_k}\Psi_T(T)(E_k, X, Y, Z).$$

Accordingly, the above operator δ on semi-Riemannian manifolds is a formal analogue of the adjoint operator to the exterior differential d on a Riemannian manifold. See Besse [1], for example.

The semi-Riemannian manifold (M, g) is said to have *harmonic-like curvature* for T if $\delta(\Psi_T) = 0$. In particular, if $T = R$, then we say that (M, g) has *harmonic curvature*(See Besse [1]).

Now, the concept of Ricci-like tensors for the curvature-like tensor on the semi-Riemannian manifold is introduced. Let M be an n-dimensional semi-Riemannian manifold with semi-Riemannian metric g and curvature like tensor T with components T_{ijkl}. The tensor $\mathrm{Ric}(T)$ associated with the curvature-like tensor T is defined by

$$\mathrm{Ric}(T)(X, Y) = \mathrm{trace}\{Z \to T(Z, X)Y\},$$

where $T(Z, X)Y$ is a vector field defined by $T(X, Y, Z, W) = g(T(X, Y)Z, W)$ for any vector fields X, Y, Z and W. Then $\mathrm{Ric}(T)$ is called the *Ricci-like tensor* for T. By the definition, the Ricci-like tensor $\mathrm{Ric}(T)$ of T is a symmetric tensor of type (0,2) and its components T_{ij} are given by $T_{ij} = \sum_k \epsilon_k T_{kijk}$. Moreover, by (2.3) we know that $T_{ij} = T_{ji}$. The scalar-like curvature t associated with T is defined by $t = \mathcal{C}_{12}(\mathrm{Ric}(T)) = \sum_{j,k} \epsilon_{jk} T_{kjjk}$. If we follow new extended formulas in a semi-Riemannian manifold, we are able to show the following theorem in [12], which gives an extension of the paper [14] discussed in its Riemannian version,

Theorem 2.1. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. If any two assertions in the following are satisfied, then another one holds true*:

(a) *T satisfies the second Bianchi identity,*
(b) *The conformal curvature-like tensor $B = B(T, U)$ satisfies the second Bianchi identity,*
(c) *U is the Weyl tensor.*

3. Weyl tensors

Let M be an $n(\geq 2)$-dimensional semi-Riemannian manifold of index $2s$, $0 \leq s \leq n$, with Riemannian connection ∇ and let R(resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now let T be a curvature-like tensor and let U be a symmetric tensor of type (0,2) such that $2\mathcal{C}_{12}(\nabla U) = \mathcal{C}_{23}(\nabla U)$. In this section we are going to prove some lemmas concerned with such a symmetric tensor U. From the conformal curvature-like tensor $B = B(T, U)$ given in (2.6), we know that $B = B(T, U)$ is the curvature-like tensor for the pair (T, U). Moreover, we have

$$\mathrm{Ric}(B) = \mathrm{Ric}(T) - U. \tag{3.1}$$

In fact, putting $i = l$, multiplying ϵ_i and summing up with respect to the index l in (2.6) we get $B_{jk} = T_{jk} - U_{jk}$, where T_{jk} and B_{jk} are components of the Ricci-like tensors $\mathrm{Ric}(T) = \mathcal{C}_{14}(T)$ and $\mathrm{Ric}(B) = \mathcal{C}_{14}(B)$.

Now let us put $t = \mathcal{C}_{12}(\mathrm{Ric}(T))$, $b = \mathcal{C}_{12}(\mathrm{Ric}(B))$ and $u = \mathcal{C}_{12}(U)$. Then by (3.1) we have

$$b = t - u. \tag{3.2}$$

Remark 3.1. For the conformal curvature tensor C, the Ricci-like tensor $\mathrm{Ric}(C)$ satisfies $\mathrm{Ric}(C) = 0$.

When the symmetric tensor U in D^2M is the Weyl tensor, we are going to prove some lemmas, which will be used in Section 5, as follows:

Lemma 3.1. *Let M be an $n(\geq 2)$-dimensional semi-Riemannian manifold. If U is the Weyl tensor, we obtain*

$$\sum_r \epsilon_r (R_{rikl} U_{rj} + R_{rilj} U_{rk} + R_{rijk} U_{rl}) = 0. \tag{3.3}$$

Proof. By the definition (2.5) we have

$$U_{ijk} - U_{ikj} = \frac{1}{2(n-1)}(u_k\epsilon_i\delta_{ij} - u_j\epsilon_i\delta_{ik}).$$

Differentiating covariantly, we get

$$U_{ijkl} - U_{ikjl} = \frac{1}{2(n-1)}(u_{kl}\epsilon_i\delta_{ij} - u_{jl}\epsilon_i\delta_{ik}). \tag{3.4}$$

Interchanging the indices k and l and substituting the resulting equation from the above one, we obtain

$$U_{ijkl} - U_{ijlk} + U_{iljk} - U_{ikjl} = \frac{1}{2(n-1)}(u_{jk}\epsilon_i\delta_{il} - u_{jl}\epsilon_i\delta_{ik}),$$

where we have used the property u_{ij} is symmetric with respect to i and j, because u is the function. So we have

The left hand side

$$\begin{aligned}
&(U_{ijkl} - U_{ijlk}) + (U_{iljk} - U_{ilkj}) + (U_{ilkj} - U_{ikjl}) \\
&\quad = (U_{ijkl} - U_{ijlk}) + (U_{iljk} - U_{ilkj}) \\
&\qquad + \left[\left\{U_{iklj} + \frac{1}{2(n-1)}(u_{kj}\epsilon_i\delta_{il} - u_{lj}\epsilon_i\delta_{ik})\right\} - U_{ikjl}\right] \\
&\quad = -\sum_r \epsilon_r(R_{lkir}U_{rj} + R_{lkjr}U_{ir}) - \sum_r \epsilon_r(R_{kjir}U_{rl} + R_{kjlr}U_{ir}) \\
&\qquad - \sum_r \epsilon_r(R_{jlir}U_{rk} + R_{jlkr}U_{ir}) + \frac{1}{2(n-1)}(u_{kj}\epsilon_i\delta_{il} - u_{lj}\epsilon_i\delta_{ik}) \\
&\quad = -\sum_r (R_{lkir}U_{rj} + R_{jlir}U_{rk} + R_{kjir}U_{rl}) + \frac{1}{2(n-1)}(u_{kj}\epsilon_i\delta_{il} - u_{lj}\epsilon_i\delta_{ik}),
\end{aligned}$$

where the second equality follows from (3.4) and the third equality can be derived from the Ricci identity for the tensor U_{ij}. Accordingly, we have (3.3), which completes the proof. □

Lemma 3.2. *Let M be an $n(\geq 2)$-dimensional semi-Riemannian manifold and let $B = B(R, U)$ be the conformal curvature-like tensor for the Riemannian curvature tensor R and any symmetric tensor U in D^2M. If U is the Weyl tensor, then we obtain*

$$\sum_r \epsilon_r(B_{rikl}U_{rj} + B_{rilj}U_{rk} + B_{rijk}U_{rl}) = 0, \tag{3.5}$$

$$\sum_r \epsilon_r B_{kr}U_{rl} = \sum_r \epsilon_r B_{lr}U_{rk}. \tag{3.6}$$

Proof. By the assumption of this Lemma, the components of B are given by

$$B_{ijkl} = R_{ijkl} - \frac{1}{n-2}(\epsilon_i U_{jk}\delta_{il} - \epsilon_i U_{jl}\delta_{ik} + \epsilon_j U_{il}\delta_{jk} - \epsilon_j U_{ik}\delta_{jl}) + \frac{u}{(n-1)(n-2)}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{3.7}$$

Substituting B_{ijkl} into the left hand side of (3.5) and using (3.3), we get the equation (3.5). Since the tensor U_{ij} is symmetric and the tensor B_{ijkl} is skew-symmetric indices i and j, we have $\sum_{r,s}\epsilon_{rs}B_{rskl}U_{rs} = 0$. Putting $i = j$, multiplying ϵ_i and summing up with respect to i in (3.5), from the above property together with $B_{ij} = \sum_r \epsilon_r B_{rijr}$, we get (3.6). This completes the proof. □

4. Conformally symmetric

Let M be an $n(\geq 2)$-dimensional semi-Riemannian manifold of index $2s$, $0 \leq s \leq n$, with Riemannian connection ∇ and let R(resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now, let U be a symmetric tensor of type (0,2) such that $2\mathcal{C}_{12}(\nabla U) = \mathcal{C}_{23}(\nabla U)$. We put $u = \mathcal{C}_{12}U$. For such a pair (R, U) we define a tensor $B = B(R, U)$ with components B_{ijkl} with respect to the field $\{E_j\}$ of orthonormal frames by

$$B_{ijkl} = R_{ijkl} - \frac{1}{n-2}(\epsilon_i U_{jk}\delta_{il} - \epsilon_i U_{jl}\delta_{ik} + \epsilon_j U_{il}\delta_{jk} - \epsilon_j U_{ik}\delta_{jl}) + \frac{u}{(n-1)(n-2)}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{4.1}$$

We have then

$$\mathrm{Ric}(B) = S - U. \tag{4.2}$$

In fact, putting $i = l$, multiplying ϵ_i and summing up with respect to the index i in (4.1) we get $B_{jk} = S_{jk} - U_{jk}$, where B_{jk} are components of the Ricci-like tensor $\mathrm{Ric}(B) = \mathcal{C}_{14}(B)$.

Now let us put $b = \mathcal{C}_{12}(\mathrm{Ric}(B)) = \mathrm{Tr}(\mathrm{Ric}(B))$ and $u = \mathcal{C}_{12}(U) = \mathrm{Tr}U$. Then by (4.2) we have

$$b = r - u. \tag{4.3}$$

By Theorem 2.1 if the curvature-like tensor $B = B(R, U)$ for the pair (R, U) satisfies the second Bianchi identity, then U is the Weyl tensor. Of course the Ricci tensor S is the Weyl tensor. We remark that the restriction $n \geq 4$ of the dimension is here necessary.

The semi-Riemannian manifold M is said to be *conformally symmetric*, if the conformal curvature-like tensor B is parallel, i.e., if $\nabla B = 0$. Concerned with such a conformal symmetry of the conformal curvature-like tensor B we prove the following:

Lemma 4.1. *Let M be an* $n(\geq 4)$*-dimensional semi-Riemannian manifold. If M is conformally symmetric for B, then*

$$\sum_r \epsilon_r (B_{rikl}U_{rjh} + B_{rilj}U_{rkh} + B_{rijk}U_{rlh}) = 0. \tag{4.4}$$

Proof. The Riemannian curvature tensor R satisfies the second Bianchi identity. On the other hand, since the conformal curvature-like tensor $B(R, U)$ is parallel, it also satisfies the second Bianchi identity by Proposition 2.1. Accordingly, the symmetric tensor U is the Weyl tensor by Theorem 2.1, because of $n \geq 4$. By Lemma 3.2 we have

$$\sum_r \epsilon_r (B_{rikl}U_{rj} + B_{rilj}U_{rk} + B_{rijk}U_{rl}) = 0,$$

from which together with the assumption that B is parallel we have the equation (4.4). This completes the proof. □

Now we have $\sum_{r,s} \epsilon_{rs} B_{rskl}U_{rsh} = 0$, because B_{ijkl} is skew-symmetric with respect to i and j and U_{ijh} is symmetric with respect to i and j. Putting $i = j$, multiplying ϵ_i, summing up with respect to i in (4.4) and applying the above property to the obtained equation, we have

$$\sum_r \epsilon_r B_{kr}U_{rlh} = \sum_r \epsilon_r B_{lr}U_{rkh}. \tag{4.5}$$

Since U is the Weyl tensor, it satisfies $\sum_r \epsilon_r U_{krr} = u_k/2$. Putting $l = h$, multiplying ϵ_l, summing up with respect to l in (4.4) and applying the property $\sum_r \epsilon_r U_{krr} = u_k/2$, we have

$$\sum_{r,s} \epsilon_{rs}(B_{riks}U_{jrs} - B_{rijs}U_{krs}) = -\frac{1}{2}\sum_r \epsilon_r B_{rijk}u_r. \tag{4.6}$$

Putting $i = j$, multiplying ϵ_i and summing up with respect to i in (4.6), we have

$$2\sum_{r,s} \epsilon_{rs} B_{rs}U_{krs} = \sum_r \epsilon_r B_{kr}u_r, \tag{4.7}$$

because we see $\sum_{rst} \epsilon_{rst} B_{rtks}U_{trs} = 0$.

Summing up the above formulas, we prove the following which will be useful in Section 5.

Lemma 4.2. *Let M be an* $n(\geq 4)$*-dimensional semi-Riemannian manifold. If M is conformally symmetric for the conformal curvature-like tensor* $B(R, U)$*, then we obtain*

$$2(n-1)\sum_r \epsilon_r (B_{ir}U_{jkr} - B_{kr}U_{jir}) = \sum_r \epsilon_{rj}(B_{ir}u_r\delta_{jk} - B_{kr}u_r\delta_{ij}). \tag{4.8}$$

Proof. Since M is conformally symmetric for the conformal curvature-like tensor $B(R, U)$, we have $B_{ijklh} = 0$ and $B_{ijklhp} = 0$. Accordingly, we have by (4.1)

$$\begin{aligned} R_{ijklh} = &\frac{1}{n-2}(\epsilon_j U_{ilh}\delta_{jk} - \epsilon_j U_{ikh}\delta_{jl} + \epsilon_i U_{jkh}\delta_{il} - \epsilon_i U_{jlh}\delta_{ik}) \\ &- \frac{1}{(n-1)(n-2)}u_h\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \end{aligned} \tag{4.9}$$

On the other hand, by the Ricci identity, we get

$$\sum_r \epsilon_r (R_{phir} B_{rjkl} + R_{phjr} B_{irkl} + R_{phkr} B_{ijrl} + R_{phlr} B_{ijkr}) = 0.$$

Differentiating covariantly and taking account of $B_{ijklh} = 0$, we have

$$\sum_r \epsilon_r (R_{phirq} B_{rjkl} + R_{phjrq} B_{irkl} + R_{phkrq} B_{ijrl} + R_{phlrq} B_{ijkr}) = 0.$$

By (4.9) and the above equation we have

$$\begin{aligned}
&\sum_r \epsilon_r [\{(\epsilon_h U_{prq} \delta_{hi} - \epsilon_h U_{piq} \delta_{hr} + \epsilon_p U_{hiq} \delta_{pr} - \epsilon_p U_{hrq} \delta_{pi}) \\
&\quad + u_q \epsilon_{hp} (\delta_{pr} \delta_{hi} - \delta_{pi} \delta_{hr})/(n-1)\} B_{rjkl} + \{(\epsilon_h U_{prq} \delta_{hj} \\
&\quad - \epsilon_h U_{pjq} \delta_{hr} + \epsilon_p U_{hjq} \delta_{pr} - \epsilon_p U_{hrq} \delta_{pj}) + u_q \epsilon_{hp} (\delta_{pr} \delta_{hj} - \delta_{pj} \delta_{hr})/(n-1)\} B_{irkl} \\
&\quad + \{(\epsilon_h U_{prq} \delta_{hk} - \epsilon_h U_{pkq} \delta_{hr} + \epsilon_p U_{hkq} \delta_{pr} - \epsilon_p U_{hrq} \delta_{pk}) \\
&\quad + u_q \epsilon_{hp} (\delta_{pr} \delta_{hk} - \delta_{pk} \delta_{hr})/(n-1)\} B_{ijrl} + \{(\epsilon_h U_{prq} \delta_{hl} - \epsilon_h U_{plq} \delta_{hr} \\
&\quad + \epsilon_p U_{hlq} \delta_{pr} - \epsilon_p U_{hrq} \delta_{pl}) + u_q \epsilon_{hp} (\delta_{pr} \delta_{hl} - \delta_{pl} \delta_{hr})/(n-1)\} B_{ijkr}] = 0.
\end{aligned} \tag{4.10}$$

Putting $h = i$, multiplying ϵ_i and summing up with respect to i and taking account of the first Bianchi identity, we have

$$\begin{aligned}
&\sum_r \epsilon_r \left\{ (n-2) B_{rjkl} U_{prq} - \sum_s \epsilon_{sp} (B_{rjsl} \delta_{pk} + B_{sjkr} \delta_{pl}) U_{rsq} \right. \\
&\quad \left. + (B_{rpkl} U_{rjq} + B_{rjpl} U_{rkq} + B_{rjkp} U_{rlq}) \right\} + (B_{jl} U_{pkq} - B_{jk} U_{plq}) \\
&\quad - \frac{1}{n-1} u_q \epsilon_p (B_{jl} \delta_{pk} - B_{jk} \delta_{pl}) = 0.
\end{aligned}$$

Taking account of (4.4), the above equation is deformed as

$$\begin{aligned}
&\sum_r \epsilon_r \left\{ (n-1) B_{rjkl} U_{prq} - \sum_s \epsilon_{sp} (B_{rjsl} \delta_{pk} + B_{sjkr} \delta_{pl}) U_{rsq} + B_{rpkl} U_{rjq} \right\} \\
&\quad + (B_{jl} U_{pkq} - B_{jk} U_{plq}) - \frac{1}{n-1} u_q \epsilon_p (B_{jl} \delta_{pk} - B_{jk} \delta_{pl}) = 0.
\end{aligned} \tag{4.11}$$

Putting $l = q$ in the above equation, multiplying ϵ_l and summing up with respect to l, we get

$$\sum_{r,s} \epsilon_{rs} \Big\{ (n-2)B_{rjks}U_{prs} - \sum_t \epsilon_t U_{rst} \epsilon_p (B_{rjst}\delta_{pk} + B_{sjkr}\delta_{pt})$$

$$+ (B_{rpks}U_{rjs} + B_{rjps}U_{rks} + B_{rjkp}U_{rss}) \Big\} + \sum_r \epsilon_r \{ (B_{jr}U_{pkr} - B_{jk}U_{prr})$$

$$- \frac{1}{n-1} u_r \epsilon_p (B_{jr}\delta_{pk} - B_{jk}\delta_{pr}) \} = 0. \tag{4.12}$$

Taking account of the fact that B_{ijkl} is skew symmetric with respect to k and l and using (2.5), we get the following relation:

$$\sum_{r,s,t} \epsilon_{rst} B_{jrst} U_{rst} = \frac{1}{2(n-1)} \sum_r \epsilon_r B_{jr} u_r. \tag{4.13}$$

In fact, we get

$$\text{The left hand side} = \frac{1}{2} \sum_{r,s,t} \epsilon_{rst} B_{jrst} (U_{rst} - U_{rts})$$

$$= \frac{1}{4(n-1)} \sum_{s,t} \epsilon_{st} B_{jrst} (u_T \delta_{rs} - u_s \delta_{rt})$$

$$= \frac{1}{4(n-1)} \sum_t \epsilon_t (B_{jt}u_t + B_{jt}u_t)$$

Making use of (4.13) and calculating straightforwardly, we can obtain

$$\sum_r \epsilon_{rs} \left\{ (n-2)B_{rjks}U_{prs} - B_{rjks}U_{rsp} + B_{rpks}U_{jrs} + B_{rjps}U_{krs} \right\}$$

$$+ \sum_r \epsilon_r (\frac{1}{2} B_{rjkp} u_r + B_{jr} U_{pkr})$$

$$- \frac{1}{2(n-1)} \left\{ (n-3) B_{jk} u_p + \sum_r \epsilon_{rp} B_{jr} u_r \delta_{pk} \right\} = 0. \tag{4.14}$$

Interchanging indices j and k in (4.14) and subtracting the resulting equation from the original one, using (2.5) and (4.6), we have

$$(n-1) \sum_r \epsilon_r (B_{pjkr} + B_{jkpr} - B_{pkjr}) u_r + 2(n-1) \sum_r \epsilon_r (B_{jr} U_{pkr}$$

$$- B_{kr} U_{pjr}) - \epsilon_p \sum_r \epsilon_r u_r (B_{jr}\delta_{pk} - B_{kr}\delta_{pj}) = 0.$$

Making use of the first Bianchi identity for B, we have (4.8). This completes the proof. □

5. Scalar-like curvatures

Let M be an $n(\geq 4)$-diemnsional semi-Riemannian manifold of index s $(0 \leq s \leq n)$ equipped with semi-Riemannian metric g and Riemannian connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M. Let U be the symmetric tensor in D^2M and let $B = B(R, U)$ be the conformal curvature-like tensor. Then $u = \mathrm{Tr}U$ is called the *scalar-like curvature* for B.

Now let C be the conformal curvature tensor defined on M. Glodeck [5] and Tanno [16] proved that any non-conformally flat conformal symmetric Riemannian manifold has the constant scalar curvature. We put $b = \mathrm{Tr}(\mathrm{Ric}(B))$. If $\nabla B = 0$, then the function b is constant on M. In this section we prove the following

Proposition 5.1. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold and let U be the symmetric tensor in D^2M and let $B = B(R, U)$ be the conformal curvature-like tensor. If $\nabla B = 0$, then $b\langle\nabla u, \nabla u\rangle = 0$.*

In order to prove this proposition we verify some lemmas step by step as follows:

Lemma 5.1. *Under the situation of* Proposition 5.1, *we have*

$$\sum_r \epsilon_r B_{ir} U_{rj} = \sum_r \epsilon_r B_{jr} U_{ri}, \tag{5.1}$$

$$\sum_r \epsilon_r B_{ir} U_{rjk} = \sum_r \epsilon_r B_{jr} U_{rik}. \tag{5.2}$$

Proof. Since B is parallel, it satisfies the second Bianchi identity. Accordingly, by Theorem 2.1, U is the Weyl tensor and it satisfies (3.5). Putting $i = j$ in (3.5), multiplying ϵ_i, and summing up with respect to the index i, we get

$$\sum_{r,s} \epsilon_{rs} B_{rskl} U_{rs} - \sum_r \epsilon_r B_{lr} U_{rk} + \sum_r \epsilon_r B_{kr} U_{rl} = 0,$$

the first term of which vanishes identically on M. Thus we get

$$\sum_r \epsilon_r B_{ir} U_{rj} = \sum_r \epsilon_r B_{jr} U_{ri}.$$

Accordingly, (5.1) is satisfied. Because of $\nabla B = 0$, we have $\nabla \mathrm{Ric}(B) = 0$, which means that $B_{ijk} = 0$. So Lemma 5.1 is proved. □

Lemma 5.2. *Under the situation of* Proposition 5.1, *we have*

$$\sum_r \epsilon_r B_{ir} u_r = b u_i, \tag{5.3}$$

$$\langle\nabla u, \nabla u\rangle B_{ij} = b u_i u_j. \tag{5.4}$$

Proof. By (2.5) and (4.8), we have

$$\sum_r \epsilon_r (B_{ir}U_{jkr} - B_{kr}U_{ijr}) - \frac{1}{2(n-1)}\left(\sum_r \epsilon_r B_{ir}u_r\epsilon_j\delta_{jk} - \sum_r \epsilon_r B_{kr}u_r\epsilon_i\delta_{ij}\right)$$
$$= \sum_r \epsilon_r B_{ir}\left\{U_{jrk} + \frac{1}{2(n-1)}(u_r\epsilon_j\delta_{jk} - u_k\epsilon_j\delta_{jr})\right\}$$
$$- \sum_r \epsilon_r B_{kr}\left\{U_{jri} + \frac{1}{2(n-1)}(u_r\epsilon_j\delta_{ji} - u_i\epsilon_j\delta_{jr})\right\}$$
$$- \frac{1}{2(n-1)}\left(\sum_r \epsilon_r B_{ir}u_r\epsilon_j\delta_{jk} - \sum_r \epsilon_r B_{kr}u_r\epsilon_i\delta_{ij}\right)$$
$$= \frac{1}{2(n-1)}(-B_{ij}u_k + B_{kj}u_i) = 0,$$

where the second equality follows from (5.2). Accordingly, we can obtain

$$B_{ij}u_k = B_{kj}u_i, \tag{5.5}$$

from which we have (5.3) and

$$\sum_r \epsilon_r u_r u_r B_{ij} = \sum_r \epsilon_r B_{jr}u_r u_i$$

This implies (5.4), which proves our assertion. □

Lemma 5.3. *Under the situation of* Proposition 5.1, *we have*

$$2b\langle\nabla u, \nabla u\rangle \sum_r \epsilon_r U_{rij}u_r = b\langle\nabla u, \nabla u\rangle u_i u_j. \tag{5.6}$$

Proof. By (4.7), (5.3) and (5.4), we have

$$2b\sum_{r,s}\epsilon_{rs}U_{irs}u_r u_s = b\sum_r \epsilon_r u_r u_r u_i.$$

On the other hand, the left side of the above equation can be reformed as

$$2b\sum_{r,s}\epsilon_{rs}\left\{U_{rsi} + \frac{1}{2(n-1)}(u_s\epsilon_r\delta_{ri} - u_i\epsilon_r\delta_{rs})\right\}u_r u_s = 2b\sum_{r,s}\epsilon_{rs}U_{rsi}u_r u_s$$

by (2.5). From this together with the above equations, we get

$$2b\sum_{r,s}\epsilon_{rs}U_{rsi}u_r u_s = b\sum_r \epsilon_r u_r u_r u_i. \tag{5.7}$$

By (5.2) and (5.4), we have

$$b\sum_r \epsilon_r u_i U_{rjk} u_r = b\sum_r \epsilon_r u_j U_{rik} u_r.$$

Thus we have

$$2b\sum_s \epsilon_s u_s u_s \sum_r \epsilon_r U_{rjk} u_r = 2b\sum_{r,s} \epsilon_{rs} u_j U_{rsk} u_r u_s = b\left(\sum_s \epsilon_s u_s u_s\right) u_j u_k,$$

where the last equality is derived from (5.7). This completes the proof. □

Let M' be the subset of M consisting of points x at which $b\langle\nabla u, \nabla u\rangle(x) \neq 0$. By (4.2) if $U = S$, then $\mathrm{Ric}(B) = 0$ and hence $b = \mathrm{Tr}(\mathrm{Ric}(B)) = 0$. This means that $U \neq S$ on M'. Now let us consider a function f on M' defined by $b/\langle\nabla u, \nabla u\rangle$. We denote by u_{ij} the components of the tensor $\nabla\nabla u$.

Lemma 5.4. *Under the situation of Proposition 5.1, we have on M'*

$$u_{ij} = hu_i u_j, \tag{5.8}$$

where h is a differentiable function defined on M'.

Proof. By Lemma 5.2, we have $B_{ij} = fu_i u_j$. Differentiating covariantly this equation, and taking account of $\nabla B = 0$, we get

$$f_k u_i u_j + f(u_{ik} u_j + u_i u_{jk}) = 0.$$

Putting $j = k$, multiplying ϵ_k and summing up with respect to k, we get the fact that $\sum_r \epsilon_r u_{ir} u_r$ is proportional to u_i, since the function f has no zero points on M'. Transvecting $u_i u_j$ to the above equation, we obtain the fact that f_k is proportional to u_k. It implies that $u_{jk} = hu_j u_k$ on M'. This completes the proof. □

Under such a situation, applying B_{ij} to the Ricci identity and using $\nabla B = 0$, we have

$$\sum_r \epsilon_r (R_{lkir} B_{rj} + R_{lkjr} B_{ir}) = 0.$$

By Lemma 5.2 we have $B_{ij} = fu_i u_j$ and hence, from the above two equations we get

$$\sum_r \epsilon_r (R_{lkir} u_r u_j + R_{lkjr} u_i u_r) = 0$$

on M'. Thus, we obtain $\sum_{r,s} \epsilon_{rs}(R_{lkir} u_r u_s u_s + R_{lksr} u_i u_r u_s) = 0$. Since R_{lksr} is skew-symmetric with respect to indices r and s, the second term is zero and hence it turns out to be $\sum_r \epsilon_r R_{lkir} u_r \langle \nabla u, \nabla u \rangle = 0$. So we get on M'

$$\sum_r \epsilon_r R_{rjkl} u_r = 0. \tag{5.9}$$

Proof of Proposition 5.1. In order to prove Proposition 5.1, it suffices to show that the subset M' is empty. Suppose that M' is not empty. Differentiating (5.9) covariantly, we get on M'

$$\sum_r \epsilon_r (R_{rjkl} u_{ri} + R_{rjkli} u_r) = 0.$$

By (5.8) and (5.9) the first term vanishes identically and so it yields that

$$\sum_r \epsilon_r R_{rjkli} u_r = 0.$$

By (4.9) we have

$$\sum_r \epsilon_r (\epsilon_j U_{rlh} \delta_{jk} - \epsilon_j U_{rkh} \delta_{jl}) u_r + U_{jkh} u_l - U_{jlh} u_k - \frac{1}{n-1} u_h \epsilon_j (\delta_{jk} u_l - \delta_{jl} u_k) = 0,$$

since B is parallel. By (5.6) the above equation is reformed as

$$\epsilon_j u_l u_h \delta_{jk} - \epsilon_j u_k u_h \delta_{jl} + 2U_{jkh} u_l - 2U_{jlh} u_k - \frac{2}{n-1} u_h \epsilon_j (u_l \delta_{jk} - u_k \delta_{jl}) = 0,$$

and hence we have

$$(n-3) u_h \epsilon_j (u_l \delta_{jk} - u_k \delta_{jl}) + 2(n-1)(U_{jkh} u_l - U_{jlh} u_k) = 0.$$

Putting $j = k$, multiplying ϵ_j and summing up with respect to j in the above equation and by (5.6), we get

$$(n-1)(n-2) u_h u_l = 0,$$

which means that $\langle \nabla u, \nabla u \rangle = 0$ on M', a contradiction. Thus the subset M' is empty. □

By Proposition 5.1, we are able to generalize a theorem due to Glodeck [5] and Tanno [16] as follows:

Theorem 5.1. *Let (M, g) be an $n(\geq 4)$-dimensional semi-Riemannian manifold. Let U be a symmetric tensor in $D^2 M$ with $u = \mathrm{Tr} U$ and let $B = B(R, U)$ be the conformal curvature-like tensor. If $\nabla B = 0$, then the scalar product $\langle \nabla u, \nabla u \rangle$ vanishes identically.*

Proof. Since B is parallel, $b = \mathrm{Tr}(\mathrm{Ric}(B))$ is constant. First we suppose $b \neq 0$. Then by Proposition 5.1 we get $\langle \nabla u, \nabla u \rangle = 0$.

Next we suppose that $b = 0$. We put $U(k) = U + kg$, where k is a positive constant. Then $U(k)$ is also symmetric tensor in D^2M. Now we put $B(k) = B(R, U(k))$. Then it can be another conformal curvature-like tensor on M defined in such a way that

$$B(k)_{ijkl} = B_{ijkl} - \frac{k}{n-1}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

Hence we have

$$\mathrm{Ric}(B(k)) = S - U(k) = S - U - kg,$$

and

$$b(k) = \mathrm{Tr}(\mathrm{Ric}(B(k))) = b - nk = -nk \neq 0.$$

Also we know that if B is parallel, so is $B(k)$. Accordingly, we are able to apply Proposition 5.1 to such a situation $B(k) = B(R, U(k))$, so we have $b(k)\langle\nabla u(k), \nabla u(k)\rangle = 0$, where $u(k) = \mathrm{Tr}(U(k))$. Since $b(k)$ is not zero, we have $\langle\nabla u(k), \nabla u(k)\rangle = 0$. By the continuity, $\nabla u(k)$ converges to $\nabla u(0)$ as k tends to +0 and hence we have $\langle\nabla u(0), \nabla u(0)\rangle = 0$. It means that the scalar product $\langle\nabla u, \nabla u\rangle = 0$. It completes the proof. □

Then by this theorem we get the following which will be useful in the proof of our Main Theorem.

Corollary 5.1. *Let (M, g) be an $n(\geq 4)$-dimensional semi-Riemannian manifold. Let U be a symmetric tensor in D^2M with $u = \mathrm{Tr}U$ and let $B = B(R, U)$ be the conformal curvature-like tensor. If ∇u is not null and if $\nabla B = 0$, then the scalar-like curvature u is constant.*

6. Proof of main theorem

In this section we prove the main theorem stated in the introduction. Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold of index s $(0 \leq s \leq n)$ equipped with semi-Riemannian metric g and Riemannian connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M. Let U be the symmetric tensor in D^2M such that $\nabla u = \nabla(\mathrm{Tr}U)$ is not null and let $B = B(R, U)$ be the conformal curvature-like tensor. Now we are going to verify our main Theorem by using several steps given in previous sections.

We assume that B is parallel. Then B satisfies the second Bianchi identity and hence by Theorem 2.1, U is the Weyl tensor.

On the other hand, the scalar-like curvature $u = \mathrm{Tr}U$ for B is constant by Theorem 5.1. This means that the symmetric tensor U becomes the Codazzi tensor. Namely it satisfies

$$U_{ijk} = U_{ikj}. \tag{6.1}$$

By (4.6) and the curvature-like properties of the conformal curvature-like tensor B, we have

$$\sum_{r,s} \epsilon_{rs} B_{riks} U_{rsj} = \sum_{r,s} \epsilon_{rs} B_{rijs} U_{rsk} = \sum_{r,s} \epsilon_{rs} B_{rjis} U_{rsk}, \tag{6.2}$$

which means that $\sum_{r,s} \epsilon_{rs} B_{rijs} U_{rsk}$ is symmetric in all indices i, j and k. On the other hand, by (4.12) we have

$$\sum_{r,s} \epsilon_{rs}\{(n-2)B_{rjks}U_{irs} - B_{rjks}U_{rsi} + B_{riks}U_{jrs} + B_{rijs}U_{krs}\} + \sum_r \epsilon_r B_{jr} U_{ikr} = 0,$$

where we have used the scalar-like curvature u is constant. Combining the above two equations we have

$$(n-1)\sum_{r,s} \epsilon_{rs} B_{rjks} U_{irs} + \sum_r \epsilon_r B_{jr} U_{ikr} = 0. \tag{6.3}$$

By Lemmas 5.1 and (6.1), we have the following

Lemma 6.1. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold. Let U be the symmetric tensor U in D^2M and let $B = B(R, U)$ be the conformal-like curvature tensor. Assume that ∇u is not null. If B is parallel, then we have*

$$\sum_r \epsilon_r B_{ir} U_{rjk} = \sum_r \epsilon_r B_{jr} U_{rik} = \sum_r \epsilon_r B_{kr} U_{rij}. \tag{6.4}$$

Putting $j = k$ in (6.3), multiplying ϵ_j and summing up with respect to the index j, we get

$$(n-1)\sum_{r,s} \epsilon_{rs} B_{rs} U_{irs} + \sum_{r,s} \epsilon_{rs} B_{rs} U_{irs} = 0,$$

and hence we get

$$\sum_{r,s} \epsilon_{rs} B_{rs} U_{irs} = 0. \tag{6.5}$$

Next we prove

Lemma 6.2. *Under the situation of* Lemma 6.1, *we have*

$$n\sum_r \epsilon_r B_{ir} U_{rjk} = bU_{ijk}, \tag{6.6}$$

where $b = \operatorname{Tr}\operatorname{Ric}(B) = \sum_i \epsilon_i B_{ii}$ and $B_{ij} = \sum_k \epsilon_k B_{ikkj}$.

Proof. Under the above situation, applying B_{ij} to the Ricci identity, we have

$$\sum_r \epsilon_r (R_{lkir} B_{rj} + R_{lkjr} B_{ir}) = 0.$$

Since B is parallel, we get

$$\sum_r \epsilon_r (R_{lkirh} B_{rj} + R_{lkjrh} B_{ir}) = 0.$$

By (4.1) we have

$$\sum_r \epsilon_r \{(\epsilon_k U_{lrh}\delta_{jk} - \epsilon_k U_{ljh}\delta_{kr} + \epsilon_l U_{jkh}\delta_{lr} - \epsilon_l U_{krh}\delta_{lj}) B_{ri} \\ + (\epsilon_k U_{lrh}\delta_{ik} - \epsilon_k U_{lih}\delta_{kr} + \epsilon_l U_{ikh}\delta_{lr} - \epsilon_l U_{krh}\delta_{li}) B_{rj}\} = 0$$

and hence we have

$$\sum_r \epsilon_{ri} B_{jr} U_{lrh}\delta_{ik} - \sum_r \epsilon_{ri} B_{jr} U_{krh}\delta_{il} + \sum_r \epsilon_{rj} B_{ir} U_{lrh}\delta_{jk} - \sum_r \epsilon_{rj} B_{ir} U_{krh}\delta_{lj} \\ - B_{jk} U_{lih} + B_{jl} U_{kih} - B_{ik} U_{ljh} + B_{il} U_{kjh} = 0. \tag{6.7}$$

Putting $j = k$ in (6.7), multiplying ϵ_j and summing up with respect to the index j, we get the conclusion. This completes the proof. □

Now let us denote by M'' the subset in M consisting of points x in M'' at which $\nabla R(x) \neq 0$. Then on such an open subset M'' we are able to prove the following lemma.

Lemma 6.3. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold with generalized non-null stress energy tensor. Let U be the symmetric tensor in D^2M and let $B = B(R, U)$ be the conformal curvature-like tensor. If B is parallel, then the generalized non-null stress energy tensor vanishes, that is*

$$B_{ij} = \frac{b}{n} g_{ij} = \frac{b}{n}\epsilon_i \delta_{ij} \tag{6.8}$$

on M'', where b denotes $Tr(Ric(B))$.

Proof. On the subset M'', substituting (6.6) into (6.7), we have

$$b\{\epsilon_i U_{jlh}\delta_{ik} - \epsilon_i U_{jkh}\delta_{il} + \epsilon_j U_{ilh}\delta_{jk} - \epsilon_j U_{ikh}\delta_{lj}\}/n - B_{jk} U_{lih} \\ + B_{jl} U_{kih} - B_{ik} U_{ljh} + B_{il} U_{kjh} = 0.$$

Transvecting $\epsilon_i \epsilon_k B_{ik}$ to the above equation, summing up with respect to i and k and taking account of (6.5) and (6.6), we have

$$(b^2/n - \langle \mathrm{Ric}(B), \mathrm{Ric}(B)\rangle) U_{ljh} = 0,$$

where $\langle \mathrm{Ric}(B), \mathrm{Ric}(B)\rangle = \sum_i \epsilon_i \epsilon_k B_{ik} B_{ik}$.

On the other hand, we know that

$$\begin{aligned}&\langle \mathrm{Ric}(B) - bg/n, \mathrm{Ric}(B) - bg/n\rangle \\ &\quad = \sum_{i,j}\epsilon_i\epsilon_j B_{ij}B_{ij} - \frac{2b}{n}\sum_{i,j}\epsilon_i\epsilon_j g_{ij}B_{ij} + \frac{b^2}{n^2}\sum_{i,j}\epsilon_i\epsilon_j g_{ij}g_{ij} \\ &\quad = \langle \mathrm{Ric}(B), \mathrm{Ric}(B)\rangle - \frac{b^2}{n}.\end{aligned}$$

Then from these equations we assert the following

$$b^2/n - \langle \mathrm{Ric}(B), \mathrm{Ric}(B)\rangle = 0 \text{ or } \nabla U = 0.$$

Then it can be easily seen that $U_{ljk} = 0$ if and only if $\nabla R = 0$, because $T = R$ in (1.3). Accordingly, under the assumption of Lemma 6.3 it satisfies $b^2/n - \langle \mathrm{Ric}(B), \mathrm{Ric}(B)\rangle = 0$ on $M'' = \{x \in M | \nabla R \neq 0\}$. Then the generalized non-null tensor implies $B_{ij} = b\epsilon_i\delta_{ij}/n$ on M''. It completes the proof. □

Now we are going to prove the semi-Riemannian version of Theorems due to Derdzinski and Roter [3], and Miyazawa [7].

Now by (6.3) and Lemma 6.2 we have

$$(n-1)\sum_r \epsilon_r\epsilon_s B_{rjks}U_{irs} + aU_{ijk} = 0, \qquad a = b/n. \tag{6.9}$$

Then (4.11) together with (6.8) and (6.9) imply that

$$\begin{aligned}\sum_r \epsilon_r B_{rpkl}U_{jqr} &= \sum_{r,s}\epsilon_r\epsilon_{sp}(B_{rjsl}\delta_{pk} + B_{sjkr}\delta_{pl})U_{rsq} \\ &\qquad - (n-1)\sum \epsilon_r B_{rjkl}U_{prq} - (B_{jl}U_{pkq} - B_{jk}U_{plq}) \\ &= -(n-1)\sum_r \epsilon_r B_{rjkl}U_{pqr} - a(\epsilon_j\delta_{jl}U_{pkq} - \epsilon_j\delta_{jk}U_{plq}) \\ &\qquad + \frac{a}{n-1}\left(\sum_p \epsilon_p U_{jlq}\delta_{pk} - \sum_p \epsilon_p U_{jkq}\delta_{pl}\right).\end{aligned}$$

where we have used the fact that the scalar-like curvature u is constant in Corollary 5.1. Repeating this equation, we get

$$\begin{aligned}\sum_r B_{rpkl}U_{jqr} &= (n-1)\{(n-1)\sum_r B_{rpkl}U_{jqr} + a(\epsilon_p\delta_{pl}U_{jkq} - \epsilon_p\delta_{pk}U_{jlq}) \\ &\qquad - \frac{a}{n-1}(\epsilon_j\delta_{jk}U_{plq} - \epsilon_j\delta_{jl}U_{pkq})\} - a(\epsilon_j\delta_{jl}U_{pkq} - \epsilon_j\delta_{jk}U_{plq}) \\ &\qquad + \frac{a}{n-1}(\epsilon_p\delta_{pk}U_{qjl} - \epsilon_p\delta_{pl}U_{jkq})\end{aligned}$$

from which it follows that

$$\sum_r \epsilon_r B_{rpkl} U_{jqr} = -\frac{a}{n-1}\{\epsilon_p \delta_{pl} U_{jkq} - \epsilon_p \delta_{pk} U_{jlq}\}. \tag{6.10}$$

Now we are going to prove the following:

Theorem 6.1. *Let M be an n(≥4)-dimensional semi-Riemannian manifold with generalized non-null stress energy tensor. Let U be the symmetric tensor in D^2M and let $B = B(R, U)$ be the conformal curvature-like tensor. Assume that ∇u is non-null vector. If B is parallel, then M is locally symmetric, conformally flat or $\langle \nabla U, \nabla U \rangle = 0$.*

Proof. By Theorem 5.1 the formula (4.10) can be written as follows:

$$\begin{aligned}
&\sum_r \epsilon_r \{\epsilon_h \delta_{ih} U_{prq} B_{rjkl} + \epsilon_h \delta_{jh} U_{prq} B_{irkl} + \epsilon_h \delta_{kh} U_{prq} B_{ijrl} + \epsilon_h \delta_{lh} U_{prq} B_{ijkr}\} \\
&- \{U_{piq} B_{hjkl} + U_{pjq} B_{ihkl} + U_{pkq} B_{ijhl} + U_{plq} B_{ijkh}\} \\
&+\{U_{hiq} B_{pjkl} + U_{kjq} B_{ipkl} + U_{hkq} B_{ijpl} + U_{hlq} B_{ijkp}\} \\
&- \left\{ \left(\sum_r \epsilon_r U_{hrq} B_{rjkl} \right) \epsilon_p \delta_{pi} + \left(\sum_r \epsilon_r U_{hrq} B_{irkl} \right) \epsilon_p \delta_{pj} \right. \\
&\left. + \left(\sum_r \epsilon_r U_{hrq} B_{ijrl} \right) \epsilon_p \delta_{pk} + \left(\sum_r \epsilon_r U_{hrq} B_{ijkr} \right) \epsilon_p \delta_{pl} \right\} = 0.
\end{aligned} \tag{6.11}$$

Now first let us calculate term by term in the left side as follows:

$$\sum_r \epsilon_r U_{hrq} B_{rjkl} = \sum_r \epsilon_r B_{rjkl} U_{hqr} = -\frac{a}{n-1}\{\epsilon_j \delta_{jl} U_{hkq} - \epsilon_j \delta_{jk} U_{hlq}\},$$

$$\sum_r \epsilon_r U_{hrq} B_{irkl} = -\sum_r \epsilon_r B_{rikl} U_{hqr} = \frac{a}{n-1}\{\epsilon_i \delta_{il} U_{hkq} - \epsilon_i \delta_{ik} U_{hlq}\},$$

$$\sum_r \epsilon_r U_{hrq} B_{ijrl} = \sum_r \epsilon_r B_{rlij} U_{hqr} = -\frac{a}{n-1}\{\epsilon_l \delta_{lj} U_{hiq} - \epsilon_l \delta_{li} U_{hjq}\},$$

$$\sum_r \epsilon_r U_{hrq} B_{ijkr} = -\sum_r \epsilon_r B_{rkij} U_{hqr} = \frac{a}{n-1}\{\epsilon_k \delta_{kj} U_{hiq} - \epsilon_k \delta_{ki} U_{hjq}\},$$

and

$$\sum_r \epsilon_r U_{prq} B_{rjkl} = \sum_r \epsilon_r B_{rjkl} U_{pqr} = -\frac{a}{n-1}\{\epsilon_j \delta_{jl} U_{pkq} - \epsilon_j \delta_{jk} U_{plq}\},$$

$$\sum_r \epsilon_r U_{prq} B_{irkl} = -\sum_r \epsilon_r B_{rikl} U_{pqr} = \frac{a}{n-1}\{\epsilon_i \delta_{il} U_{pkq} - \epsilon_i \delta_{ik} U_{plq}\},$$

$$\sum_r \epsilon_r U_{prq} B_{ijrl} = \sum_r \epsilon_r B_{rlij} U_{pqr} = -\frac{a}{n-1}\{\epsilon_l \delta_{lj} U_{piq} - \epsilon_l \delta_{li} U_{pjq}\},$$

$$\sum_r \epsilon_r U_{prq} B_{ijkr} = -\sum_r \epsilon_r B_{rkij} U_{pqr} = \frac{a}{n-1}\{\epsilon_k \delta_{kj} U_{piq} - \epsilon_k \delta_{ki} U_{pjq}\},$$

where we have used the formula (6.10).

Substituting these formulas into (6.11), we have

$$\begin{aligned}
&-\{U_{piq}B_{hjkl} + U_{pjq}B_{ihkl} + U_{pkq}B_{ijhl} + U_{plq}B_{ijkh}\}\\
&\quad + \{U_{hiq}B_{pjkl} + U_{kjq}B_{ipkl} + U_{hkq}B_{ijpl} + U_{hlq}B_{ijkp}\}\\
&\quad + \frac{a}{n-1}\{(\epsilon_j\delta_{jl}U_{hkq} - \epsilon_j\delta_{jk}U_{hlq})\epsilon_p\delta_{pi} - (\epsilon_i\delta_{il}U_{hkq} - \epsilon_i\delta_{ik}U_{hlq})\epsilon_p\delta_{pj}\\
&\quad + (\epsilon_l\delta_{lj}U_{hiq} - \epsilon_l\delta_{li}U_{hjq})\epsilon_p\delta_{pk} - (\epsilon_k\delta_{kj}U_{hiq} - \epsilon_k\delta_{ki}U_{hjq})\epsilon_p\delta_{pl}\}\\
&\quad + \frac{a}{n-1}\{-(\epsilon_j\delta_{jl}U_{pkq} - \epsilon_j\delta_{jk}U_{plq})\epsilon_h\delta_{ih} + (\epsilon_i\delta_{il}U_{pkq} - \epsilon_i\delta_{ik}U_{plq})\epsilon_h\delta_{jh}\\
&\quad - (\epsilon_l\delta_{lj}U_{piq} - \epsilon_l\delta_{li}U_{pjq})\epsilon_h\delta_{kh} + (\epsilon_k\delta_{kj}U_{piq} - \epsilon_k\delta_{ki}U_{pjq})\epsilon_h\delta_{hl}\} = 0,
\end{aligned} \tag{6.12}$$

where we have used the formula (6.10).

Now let us transvect U_{piq} to the first part of the left side of (6.12). Then each term of the first part can be given respectively as follows:

$$-\sum_{p,i,q}\epsilon_{piq}U_{piq}U_{piq}B_{hjkl} = -\langle\nabla U, \nabla U\rangle B_{hjkl},$$

$$-\sum_{p,i,q}\epsilon_{piq}B_{ihkl}U_{piq}U_{pjq} = \frac{a}{n-1}\epsilon_{pq}\{\epsilon_h\delta_{hl}U_{pkq} - \epsilon_h\delta_{hk}U_{plq}\}U_{pjq},$$

$$-\sum_{p,i,q}\epsilon_{piq}B_{ijhl}U_{piq}U_{pkq} = \frac{a}{n-1}\epsilon_{pq}\{\epsilon_j\delta_{jl}U_{phq} - \epsilon_j\delta_{jh}U_{plq}\}U_{pkq},$$

$$-\sum_{p,i,q}\epsilon_{piq}B_{ijkh}U_{piq}U_{plq} = \frac{a}{n-1}\epsilon_{pq}\{\epsilon_j\delta_{jh}U_{pkq} - \epsilon_j\delta_{jh}U_{phq}\}U_{plq},$$

and

$$\sum_{p,i,q}\epsilon_{piq}B_{pjkl}U_{piq}U_{hiq} = -\frac{a}{n-1}\epsilon_{iq}\{\epsilon_j\delta_{jl}U_{pik} - \epsilon_j\delta_{jk}U_{ilq}\}U_{hiq},$$

$$\sum_{p,i,q}\epsilon_{piq}U_{kjq}B_{ipkl}U_{piq} = 0,$$

$$\sum_{p,i,q}\epsilon_{piq}B_{ijpl}U_{ipq}U_{hkq} = \frac{a}{n-1}\epsilon_q U_{ljq}U_{hkq},$$

$$\sum_{p,i,q}\epsilon_{piq}B_{ijkp}U_{piq}U_{hlq} = -\frac{a}{n-1}\epsilon_q U_{jkq}U_{hlq}.$$

By transvecting U_{piq} to the second part of (6.12), we have

$$\frac{a}{n-1}\left\{-\left(\sum_q \epsilon_q U_{jlq}U_{hkq} - \sum_q \epsilon_q U_{jkq}U_{hlq}\right)\right.$$
$$+\left(\sum_{i,q}\epsilon_{iq}\delta_{jl}U_{kiq}U_{hiq} - \sum_q \epsilon_q U_{klq}U_{hjq}\right)$$
$$\left.-\left(\sum_{i,q}\epsilon_{iq}\epsilon_k\delta_{kj}U_{liq}U_{hiq} - \sum_q \epsilon_q U_{lkq}U_{hjq}\right)\right\},$$

where we have used $u_q = \sum_i \epsilon_i U_{iiq} = 0$ in Corollary 5.1, which means that the scalar-like curvature u is constant, in the calculation of the first term of the second part.

Finally, the transvection U_{piq} to the third part of (6.12) gives

$$\frac{a}{n-1}\left\{-\left(\epsilon_j\delta_{jl}\sum_{p,q}\epsilon_{pq}U_{phq}U_{pkq} - \epsilon_j\delta_{jk}\sum_{p,q}\epsilon_{pq}U_{phq}U_{plq}\right)\right.$$
$$-\left(\epsilon_l\delta_{lj}\sum_{p,i,q}\epsilon_{piq}U_{piq}U_{piq} - \sum_{p,q}\epsilon_{pq}U_{plq}U_{pjq}\right)\epsilon_h\delta_{kh}$$
$$\left.+\left(\epsilon_k\delta_{kj}\sum_{p,i,q}\epsilon_{piq}U_{piq}U_{piq} - \sum_{p,q}\epsilon_{pq}U_{pkq}U_{pjq}\right)\epsilon_h\delta_{hl}\right\}.$$

Summing up all of these formulas, the transvection U_{piq} to (6.12) implies

$$\langle\nabla U, \nabla U\rangle B_{hjkl} = \frac{a}{n-1}\langle\nabla U, \nabla U\rangle\epsilon_{hj}(\delta_{hl}\delta_{jk} - \delta_{jl}\delta_{hk}), \tag{6.13}$$

where we have put $\langle\nabla U, \nabla U\rangle = \sum_{p,i,q}\epsilon_{piq}U_{piq}U_{piq}$.

Now let us consider only two cases.

For the first let us consider the open set $M - M''$. Then on such an open set we know that M is locally symmetric, that is, $\nabla R = 0$.

Next we consider on the open set $M'' = \{x\in M|\nabla R(x) \neq 0\}$. Then let us continue our discussion on such an open set M''. When $\langle\nabla U, \nabla U\rangle \neq 0$, we know from (6.13) that

$$B_{ijkl} = \frac{a}{n-1}\epsilon_i\epsilon_j(\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}).$$

Then for a curvature-like tensor $B = B(R, U)$ with components B_{ijkl} we have

$$R_{ijkl} = \frac{(n-2)a-u}{(n-1)(n-2)}\epsilon_i\epsilon_j(\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik})$$
$$+\frac{1}{n-2}(\epsilon_i S_{jk}\delta_{il} - \epsilon_i S_{jl}\delta_{ik} + \epsilon_j S_{il}\delta_{jk} - \epsilon_j\delta_{ik}\delta_{jl})$$

$$- \frac{a}{n-2}(\epsilon_i \delta_{jk}\delta_{il} - \epsilon_i \delta_{jl}\delta_{ik} + \epsilon_j \delta_{il}\delta_{jk} - \epsilon_j \delta_{ik}\delta_{jl}),$$

where we have used the fact that

$$B_{jk} = a g_{jk} = S_{jk} - U_{jk}.$$

Then the conformal curvature tensor C with component C_{ijkl} is given by

$$C_{ijkl} = \left\{ \frac{(n-2)a - u}{(n-1)(n-2)} - \frac{2a}{n-2} + \frac{r}{(n-1)(n-2)} \right\} \epsilon_i \epsilon_j (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) = 0.$$

That is, M is conformally flat. □

Remark 6.1. When M is a Riemannian manifold, the result $\langle \nabla U, \nabla U \rangle = 0$ mentioned in Theorem 6.1 implies that the symmetric tensor U is parallel on M, that is $\nabla U = 0$. Then the assumption of conformally symmetry $\nabla B = 0$ implies that $\nabla R = 0$, that is M is locally symmetric.

Corollary 6.1. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold. If ∇r is not null, where r is the scalar curvature, and if the conformal curvature tensor C is parallel, then M is locally symmetric or conformally flat.*

Proof. Let M^c be the subset of M consisting of points x at which $C(x) \neq 0$.

Now first let us consider our proof on such an open set M^c. We put $B = B(k) = B(R, U(k))$, where $U(k) = S + kg$, k is a constant. So we have

$$\begin{aligned} B_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\{\epsilon_i(U_{jk}\delta_{il} - U_{jl}\delta_{ik}) + \epsilon_j(U_{il}\delta_{jk} - U_{ik}\delta_{jl})\} \\ &+ \frac{1}{(n-1)(n-2)} u \epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \end{aligned}$$

where $U_{jk} = S_{jk} + k\epsilon_j \delta_{jk}$ and $u = u(k) = \mathrm{Tr} U(k) = r + nk$. We have then

$$B_{ijkl} = C_{ijkl} - \frac{k}{n-1}\epsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \qquad B_{jk} = C_{jk} - k\epsilon_j \delta_{jk} = -k\epsilon_j \delta_{jk},$$

because of $\mathrm{Ric}(C) = 0$. So we see that $b(k) = -nk \neq 0$. This means that the generalized stress energy tensor is non-null. Since C is parallel, so is also B. Moreover, $\nabla u = \nabla r$ is non-null by the assumption. Then by Theorem 6.1, M is locally symmetric, conformally flat or $\langle \nabla S, \nabla S \rangle = 0$. But on M^c the locally symmetry of M implies that the Ricci tensor S is parallel, that is $\nabla S = 0$ on M^c.

On the complement $M - M^c$ we have $C = 0$, that is, M is conformally flat. If $M - M^c$ is empty, then it satisfies $\nabla R = 0$ non M. This completes the proof. □

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