Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor

Young Jin Suh

Kyungpook National University, Department of Mathematics, Taegu, 702-701, Republic of Korea

ARTICLE INFO

Article history:
Received 16 July 2009
Accepted 19 June 2010
Available online 28 June 2010

MSC:
primary 53C40
secondary 53C15

Keywords:
Real hypersurfaces
Complex two-plane Grassmannians
Commuting Ricci tensor
Hopf hypersurfaces
Totally geodesic

ABSTRACT

In this paper, first we introduce the full expression for the Ricci tensor of a real hypersurface \( M \) in complex two-plane Grassmannians \( G_2(C^{m+2}) \) from the equation of Gauss. Next we prove that a Hopf hypersurface in complex two-plane Grassmannians \( G_2(C^{m+2}) \) with commuting Ricci tensor is locally congruent to a tube of radius \( r \) over a totally geodesic \( G_2(C^{m+1}) \). Finally it can be verified that there do not exist any Hopf Einstein hypersurfaces in \( G_2(C^{m+2}) \).

© 2010 Elsevier B.V. All rights reserved.

0. Introduction

In the geometry of real hypersurfaces in complex space forms \( M_m(c) \) or in quaternionic space forms \( Q_m(c) \) Kimura [1,2] (resp. Pérez and the author [3]) considered real hypersurfaces in \( M_m(c) \) (resp. in \( Q_m(c) \)) with commuting Ricci tensor, that is, \( S\phi = \phi S \) (resp. \( S\phi_h = \phi S \) \( i = 1, 2, 3 \)) where \( S \) and \( \phi \) (resp. \( S \) and \( \phi_h \), \( i = 1, 2, 3 \)) denote the Ricci tensor and the structure tensor of real hypersurfaces in \( M_m(c) \) (resp. in \( Q_m(c) \)).

In [1,2], Kimura has classified that a Hopf hypersurface \( M \) in complex projective space \( P_m(C) \) with commuting Ricci tensor is locally congruent of type (A), to a tube over a totally geodesic \( P_k(C) \), of type (B), to a tube over a complex quadric \( Q_{m-1} \), \( \cot^2 \theta = m - 2 \), of type (C), to a tube over \( P_1(C) \times P_{(m-1)/2}(C) \), \( \cot^2 \theta = \frac{1}{m-2} \) where \( m \) is odd, of type (D), to a tube over a complex two-plane Grassmanian \( G_2(C^5) \), \( \cot^2 \theta = \frac{3}{2} \) with \( m = 9 \), of type (E), to a tube over a Hermitian symmetric space \( SO(10)/U(5) \), \( \cot^2 \theta = \frac{3}{2} \) with \( m = 15 \).

The notion of Hopf hypersurfaces means that the structure vector \( \bar{e} \) defined by \( \bar{e} = -JN \) satisfies \( A\bar{e} = \alpha \bar{e} \), where \( J \) denotes a \( \bar{K} \)-ähler structure of \( P_m(C) \), \( N \) and \( A \) a unit normal and the shape operator of \( M \) in \( P_m(C) \) (see [4]).

On the other hand, for in a quaternionic projective space \( Q^p \) Pérez and the author [3] have classified real hypersurfaces in \( Q^p \) with commuting Ricci tensor \( S\phi_h = \phi S \) \( i = 1, 2, 3 \), where \( S \) (resp. \( \phi_h \)) denotes that the Ricci tensor (resp. the structure tensor) of \( M \) in \( Q^p \) is locally congruent of type \( A_1, A_2 \), that is, to a tube over \( Q^p \) with radius \( 0 < r < \frac{\pi}{4} \), \( k \in \{0, \ldots, m - 1\} \). The almost contact structure vector fields \( \{\xi_1, \xi_2, \xi_3\} \) are defined by \( \xi_i = -JN, i = 1, 2, 3 \), where \( \xi_i, i = 1, 2, 3 \), denote a quaternionic \( \bar{K} \)-ähler structure of \( Q^p \) and \( N \) a unit normal field of \( M \) in \( Q^p \). Moreover, Pérez and the present author [5]
have considered the notion of $\nabla_{\xi} R = 0$, $i = 1, 2, 3$, where $R$ denotes the curvature tensor of a real hypersurface $M$ in $\mathbb{Q}P^m$, and proved that $M$ is locally congruent to a tube of radius $\frac{r}{2}$ over $\mathbb{Q}P^k$.

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor is not so simple and will be quite different from the cases mentioned above.

So in this paper we consider a real hypersurface $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $S\phi = \phi S$, where $S$ and $\phi$ denote the Ricci tensor and the structure tensor of $M$ in $G_2(\mathbb{C}^{m+2})$, respectively. The curvature tensor $R(X, Y)Z$ of $M$ in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $R(X, Y)Z$ of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for any vector fields $X$, $Y$ and $Z$ on $M$. Then by contraction and using the geometric structure $J_iJ_i = \delta_{i\bar{j}}$, $i = 1, 2, 3$, connecting the Kähler structure $J$ and the quaternionic Kähler structure $J_i$, $i = 1, 2, 3$, we can derive the Ricci tensor $S$ given by (see Section 3)

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_{4m-1}\}$ denotes a basis of the tangent space $T_xM$ of $M$, $x \in M$, in $G_2(\mathbb{C}^{m+2})$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\xi$ not containing $J$ (see [6,7]). So, for in $G_2(\mathbb{C}^{m+2})$ we have two natural geometrical conditions for real hypersurfaces: that $[\xi] = \text{Span} \{\xi\}$ or $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of geometric conditions Berndt and the present author [6] have proved the following:

**Theorem A.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.

When the structure vector field $\xi$ of $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant under the shape operator $A$, $M$ is said to be a Hopf hypersurface. In such a case the integral curves of the structure vector field $\xi$ are geodesics (see [7]). The flow generated by the integral curves of the structure vector field $\xi$ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a geodesic Reeb flow.

On the other hand, we say that the Reeb vector field is killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field $\xi$, which gives a characterization of real hypersurfaces of type (A) in Theorem A. Moreover, it was verified in [8] that $\mathcal{L}_\xi g = 0$ is equivalent to $\mathcal{L}_\xi A = 0$ for the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$.

When the Ricci tensor $S$ of $M$ in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor $\phi$, we say that $M$ has a commuting Ricci tensor. In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ belongs to either the three-dimensional distribution $\mathcal{D}^\perp$ or to the orthogonal complement $\mathcal{D}$ such that $T_xM = \mathcal{D} \oplus \mathcal{D}^\perp$. The case (A) in Theorem A is just the case where the one-dimensional distribution $[\xi]$ belongs to the distribution $\mathcal{D}^\perp$. Of course they satisfy that the Reeb vector $\xi$ is Killing, that is, the structure tensor $\phi$ commutes with the shape operator $A$. But it is not difficult to check that the Ricci tensor $S$ of real hypersurfaces of type (B) mentioned in Theorem A cannot commute with the structure tensor $\phi$. Moreover, in Section 5 we can check that any real hypersurface of type (A) in Theorem A has a commuting Ricci tensor.

In this paper we consider such a converse problem and want to give a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $S\phi = \phi S$ as follows:

**Theorem.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geq 3$. Then $M$ is locally congruent to a tube of radius $r$ over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

On the other hand, it is known that the Ricci tensor $S$ of an Einstein hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is given by $S = \phi g$ for a constant $\phi$ and a Riemannian metric $g$ defined on $M$. Naturally the Ricci tensor $S$ commutes with the structure tensor $\phi$, that is, $S\phi = \phi S$. So by virtue of our theorem mentioned above it becomes a hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$. But by Proposition C in Section 5 it can be easily checked that any tubes of radius $r$ over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ cannot be Einstein (see [9]). Then, as an application of our theorem in the direction of mathematical physics, we assert the following:

**Corollary.** There do not exist any Hopf Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

In Section 2 we recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and in Section 3 we will show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. The formula for the Ricci tensor $S$ and its covariant derivative $\nabla S$ will be shown explicitly in this section. In Sections 4 and 5 we will give a complete proof of the main theorem according to the geodesic Reeb flow satisfying $\xi \in \mathcal{D}$ or the geodesic Reeb flow satisfying $\xi \in \mathcal{D}^\perp$.

1. **Riemannian geometry of $G_2(\mathbb{C}^{m+2})$**

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$; for details we refer the reader to [10,6,7,11]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m + 2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the
homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(C^{m+2})$ becomes analytic. Denote by $g_0$ and $t$ the Lie algebra of $G$ and $K$, respectively, and by $m$ the orthogonal complement of $t$ in $g_0$ with respect to the Cartan–Killing form $B$ of $g_0$. Then $g_0 = t \oplus m$ is an $Ad(K)$-invariant reductive decomposition of $g_0$. We put $o = eK$ and identify $T_o G_2(C^{m+2})$ with $m$ in the usual manner. Since $B$ is negative definite on $g_0$, its negative restricted to $m \times m$ yields a positive definite inner product on $m$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(C^{m+2})$. In this way $G_2(C^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximal sectional curvature of $(G_2(C^{m+2}), g)$ is 8.

The Lie algebra $t$ has the direct sum decomposition $t = \text{su}(m) \oplus \text{su}(2) \oplus \mathbb{R}$, where $\mathbb{R}$ is the center of $t$. Viewing $t$ as the holonomy algebra of $G_2(C^{m+2})$, the center $\mathbb{R}$ induces a Kähler structure $J$ and the $\text{su}(2)$ part a quaternionic Kähler structure $\tilde{J}$ on $G_2(C^{m+2})$. If $J_1$ is any almost Hermitian structure in $\tilde{J}$, then $J_1 = J_1J$ and $J_1J$ is a symmetric endomorphism with $(J_1J)^2 = I$ and $\text{tr}(J_1J) = 0$. This fact will be used in later sections.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{g}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{g}$ such that $J_\nu J_{\nu + 1} = J_{\nu + 1} J_\nu = - J_\nu J_\nu$, where the index is taken modulus $3$. Since $\tilde{J}$ is parallel with respect to the Riemannian connection $\nabla$ of $(G_2(C^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\tilde{J}$ three local 1-forms $q_1, q_2, q_3$ such that

$$\tilde{\nabla}_J J = q_{\nu + 2}(X) J_{\nu + 1} - q_{\nu + 1}(X) J_{\nu + 2}$$

(1.1)

for all vector fields $X$ on $G_2(C^{m+2})$.

Let $p \in G_2(C^{m+2})$ and $W$ a subspace of $T_p G_2(C^{m+2})$. We say that $W$ is a quaternionic subspace of $T_p G_2(C^{m+2})$ if $JW \subset W$ for all $J \in J_p$. And we say that $W$ is a totally complex subspace of $T_p G_2(C^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{g}$ of $J_p$ such that $JW \subset W$ for all $J \in \mathfrak{g}$ and $\mathfrak{g} \perp W$ for all $J \in \mathfrak{g}$. Here, the orthogonal complement of $\mathfrak{g}$ in $J_p$ is taken with respect to the bundle metric and orientation on $\tilde{J}$ for which any local oriented orthonormal frame field of $\tilde{J}$ is a canonical local basis of $\mathfrak{g}$. A quaternionic (resp. totally complex) submanifold of $G_2(C^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(C^{m+2})$.

The Riemannian curvature tensor $R$ of $G_2(C^{m+2})$ is locally given by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ + \sum_{\nu = 1}^{3} \{g(J_{\nu}Y, Z)J_{\nu}X - g(J_{\nu}X, Z)J_{\nu}Y - 2g(J_{\nu}X, Y)J_{\nu}Z\}$$

(1.2)

where $J_1, J_2, J_3$ is any canonical local basis of $\tilde{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(C^{m+2})$

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let $M$ be a real hypersurface in $G_2(C^{m+2})$, that is, a submanifold in $G_2(C^{m+2})$ with real codimension 1. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$.

The Kähler structure $J$ of $G_2(C^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $J$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$. Using the above expression (1.2) for the curvature tensor $R$, the Gauss and the Codazzi equations are respectively given by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ$$

$$+ \sum_{\nu = 1}^{3} \{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\}$$

$$+ \sum_{\nu = 1}^{3} \{g(\phi_\nu JY, Z)\phi_\nu JX - g(\phi_\nu JX, Z)\phi_\nu JY\} - \sum_{\nu = 1}^{3} \{\eta(\nu)\eta(J_{\nu}Y)\phi_\nu X - \eta(X)\eta(J_{\nu}X)\phi_\nu Y\}$$

$$- \sum_{\nu = 1}^{3} \{\eta(X)g(\phi_\nu Y, Z) - \eta(Y)g(\phi_\nu X, Z)\} \xi_{\nu} + g(AY, Z)AX - g(AX, Z)AY$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \sum_{\nu = 1}^{3} \{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_{\nu}\}$$

$$+ \sum_{\nu = 1}^{3} \{\eta_\nu(\phi X)\phi_\nu Y - \eta_\nu(\phi Y)\phi_\nu X\} + \sum_{\nu = 1}^{3} \{\eta(X)\eta(J_{\nu}Y) - \eta(Y)\eta(J_{\nu}X)\} \xi_{\nu},$$

where $R$ denotes the curvature tensor of $M$ in $G_2(C^{m+2})$. 
The following identities can be proved in a straightforward manner and will be used frequently in subsequent calculations (see [12,9,8,11]):

\[
\begin{align*}
\phi_{v+1}\xi_v &= -\xi_{v+2}, & \phi_v\xi_{v+1} &= \xi_{v+2}, \\
\phi\xi_v &= \phi_0, & \eta_0(\phi X) &= \eta_0(\phi X), \\
\phi_v\phi_{v+1}X &= \phi_{v+2}X + \eta_{v+1}(X)\xi_v, & \phi_v\phi_{v+1}X &= -\phi_{v+2}X + \eta_v(X)\xi_{v+1}.
\end{align*}
\]

(2.1)

Now let us put

\[
JX = \phi X + \eta(X)N, & J_0X = \phi_0X + \eta_0(X)N
\]

for any tangent vector \(X\) of a real hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\), where \(N\) denotes a normal vector of \(M\) in \(G_2(\mathbb{C}^{m+2})\). Then from this and the formulas (1.1) and (2.1) we have that

\[
\begin{align*}
(\nabla_X\phi)Y &= \eta(Y)AX - g(AX, Y)\xi, & \nabla_X\xi &= \phi AX, \\
\nabla_{\phi v}X &= q_{v+2}(X)\xi_{v+1} - q_{v+1}(X)\xi_{v+2} + \phi_v AX, \\
(\nabla_X\phi_v)Y &= -q_{v+1}(X)\phi_{v+2}Y + q_{v+2}(X)\phi_{v+1}Y + \eta_0(Y)AX - g(AX, Y)\xi_v.
\end{align*}
\]

(2.2)

(2.3)

(2.4)

Summing these formulas, we find the following:

\[
\begin{align*}
\nabla_X(\phi_0\xi) &= \nabla_X(\phi_0\xi), \\
&= (\nabla_X\phi)\xi_v + \phi(\nabla_X\xi_v) \\
&= q_{v+2}(X)\phi_{v+1}\xi_v - q_{v+1}(X)\phi_{v+2}\xi_v + \phi_0 AX - g(AX, \xi)\xi_v + \eta(\xi)AX.
\end{align*}
\]

(2.5)

Moreover, from \(\mathcal{J}v = \mathcal{J}_0\mathcal{J}, \nu = 1, 2, 3\), it follows that

\[
\phi_0\phi_vX = \phi_0\phi_X + \eta_0(X)\xi_v - \eta(X)\xi_v.
\]

(2.6)

3. Proof of main theorem

In this section let us consider a real hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\) with commuting Ricci tensor, that is, \(S\phi = \phi S\).

Now let us contract \(Y\) and \(Z\) in the equation of Gauss in Section 2. Then the Ricci tensor \(S\) of a real hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\) is given by

\[
SX = \sum_{i=1}^{4m-1} R(X, e_i) e_i
\]

\[
= (4m + 10)X - 3\eta(X)\xi_v - 3 \sum_{v=1}^{3} \eta_v(X)\xi_v + 3 \sum_{v=1}^{3} \{\text{Tr} \phi_0\phi_v\phi_0 X - (\phi_0\phi)^2 X\}
\]

\[
- \sum_{v=1}^{3} \eta_v(\xi)\phi_0\phi_v X - \eta(X)\phi_0\phi_v \xi_v - \sum_{v=1}^{3} \{\text{Tr} \phi_0\phi_v\eta(X) - \eta_0(\phi_0\phi_v)X\} + hAX - A^2X,
\]

(3.1)

where \(h\) denotes the trace of the shape operator \(A\) of \(M\) in \(G_2(\mathbb{C}^{m+2})\). From the formula \(\mathcal{J}v = \mathcal{J}_0\mathcal{J}, \text{Tr} \mathcal{J}v = 0, \nu = 1, 2, 3\), we calculate the following for any basis \(\{e_1, \ldots, e_{4m-1}, N\}\) of the tangent space of \(G_2(\mathbb{C}^{m+2})\):

\[
0 = \text{Tr} \mathcal{J}v
\]

\[
= \sum_{k=1}^{4m-1} g(\mathcal{J}v e_k, e_k) + g(\mathcal{J}v N, N)
\]

\[
= \text{Tr} \phi_0\phi_v - \eta_0(\xi) - g(J_0N, JN)
\]

\[
= \text{Tr} \phi_0\phi_v - 2\eta_0(\xi)
\]

(3.2)

and

\[
(\phi_0\phi)^2 X = \phi_0\phi(\phi_0\phi X - \eta_0(X)\xi_v + \eta(X)\xi_v)
\]

\[
= \phi_0(-\phi_0 X + \eta(\phi_0 X)\xi_v + \eta(X)\phi_0^2\xi_v
\]

\[
= \phi_0 X - \eta_0(X)\xi_v + \eta(\phi_0 X)\phi_0\xi_v + \eta(X)\xi_v\{-\xi_v + \eta_0(\xi)\xi_v\}.
\]

(3.3)

Substituting (3.2) and (3.3) into (3.1), we have

\[
SX = (4m + 10)X - 3\eta(X)\xi_v - 3 \sum_{v=1}^{3} \eta_v(X)\xi_v + 3 \sum_{v=1}^{3} \{\eta_v(\xi)\phi_0\phi_v X - \eta(\phi_0 X)\phi_0\xi_v - \eta(X)\eta_v(\xi)\xi_v\} + hAX - A^2X
\]

\[
= (4m + 7)X - 3\eta(X)\xi_v - 3 \sum_{v=1}^{3} \eta_v(X)\xi_v + 3 \sum_{v=1}^{3} \{\eta_v(\xi)\phi_0\phi_v X - \eta(\phi_0 X)\phi_0\xi_v - \eta(X)\eta_v(\xi)\xi_v\} + hAX - A^2X.
\]

(3.4)
Now let us take a covariant derivative of $S\phi = \phi S$. This gives that

$$(\nabla Y S)\phi X + S(\nabla Y \phi)X = (\nabla Y \phi)SX + \phi(\nabla Y S)X. \quad (3.5)$$

Then the first term of (3.5) becomes

$$(\nabla Y S)\phi X = -3g(\phi AY, \phi X)\xi - 3 \sum_{v=1}^{3} \{q_{v+2}(Y)\eta_{v+1}(\phi X) - q_{v+1}(Y)\eta_{v+2}(\phi X) + g(\phi AY, \phi X)\}\xi_v$$

$$- 3 \sum_{v=1}^{3} \eta_v(\phi X)\{q_{v+2}(Y)\xi_{v+1} - q_{v+1}(Y)\xi_{v+2} + \phi_v A\phi X\}$$

$$+ 3 \sum_{v=1}^{3} \left[ Y(\eta_v(\xi))\phi_v\phi^2 X + \eta_v(\xi)\{-q_{v+1}(Y)\eta_{v+2}\phi^2 X + q_{v+2}(Y)\phi_{v+1}\phi^2 X + \eta(\phi^2)AY - g(AY, \phi^2)\xi_v - \eta(\xi)g(AY, \phi X)\phi_v - g(\phi AY, \phi_v \phi X)\phi_v \xi_v\} + \{q_{v+1}(Y)\eta(\phi_{v+2} X) - q_{v+2}(Y)\eta(\phi_{v+1} X) - \eta_v(\phi X)\eta(AY) + \eta(\xi)g(AY, \phi X)\}\phi_v \xi_v$$

$$- \eta(\phi_v X)\{q_{v+2}(Y)\phi_{v+1} \xi_v - q_{v+1}(Y)\phi_{v+2} \xi_v + \phi_v \phi AY - \eta(\phi X)\xi_v + \eta(\xi)\phi(AY) - g(\phi AY, \phi X)\eta_v(\xi)\xi_v\} \bigg] + (\nabla h)\phi AX + h\phi(\nabla Y A)X - \phi(\nabla Y A^2)X. \quad (3.6)$$

The second term of (3.5) becomes

$$S(\nabla Y \phi)X = \eta(X) \left[ (4m + 7)AY - 3\eta(AY)\xi - 3 \sum_{v=1}^{3} \eta_v(AY)\xi_v + 3 \sum_{v=1}^{3} \{\eta_v(\xi)\phi_v\phi AY - \eta(\phi_v AY, \phi_v \xi) - \eta(AY)\eta_v(\xi)\xi_v\} \right]$$

$$+ hA^2 Y - A^2 Y - g(AY, X) \left[ (4m + 7)\xi - 3\xi - 4 \sum_{v=1}^{3} \eta_v(\xi)\xi_v + h\xi - A^2 \xi \right].$$

The first term of the right side in (3.5) becomes

$$(\nabla Y \phi)SX = \eta(SX)AY - g(AY, SX)\xi,$$

and the second term of the right side in (3.5) is given by

$$\phi(\nabla Y S) = -3\eta(X)\phi^2 AY - 3 \sum_{v=1}^{3} \{q_{v+2}(Y)\eta_{v+1}(X) - q_{v+1}(Y)\eta_{v+2}(X) + g(\phi AY, \phi X)\}\phi\xi_v$$

$$- 3 \sum_{v=1}^{3} \eta_v(X)\{q_{v+2}(Y)\phi\xi_{v+1} - q_{v+1}(Y)\phi\xi_{v+2} + \phi_v A\phi\}$$

$$+ 3 \sum_{v=1}^{3} \left[ Y(\eta_v(\xi))\phi_v\phi X + \eta_v(\xi)\{-q_{v+1}(Y)\phi_{v+2}\phi X + q_{v+2}(Y)\phi_{v+1}\phi X + \eta(\phi XY)\phi_v\phi AY - g(AY, X)\phi_v\xi_v\} \right]$$

$$+ \eta(\phi X)\phi AY + \eta(\phi X)\phi AY - g(AY, X)\phi_v\xi_v\} + \eta_v(\xi)\{\eta(X)\phi_v AY - g(AY, X)\phi_v\xi_v\}$$

$$- g(\phi AY, \phi_v X)\phi_v\xi_v + \{q_{v+1}(Y)\eta(\phi_{v+2} X) - q_{v+2}(Y)\eta(\phi_{v+1} X) - \eta_v(X)\eta(AY)\}$

$$+ \eta(\xi)g(AY, X)\phi_v\xi_v - \eta(\phi_v X)\{q_{v+2}(Y)\phi_{v+1} \xi_v - q_{v+1}(Y)\phi_{v+2} \xi_v + \phi_v \phi AY$$

$$- \eta(AY)\phi_v \xi_v + \eta(\xi)\phi AY - g(\phi AY, X)\eta_v(\xi)\phi_v \xi_v - \eta(AY)\eta_v(\xi)\phi_v \xi_v - \eta(\xi)\eta_v(\xi)\phi_\nabla \xi_v\} \bigg] + (\nabla h)\phi AX + h\phi(\nabla Y A)X - \phi(\nabla Y A^2)X. \quad (3.6)$$

Putting $X = \xi$ into (3.5) and using that the structure vector $\xi$ is principal, that is, $A\xi = \alpha\xi$, then we have

$$S(\nabla Y \phi)\xi = \left[ (4m + 7)AY - 3\eta(AY)\xi - 3 \sum_{v=1}^{3} \eta_v(AY)\xi_v \right]$$

$$+ 3 \sum_{v=1}^{3} \{\eta_v(\xi)\phi_v AY - \eta(\phi_v AY, \phi_v \xi) - \alpha\eta(Y)\eta_v(\xi)\xi_v\} + hA^2 Y - A^2 Y$$

$$- \alpha\eta(Y) \left[ 4(m + 1)\xi - 4 \sum_{v=1}^{3} \eta_v(\xi)\xi_v + (\alpha h - \alpha^2)\xi \right].$$
Moreover, the right side of (3.5) becomes
\[
(\nabla_\gamma \phi)S\xi + \phi(\nabla_\gamma S)\xi = \eta(S\xi)AY - g(AY, S\xi)\xi + \phi(\nabla_\gamma S)\xi
\]
\[
= \left[ 4(m + 1) + ha - \alpha^2 \right] - 4 \sum_{v=1}^{3} \eta_v(\xi)^2 AY - 3\eta(X)\phi^2 AY
\]
\[- \left\{ 4(m + 1)\alpha + ha\alpha^2 - \alpha^3 \right\} \eta(Y) - 4 \sum_{v=1}^{3} \eta_v(\xi)\eta_v(AY) \right] \xi
\]
\[-3 \sum_{v=1}^{3} \left[ q_v + 2(Y)\eta_{v+1}(\xi) - q_{v+1}(Y)\eta_{v+2}(\xi) + \eta_v(\phi AY)\phi\xi_v \right]
\]
\[-3 \sum_{v=1}^{3} \eta_v(\xi)\left[ q_v + 2(Y)\phi\xi_{v+1} - q_{v+1}(Y)\phi\xi_{v+2} + \phi\phi_v AY \right]
\]
\[+ \sum_{v=1}^{3} \left[ \eta_v(\xi)\left( \phi\phi_v AY - \alpha\eta(Y)\phi^2\xi_v \right) - g(\phi AY, \phi\xi_v)\phi^2\xi_v \right.\]
\[-Y(\eta_v(\xi))\phi\xi_v - \eta_v(\xi)\phi\nabla_\gamma \xi_v \right] + h\phi(\nabla_\gamma A)\xi - \phi(\nabla_\gamma A^2)\xi.\]

From this, putting \( Y = \xi \) into \( L = R \), then it follows that
\[
0 = \sum_{v=1}^{3} \left[ q_v + 2(\xi)\eta_{v+1}(\xi) - q_{v+1}(\xi)\eta_{v+2}(\xi) \right] (\phi_v X_1 + \phi_v X_2)
\]
\[+ \sum_{v=1}^{3} \eta_v(\xi) \left[ q_v + 2(\xi)(\phi_v + \phi_v + \phi_v + \phi_v) - q_{v+1}(\xi)(\phi_v + \phi_v + \phi_v + \phi_v) - \alpha\xi_v + \alpha\eta(\xi_v)(X_1 + X_2) \right]. \tag{3.6}\]

Then by comparing the \( \mathcal{D} \) and \( \mathcal{D}^\perp \) components of (3.6), we have respectively
\[
0 = \sum_{v=1}^{3} \left[ q_v + 2(\xi)\eta_{v+1}(\xi) - q_{v+1}(\xi)\eta_{v+2}(\xi) \right] \phi_v X_1 + \alpha \sum_{v=1}^{3} \eta_v(\xi)^2 X_1
\]
\[+ \sum_{v=1}^{3} \eta_v(\xi) \left[ q_v + 2(\xi)\phi_v + \phi_v + \phi_v + \phi_v X_1 \right], \tag{3.7}\]
\[
0 = \sum_{v=1}^{3} \left[ q_v + 2(\xi)\eta_{v+1}(\xi) - q_{v+1}(\xi)\eta_{v+2}(\xi) \right] \phi_v X_2
\]
\[+ \sum_{v=1}^{3} \eta_v(\xi) \left[ q_v + 2(\xi)\phi_v + \phi_v + \phi_v + \phi_v X_2 \right]. \tag{3.8}\]

Taking an inner product (3.7) with \( X_1 \), we have
\[
\alpha \sum_{v=1}^{3} \eta_v(\xi)^2 = 0. \tag{3.9}\]

Then \( \alpha = 0 \) or \( \eta_v(\xi) = 0 \) for \( v = 1, 2, 3 \). So for a non-vanishing geodesic Reeb flow we have \( \eta_v(\xi) = 0, v = 1, 2, 3 \). This means that \( \xi \in \mathcal{D} \), which gives a contradiction to our assumption \( \xi = X_1 + X_2 \). Including this, we are able to assert the following:

**Lemma 3.1.** Let \( M \) be a Hopf hypersurface in \( G_2(S^{m+2}) \) with commuting Ricci tensor. Then the Reeb vector \( \xi \) belongs either to the distribution \( \mathcal{D} \) or to the distribution \( \mathcal{D}^\perp \).
**Proof.** When the geodesic Reeb flow is non-vanishing, that is \( \alpha \neq 0 \), (3.9) gives \( \xi \in \mathcal{D} \). When the geodesic Reeb flow is vanishing, we differentiate \( A\xi = 0 \). Then by Berndt and Suh [7] we know that

\[
\sum_{v=1}^{3} \eta_v(\xi) \eta_v(\phi Y) = 0.
\]

From this, on replacing \( Y \) by \( \phi Y \), it follows that

\[
\sum_{v=1}^{3} \eta_v^2(\xi) \eta(Y) = 0.
\]

So if there are some \( Y \in \mathcal{D} \) such that \( \eta(Y) \neq 0 \), then \( \eta_v(\xi) = 0 \) for \( v = 1, 2, 3 \). This means that \( \xi \in \mathcal{D} \). If \( \eta(Y) = 0 \) for any \( Y \in \mathcal{D} \), then we know that \( \xi \in \mathcal{D}^\perp \). \( \square \)

### 4. Real hypersurfaces with geodesic Reeb flow satisfying \( \xi \in \mathcal{D} \)

Let us consider a Hopf hypersurface \( M \in G_2(C^{m+2}) \) with commuting Ricci tensor, that is, \( S\phi = \phi S \) and \( \xi \in \mathcal{D} \). From this, differentiating, we have

\[
(\nabla_Y S)\phi X + S(\nabla_Y \phi)X = (\nabla_Y \phi)SX + \phi(\nabla_Y S)X.
\] \hspace{1cm} (4.1)

In this section let us show that the distribution \( \mathcal{D} \) of \( M \in G_2(C^{m+2}) \) satisfies \( g(\mathcal{A}\mathcal{D}, \mathcal{D}^\perp) = 0 \) for the case \( \xi \in \mathcal{D} \).

Now using \( \xi \in \mathcal{D} \) in (4.1), the first term becomes

\[
(\nabla_Y S)\phi X = -3g(\phi AY, \phi X)\xi - 3 \sum_{v=1}^{3} \{q_{v+2}(Y)\eta_{v+1}(\phi X) - q_{v+1}(Y)\eta_{v+2}(\phi X) + g(\phi_v AY, \phi X)\} \xi_v
\]

\[
- 3 \sum_{v=1}^{3} \eta_v(\phi X)[q_{v+2}(Y)\xi_{v+1} - q_{v+1}(Y)\xi_{v+2} + \phi_v A\phi X]
\]

\[
+ \sum_{v=1}^{3} \left[ \{\alpha \eta(Y)\eta(\phi X) - \eta(X)\eta_v(\phi AY) - g(AY, \phi X)\} \phi_v \xi - \{q_{v+1}(Y)\eta_{v+2}(X) - q_{v+2}(Y)\eta_{v+1}(X) + \alpha \eta_v(\phi X)\eta(Y)\} \phi_v \xi
\]

\[
+ \eta_v(X)\{q_{v+2}(Y)\phi_v + q_{v+1}(Y)\phi_{v+2} + \phi_v A\phi Y - \alpha \eta(Y)\xi_v\}
\]

\[
+ (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X.
\]

The second term of (4.1) becomes

\[
S(\nabla_Y \phi)X = \eta(X)S(AY) - g(AY, X)S\xi
\]

\[
= \eta(X) \left[ (4m + 7)AY - 3\alpha \eta(Y)\xi - 3 \sum_{v=1}^{3} \eta_v(AY)\xi_v - 3 \sum_{v=1}^{3} \eta(\phi_v AY)\phi_v \xi + hA^2Y - A^2Y\right]
\]

\[
- g(AY, X) \left[ 4(m + 1)\xi + (\alpha \xi - \alpha^2)\xi\right].
\]

The first term of the right side in (4.1) becomes

\[
(\nabla_Y S)X = \eta(SX)AY - g(AY, SX)\xi
\]

\[
= 4(m + 1)\eta(X)AY + (\alpha \xi - \alpha^2)\eta(X)AY - g(AY, SX)\xi,
\]

and the second term of the right side in (4.1) is given by

\[
\phi(\nabla_Y S)X = -3\eta(\phi X)A^2Y - 3 \sum_{v=1}^{3} \{q_{v+2}(Y)\eta_{v+1}(X) - q_{v+1}(Y)\eta_{v+2}(X) + g(\phi_v AY, X)\} \phi_v \xi
\]

\[
- 3 \sum_{v=1}^{3} \eta_v(X)[q_{v+2}(Y)\phi_v \xi_{v+1} - q_{v+1}(Y)\phi_v \xi_{v+2} + \phi_v A\phi X] + \sum_{v=1}^{3} g(\phi AY, \phi_v X) \xi_v
\]

\[
- \sum_{v=1}^{3} \{q_{v+1}(Y)\eta(\phi_{v+2}X) - q_{v+2}(Y)\eta(\phi_{v+1}X) - \alpha \eta_v(X)\eta(Y)\} \xi_v
\]

\[
+ \sum_{v=1}^{3} \eta(\phi_v X)\{q_{v+2}(Y)\xi_{v+1} - q_{v+1}(Y)\xi_{v+2}\} - \phi_v \phi_v A\phi Y + \alpha \eta(Y) \phi_v \xi_v
\]

\[
+ (Yh)\phi A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X.
\]
Substituting these formulas into (4.1) and putting $X = \xi_\mu$ into the equation obtained, and next using that the structure vector $\xi$ is in $\mathcal{D}$ and (2.1), we have

$$-3g(AY, \xi_\mu) \xi + (Yh)A\phi\xi_\mu + h(\nabla_YA)\phi\xi_\mu - (\nabla_Y A^2)\phi\xi_\mu + [4(m + 1) + (h - \alpha)\alpha]g(AY, \xi_\mu)\xi$$

$$= -g(AY, S\xi_\mu)\xi - 4 \sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi_\mu) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi_\mu) + g(\phi\nu AY, \xi_\mu)\} \phi\xi_\mu$$

$$- 4 \{q_{\mu+2}(Y)\phi\xi_{\mu+1} - q_{\mu+1}(Y)\phi\xi_{\mu+2} + \phi\nu AY\} + 4 \sum_{\nu=1}^{3} g(\phi\nu AY, \phi\nu\xi_\mu)\xi_\nu$$

$$+ \alpha \eta(Y)\xi_\mu + (Yh)\phi A\xi_\mu + h\phi(\nabla_Y A)\xi_\mu - \phi(\nabla_Y A^2)\xi_\mu.$$

(4.2)

Putting $X = \xi_\mu$ into (3.4) and using $\xi \in \mathcal{D}$, we have

$$S\xi_\mu = (4m + 7)\xi_\mu - 3\xi_\mu + hA\xi_\mu - A^2\xi_\mu.$$

So the first term of the right side of (4.2) becomes

$$-g(AY, S\xi_\mu)\xi = -4(m + 1)g(AY, \xi_\mu)\xi - h\phi(A\xi_\mu, AY)\xi + g(A^2\xi_\mu, AY)\xi.$$

Then substituting this into (4.2), we have

$$4 \sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi_\mu) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi_\mu) + g(\phi\nu AY, \xi_\mu)\} \phi\xi_\nu$$

$$+ 4\{q_{\mu+2}(Y)\phi\xi_{\mu+1} - q_{\mu+1}(Y)\phi\xi_{\mu+2} + \phi\nu AY\}$$

$$+ [(8m + 5) + (h - \alpha)\alpha]g(AY, \xi_\mu)\xi + h\phi(\nabla_Y A)\xi_\mu - g(A^2\xi_\mu, AY)\xi$$

$$- 4 \sum_{\nu=1}^{3} g(\phi\nu AY, \phi\nu\xi_\mu)\xi_\nu - \alpha \eta(Y)\xi_\mu + (Yh)(A\phi - \phi A)\xi_\mu$$

$$+ h((\nabla_Y A)\phi - \phi(\nabla_Y A))\xi_\mu - ((\nabla_Y A^2)\phi - \phi(\nabla_Y A^2))\xi_\mu = 0.$$

(4.3)

From this, taking the inner product with $\xi$, we have

$$[(8m + 5) + (h - \alpha)\alpha]g(AY, \xi_\mu) + h\phi(\nabla_Y A)\xi_\mu = 0.$$

(4.4)

On the other hand, we have

$$g((\nabla_Y A)\phi\xi_\mu, \xi) = \alpha g(\phi AY, \phi\xi_\mu) - g(\lambda A\phi AY, \phi\xi_\mu).$$

$$g((\nabla_Y A^2)\phi\xi_\mu, \xi) = \alpha^2 g(\phi AY, \phi\xi_\mu) - g(A^2 A\phi AY, \phi\xi_\mu).$$

From this, together with (4.4), we have

$$[(8m + 5) + (h - \alpha)\alpha]A^2\xi_\mu + hA\phi A\phi\xi_\mu - A\phi A^2\phi\xi_\mu = 0.$$

(4.5)

Now putting $X = \xi$ in (4.1) and using $\xi \in \mathcal{D}$, then we have

$$\left[(4m + 7)AY - 3\alpha \eta(Y)\xi - 3 \sum_{\nu=1}^{3} \eta_{\nu}(AY)\xi_\nu - \sum_{\nu=1}^{3} \eta_{\nu}(\phi AY)\phi_{\nu}\xi + hA^2 Y - A^3 Y\right]$$

$$- \alpha \eta(Y) \{4(m + 1)\xi + \alpha(h - \alpha)\xi\}$$

$$= \left[[(4m + 1) + (h - \alpha)\alpha]AY - [(4m + 1) + (h - \alpha)\alpha^2] \eta(Y)\xi + (3 - \alpha h + \alpha^2)AY\right]$$

$$- (3\alpha - \alpha^2 h + \alpha^3) \eta(Y)\xi - 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi AY)\phi_{\nu}\xi_\nu + 3 \sum_{\nu=1}^{3} \eta_{\nu}(AY)\xi_\nu - h\phi A\phi AY + \phi A^2 A\phi AY.$$

(4.6)

From this, putting $Y = \xi_\mu$ and also using $\xi \in \mathcal{D}$, we have

$$2(4m + 7)A^2\xi_\mu - 2 \sum_{\nu=1}^{3} \eta_{\nu}(A^2\xi_\mu)\xi_\nu - 4 \sum_{\nu=1}^{3} \eta_{\nu}(\phi A\xi_\mu)\phi_{\nu}\xi + hA^2 \xi_\mu - A^3 \xi_\mu - h\phi A\phi A\xi_\mu + \phi A^2 A\phi A\xi_\mu = 0.$$

(4.7)
Then, by applying the structure tensor \( \phi \), we have

\[ hA\phi X - A^2\phi X = h\phi AX - \phi A^2X - 4 \sum_{v=1}^{3} \eta_{v}(X)\phi \xi_{v} + 4 \sum_{v=1}^{3} \eta_{v}(\phi X)\xi_{v}. \]  

(4.8)

Then by putting \( X = A\xi_{\mu} \) into (4.8) we have

\[ hA\phi A\xi_{\mu} - A^2\phi A\xi_{\mu} = h\phi A^2\xi_{\mu} - \phi A^2\xi_{\mu} - 4 \sum_{v=1}^{3} \eta_{v}(A\xi_{\mu})\phi \xi_{v} + 4 \sum_{v=1}^{3} \eta_{v}(\phi A\xi_{\mu})\xi_{v}. \]

From this, on applying \( \phi \) to the left side we know that

\[ h\phi A\phi A\xi_{\mu} - \phi A^2\phi A\xi_{\mu} = -hA^2\xi_{\mu} + A^2\xi_{\mu} + 4 \sum_{v=1}^{3} \eta_{v}(A\xi_{\mu})\phi \xi_{v} + 4 \sum_{v=1}^{3} \eta_{v}(\phi A\xi_{\mu})\phi \xi_{v}. \]  

(4.9)

Also, by putting \( X = \xi_{\mu} \) into (4.8) we have

\[ hA\phi \xi_{\mu} - A^2\phi \xi_{\mu} = h\phi A\xi_{\mu} - \phi A^2\xi_{\mu} - 4\phi \xi_{\mu} - 4\xi_{\mu}. \]  

(4.10)

From this, on applying \( A\phi \) to the left side and using that \( \xi \) is principal we have

\[ h\phi A\phi \xi_{\mu} - A\phi A^2\phi \xi_{\mu} = -hA^2\xi_{\mu} + A^2\xi_{\mu} + 4A\xi_{\mu} - 4A\phi \xi_{\mu}. \]  

(4.11)

Then substituting (4.11) into (4.5), we have

\[ A\phi \xi_{\mu} = \beta A\xi_{\mu}, \]  

(4.12)

where we have put \( \beta = \frac{1}{4} \{(8m + 9) + 2(h - \omega)\alpha\}. \)

On the other hand, substituting (4.9) into (4.7), we have

\[ hA^2\xi_{\mu} - A^3\xi_{\mu} = 3 \sum_{v=1}^{3} \eta_{v}(A\xi_{\mu})\xi_{v} + 4 \sum_{v=1}^{3} \eta_{v}(\phi A\xi_{\mu})\phi \xi_{v} - (4m + 7)A\xi_{\mu}. \]  

(4.13)

Now substituting (4.12) into (4.10), we have

\[ \beta(hA\xi_{\mu} - A^2\xi_{\mu}) = h\phi A\xi_{\mu} - \phi A^2\xi_{\mu} - 4\phi \xi_{\mu} - 4\xi_{\mu}. \]  

(4.14)

Then by applying the structure tensor \( \phi \) to the left side of (4.14), we have

\[ \beta(h\phi A\xi_{\mu} - \phi A^2\xi_{\mu}) = -(hA\xi_{\mu} - A^2\xi_{\mu}) + 4\xi_{\mu} - 4\phi \xi_{\mu}. \]

From this, together with (4.14), on applying the function \( \beta \) to both sides, we have

\[ \beta^2(hA\xi_{\mu} - A^2\xi_{\mu}) = \beta(h\phi A\xi_{\mu} - \phi A^2\xi_{\mu}) - 4\beta \phi \xi_{\mu} - 4\beta \xi_{\mu} \]

\[ = -(hA\xi_{\mu} - A^2\xi_{\mu}) + 4\xi_{\mu} - 4\phi \xi_{\mu} - 4\beta \phi \xi_{\mu} - 4\beta \xi_{\mu}. \]

Then we put this as follows:

\[ hA\xi_{\mu} - A^2\xi_{\mu} = \lambda \xi_{\mu} + \mu \phi \xi_{\mu}, \]  

(4.15)

where \( \lambda \) (resp. \( \mu \)) denotes \( -\frac{4(\beta - 1)}{\beta^2 + 1} \) (resp. \( \mu = -\frac{4(\beta + 1)}{\beta^2 + 1} \)). From this, together with (4.12), we have

\[ hA^2\xi_{\mu} - A^3\xi_{\mu} = \lambda A\xi_{\mu} + \mu A\phi \xi_{\mu} = (\lambda + \mu \beta)A\xi_{\mu}. \]  

(4.16)

On the other hand, by (4.13) the left side of (4.16) becomes

\[ (\lambda + \mu \beta + 4m + 7)A\xi_{\mu} = 3 \sum_{v=1}^{3} \eta_{v}(A\xi_{\mu})\xi_{v} + 4 \sum_{v=1}^{3} \eta_{v}(\phi A\xi_{\mu})\phi \xi_{v}. \]  

(4.17)

Then (4.17) gives the following for \( \xi \in \mathcal{D} \):

\[ (\lambda + \mu \beta + 4m + 7)g(A\xi_{\mu}, \phi \xi) = 4 \sum_{v=1}^{3} \eta_{v}(\phi A\xi_{\mu})g(\phi \xi, \phi \xi) 
= 4\eta_{3}(\phi A\xi_{\mu}) = -4g(A\xi_{\mu}, \phi \xi), \]

which means that \( g(A\phi \xi, \xi_{\mu}) = 0 \), because \( \lambda + \mu \beta + 4m + 11 > 0 \). Then (4.17), together with \( \lambda + \mu \beta + 4m + 7 > 0 \), gives \( g(A\xi_{\mu}, \xi_{\mu}) = 0 \). Then by Theorem A we know that \( M \) is locally congruent of type (B), that is, to a tube over a totally real and totally geodesic \( \mathbb{Q}^{p\alpha} \), \( m = 2n \), in \( G_{2}(\mathbb{C}^{m+2}) \). Concerned with such a tube, we are able to recall a proposition given
by Berndt and the present author [6] as follows:

**Proposition B.** Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathcal{D} \subset \mathcal{D}$, $A\xi = \alpha \xi$, and $\xi$ is tangent to $\mathcal{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures:

$$\alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathcal{J}\xi, \quad T_\gamma = \mathcal{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H} \mathcal{C}\xi)^\perp, \quad \mathcal{J}T_\lambda = T_\lambda, \quad \mathcal{J}T_\mu = T_\mu, \quad J\mathcal{J}T_\lambda = T_\mu.$$

Now it remains to check whether the Ricci tensor for real hypersurfaces of type (B) in Theorem A is commuting or not. So let us suppose that the Ricci tensor $S$ of type (B) is commuting, that is $S\phi = \phi S$. Then this gives (4.8). So if we consider an eigenvector $X \in T_\lambda$, by Proposition B we know that $\phi X \in T_\mu$. Then applying such a situation to (4.8), we have

$$(\lambda - \mu)(h - \lambda - \mu) = 0,$$

where the function $h$ denotes the trace of the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$.

Since $\lambda - \mu \neq 0$, we know that

$$h = \lambda + \mu = \cot r - \tan r = 2 \cot 2r.$$

By Proposition B, we also know that

$$h = -2 \tan 2r + 6 \cot 2r + (4n - 4)(\cot r - \tan r).$$

Then by comparing two formulas for the function $h$ we know that

$$\cot^2 2r = \frac{1}{2(2n - 1)}. \quad (4.18)$$

On the other hand, by putting $X = \xi_\mu, \mu = 1, 2, 3$, into (4.8) we have

$$\phi_\mu \xi + hA\phi_\xi_\mu - A^2 \phi_\xi_\mu = -3\phi_\xi_\mu + h\phi A\xi_\mu - \phi A^2 \xi_\mu.$$

In this formula, if we consider an eigenvector $\xi_\mu \in T_\beta$, then $\phi \xi_\mu \in T_\gamma, A\phi_\xi_\mu = 0, \phi A\xi_\mu = 2 \cot 2r \phi_\xi_\mu$, and $\phi A^2 \xi_\mu = (2 \cot 2r)^2 \phi_\xi_\mu$. So it follows that

$$(4 \cot^2 2r - 2h \cot 2r + 4) \phi_\mu \xi = 0,$$

where the trace $h$ is given by $h = -2 \tan 2r + 2(4n - 1) \cot 2r$. Then substituting this, we have another formula:

$$\cot^2 2r = \frac{1}{2n - 1}. \quad (4.19)$$

Then from (4.18) and (4.19) we have a contradiction. So we have shown that there do not exist any real hypersurfaces of type (B) satisfying $S\phi = \phi S$. Accordingly, we have proved that no real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor can exist for the case $\xi \in \mathcal{D}$.

**5. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathcal{D}^\perp$**

Now let us consider a Hopf hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and $\xi \in \mathcal{D}^\perp$. Now differentiating $S\phi = \phi S$ gives

$$(\nabla_\gamma S)\phi X + S(\nabla_\gamma \phi)X = (\nabla_\gamma \phi)SX + \phi(\nabla_\gamma S)X.$$

In this section by Lemma 3.1 we only discuss the geodesic Reeb flow $\xi$ belonging to the distribution $\mathcal{D}^\perp$. Since we have assumed that $\xi \in \mathcal{D}^\perp = \text{Span}(\xi_1, \xi_2, \xi_3)$, there exists a Hermitian structure $J_1 \in \mathcal{J}$ such that $JN = J_1 N$, that is, $\xi = \xi_1$. Then it follows that

$$\phi \xi_2 = \phi_2 \xi = \phi_2 \xi_1 = -\xi_3, \quad \phi \xi_3 = \phi_3 \xi_1 = -\xi_2. \quad (5.1)$$
From this, together with the expression for (3.4) and \( \xi \in \mathbb{D}^+ \), we have
\[
(4m + 1)g(AX, Y)\xi - 3 [q_3(Y)\eta_3(X) + q_2(Y)\eta_2(X)\xi_1 - q_1(Y)\eta_2(X)\xi_2 - q_1(Y)\eta_3(X)\xi_3] + 2\eta(X)\eta_2(AY)\xi_2
+ 2\eta(X)\eta_3(AY)\xi_3 + \sum_{i=1}^3 \eta_i(X)\phi_i\phi AY + (Yh)A\phi X + h(\nabla Y)\phi X - (\nabla Y)\phi X + \eta(X)[hA^2 Y - A^3 Y]
= [-g(AY, SX) - \eta_3(AY) - \eta_2(AY)]\xi + 4 [g(\phi_2AY, X)\xi_3 - g(\phi_3AY, X)\xi_2] - 3 \sum_{i=1}^3 \eta_i(X)\phi_i\phi AY
+ 4 \sum_{i=1}^3 g(\phi_i AY, \phi_i X)\xi_i + \eta_3(X)\phi_2 AY - \eta_2(AY)\phi_3 AY + (Yh)\phi AX + h(\nabla Y)A - \phi(\nabla Y^2)X.
\] (5.2)

Now putting \( X = \xi \) in (5.2), we have
\[
(4m + 1)g(AX, Y)\xi + 2\eta_2(AY)\xi_2 + 2\eta_3(AY)\xi_3 + \phi_1 AY + hA^2 Y - A^3 Y
= -g(AY, SX)\xi + 4 [g(\phi_2AY, \xi)\xi_3 - g(\phi_3AY, \xi)\xi_2] - 3\phi_1 AY + 4g(\phi AY, \phi_2 \xi)\xi_2
+ 4g(\phi AY, \phi_3 \xi)\xi_3 + h(\nabla Y)\xi - \phi(\nabla Y^2)\xi.
\]

From this, if we use the following formulas:
\[
S\xi = 4(m + 1)\xi - 4 \sum_{i=1}^3 \eta_i(\xi)\xi_i + hA\xi - A^2 \xi = (4m + h\alpha - \alpha^2)\xi
\]
and
\[
g(AY, S\xi) = \alpha(4m + h\alpha - \alpha^2)\eta(Y),
\]
then it follows that
\[
\phi_1\phi AY + hA^2 Y - A^3 Y = 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - h\phi A\phi AY + A^2\phi AY - 3\phi_1 AY.
\] (5.3)

On the other hand, by the equation of Codazzi in [6] (see page 6), we have
\[
A\phi AY = \phi Y + \sum_{i=1}^3 (\eta_i(Y)\phi_i\xi_i + \eta_i(\xi)\phi_i Y + \eta_i(\xi)\phi_i Y - 2\eta(Y)\eta_i(\xi)\phi_i Y - 2\eta_i(\xi)\eta_i(\phi Y)\xi_i + \alpha(A\phi + \phi A)Y
= \phi Y + \phi_1 Y + \phi_2 (Y)\xi_2 + \phi_3 (Y)\xi_3 + \eta_3(\phi Y)\xi_3 + \alpha(A\phi + \phi A)Y.
\] (5.4)

So for any \( Y \in \mathbb{D} \), (5.4) gives that \( A\phi AY = \phi Y + \phi_1 Y + \alpha(A\phi + \phi A)Y \). This implies
\[
\phi A^2\phi AY = \phi A(A\phi AY) = \phi A(\phi Y + \phi_1 Y)
= \phi A\phi Y + \phi A\phi_1 Y + \alpha\phi A(A\phi + \phi A)Y.
\]

From this, together with (5.3), it follows that
\[
\phi_1\phi AY + hA^2 Y - A^3 Y = 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - h(-Y + \phi_1 Y) - h\alpha(A\phi + \phi A)Y + \phi A\phi Y
+ \phi A\phi_1 Y + 3\phi_1\phi AY + \alpha\phi A(A\phi + \phi A)Y.
\] (5.5)

On the other hand, we calculate the following:
\[
S\phi Y = (4m + 7)\phi Y - 3\eta_2(\phi Y)\xi_2 - 3\eta_3(\phi Y)\xi_3 + \phi_1\phi^2 Y - \eta(\phi_2 Y)\phi_2\xi - \eta(\phi_3 Y)\phi_3\xi + h\phi A Y - A^2\phi Y,
\]
\[
\phi S Y = (4m + 7)\phi Y - 3 \sum_{i=1}^3 \eta_i(Y)\phi_i\xi_i + \phi A\phi_1 Y - \eta(\phi_2 Y)\phi_2\xi - \eta(\phi_3 Y)\phi_3\xi + h\phi A Y - \phi A^2 Y.
\]

So for any \( X \in \mathbb{D} \) the condition \( S\phi = \phi S \) implies that
\[-\phi_1 Y + hA\phi Y - A^2\phi Y = \phi_1\phi Y + h\phi A Y - \phi A^2 Y.
\]

Then by replacing \( Y \) by \( \phi Y \) for \( Y \in \mathbb{D} \) we have
\[
hA^2 Y - A^3 Y = -A\phi_1 Y + A\phi\phi_1 Y - hA\phi A\phi Y + A\phi A^2 \phi Y.
\] (5.6)

Now by using (5.4) for \( Y \in \mathbb{D} \), the terms in the right side become respectively
\[
A\phi A\phi Y = \phi^2 Y + \phi_1\phi Y + \alpha(A\phi + \phi A)Y
= -Y + \phi_1\phi Y + \alpha(A\phi + \phi A)Y
\]
and 

\[ A\phi A^2 Y = \phi A\phi Y + \phi_1 A\phi Y + \eta_2 (A\phi Y) \phi \xi_2 + \eta_3 (A\phi Y) \phi \xi_3 + \alpha (A\phi + \phi A) \phi Y. \]

From these, together with (5.5) and (5.6), we have

\[
\begin{align*}
\phi_1 \phi AY & - \phi_1 \phi Y + A\phi \phi Y + hY - h\phi_1 \phi Y - \alpha h (A\phi + \phi A) \phi Y + \phi A\phi Y + \alpha (A\phi + \phi A) \phi Y \\
& + \{ \phi_1 A\phi Y + \eta_2 (A\phi Y) \phi \xi_2 + \eta_3 (A\phi Y) \phi \xi_3 + \eta_2 (A\phi \phi Y) \phi \xi_2 + \eta_3 (A\phi \phi Y) \phi \xi_3 \} \\
& = 6\eta_2 (A\phi Y) \phi \xi_2 + 6\eta_3 (A\phi Y) \phi \xi_3 - h\phi_1 \phi Y + \phi A\phi Y - 3\phi_1 \phi AY + hY + \phi A\phi Y \\
& - \alpha h \phi (A\phi + \phi A) Y + \alpha \phi A (A\phi + \phi A) Y.
\end{align*}
\]

Then this can be rearranged as follows:

\[
\begin{align*}
\phi_1 \phi AY & - \phi_1 \phi Y + A\phi \phi Y + hY - h\phi_1 \phi Y + \alpha \phi_1 \phi Y \\
& + \{ \phi_1 A\phi Y + \eta_2 (A\phi Y) \phi \xi_2 + \eta_3 (A\phi Y) \phi \xi_3 + \eta_2 (A\phi \phi Y) \phi \xi_2 + \eta_3 (A\phi \phi Y) \phi \xi_3 \} \\
& = 6\eta_2 (A\phi Y) \phi \xi_2 + 6\eta_3 (A\phi Y) \phi \xi_3 - h\phi_1 \phi Y + \phi A\phi Y - 3\phi_1 \phi AY + \alpha \phi \phi Y
\end{align*}
\]

where we have used the following formulas obtained from (5.4):

\[ \alpha A\phi A\phi Y = -\alpha Y + \alpha \phi_1 \phi Y + \alpha^2 (A\phi + \phi A) \phi Y \]

and

\[ \alpha A\phi A\phi Y = -\alpha Y + \alpha \phi_1 \phi Y + \alpha^2 (A\phi + \phi A) Y. \]

Now let us take the inner product (5.7) with \( \xi_2 \). Then for any \( Y \in \mathcal{D} \) we have

\[
\begin{align*}
g (\phi_1 \phi AY, \xi_2) - g (\phi_1 \phi Y, A\xi_2) + g (\phi_1 \phi Y, A\xi_2) - (h - \alpha) g (\phi_1 \phi Y, \xi_2) - g (A\phi X, \phi_1 \xi_2) + \eta_2 (A\phi Y) + \eta_3 (A\phi Y) + \eta_2 (A\phi Y) \\
& = 6\eta_2 (A\phi Y) + g (\phi A\phi Y, \xi_2) - 3g (\phi A\phi Y, \xi_2) - (h - \alpha) g (\phi_1 \phi Y, \xi_2).
\end{align*}
\]

Then by a direct calculation in (5.8) for any \( Y \in \mathcal{D} \), we have

\[ \eta_3 (A\phi Y) = 2\eta_2 (A\phi Y) + \eta_3 (A\phi Y), \quad (5.9) \]

Similarly, if we take the inner product (5.7) with \( \xi_3 \), then it follows that

\[ -\eta_2 (A\phi Y) = 2\eta_2 (A\phi Y) - \eta_2 (A\phi Y), \quad (5.10) \]

for any vector field \( Y \in \mathcal{D} \). Then in this section we know that the distribution \( \mathcal{D} \) can be decomposed into two distributions \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) defined in such a way that

\[ \mathcal{D}_1 = \{ Y \in \mathcal{D} | \phi Y = \phi_1 Y \} \]

and

\[ \mathcal{D}_2 = \{ Y \in \mathcal{D} | \phi Y = -\phi_1 Y \}. \]

So first let us consider the distribution \( \mathcal{D}_1 \). The formulas (5.9) and (5.10) imply that \( \eta_v (AY) = 0 \) for any \( Y \in \mathcal{D}_1 \) and \( v = 1, 2, 3 \). Then we get our assertions on the distribution \( \mathcal{D}_1 \).

Next we consider the distribution \( \mathcal{D}_2 \). Then by (5.9) and (5.10) on such a distribution \( \mathcal{D}_2 \) we have

\[ \eta_3 (A\phi Y) = -\eta_3 (A\phi Y) \quad \text{and} \quad \eta_3 (A\phi Y) = \eta_2 (A\phi Y). \quad (5.11) \]

Substituting these formulas into (5.7), we have for any \( Y \in \mathcal{D}_2 \),

\[
\begin{align*}
\phi_1 \phi AY + \phi_1 A\phi Y &= 4\eta_2 (A\phi Y) \phi \xi_2 + 4\eta_3 (A\phi Y) \phi \xi_3 + \phi A\phi Y - 3\phi_1 \phi AY
\end{align*}
\]

From (5.4) and using \( \phi Y = -\phi_1 Y \), \( Y \in \mathcal{D}_2 \), we have

\[ A\phi AY = 0 \quad \text{and} \quad A\phi A\phi Y = 0. \]

So from this, together with (5.3) and (5.6), it follows that

\[
\begin{align*}
4\phi_1 \phi AY + hA^2 Y - A^2 Y &= 6\eta_2 (A\phi Y) \phi \xi_2 + 6\eta_3 (A\phi Y) \phi \xi_3 \\
& = 4\phi_1 \phi AY + A\phi A^2 \phi Y \\
& = 4\phi_1 \phi AY + A\phi A^2 \phi Y + \phi A\phi X + \phi_1 A\phi Y + \eta_2 (A\phi Y) \phi \xi_2 \\
& + \eta_3 (A\phi Y) \phi \xi_3 + \eta_2 (A\phi \phi Y) \phi \xi_2 + \eta_3 (A\phi \phi Y) \phi \xi_3
\end{align*}
\]

where in the first equality we have used \( A\phi A\phi Y = 0 \) and the fact that \( \phi_1 \phi AY = \phi A\phi Y + \eta (AY) \phi_1 Y = \phi_1 AY \) for any \( Y \in \mathcal{D}_2 \). Now let us consider eigenvectors \( Y, \phi Y \in \mathcal{D}_2 \) such that \( \phi Y = -\phi_1 Y \). Then we can put

\[ AY = \lambda Y + \sum_{i=1}^{3} \eta_v (AY) \phi \xi_v \]
Let \( M \) be a connected real hypersurface of \( G \). Theorem A 2

Then this also implies that

\[
\phi AY = \lambda \phi Y + \sum_{\nu=1}^{3} \eta_{\nu}(AY)\phi \xi_{\nu}.
\]

From these formulas and \((5.13)\) it follows that

\[
4 \left\{ \lambda Y + \sum_{\nu=1}^{3} \eta_{\nu}(AY)\phi_{1}\phi \xi_{\nu} \right\} + \tilde{\lambda} \left\{ Y + \sum_{\nu=1}^{3} \eta_{\nu}(AY)\phi_{1}\xi_{\nu} \right\} = 4\eta_{2}(AY)\xi_{2} + 4\eta_{3}(AY)\xi_{3} + \tilde{\lambda} \left\{ Y - \sum_{\nu=1}^{3} \eta_{\nu}(AY)\phi \xi_{\nu} \right\}.
\]

Then we have \( \lambda = 0 \) and similarly, \( \tilde{\lambda} = 0 \). So it follows that

\[
AY = 3 \sum_{\nu=1}^{3} \eta_{\nu}(AY)\xi_{\nu} = g(A\xi_{2}, Y)\xi_{2} + g(A\xi_{3}, Y)\xi_{3}.
\]

Then for \( \phi Y \in \mathcal{D}_{2} \) we know that

\[
A\phi Y = g(A\xi_{2}, \phi Y)\xi_{2} + g(A\xi_{3}, \phi Y)\xi_{3}.
\]

From this, applying \( \phi \) and \( \phi_{1} \) respectively, we have

\[
\phi A\phi Y = -g(A\xi_{2}, \phi Y)\xi_{3} + g(A\xi_{3}, \phi Y)\xi_{2}
\]

and

\[
\phi_{1}A\phi Y = g(A\xi_{2}, \phi Y)\xi_{3} - g(A\xi_{3}, \phi Y)\xi_{2}.
\]

From these formulas, together with \((5.13)\), we have

\[
\phi_{1}A\phi Y = \eta_{2}(AY)\xi_{2} + \eta_{3}(AY)\xi_{3}.
\]

Then by applying \( \phi_{1} \) we have

\[
\phi AY = \eta_{3}(AY)\xi_{2} - \eta_{2}(AY)\xi_{3}.
\]

By comparing \((5.15)\) and \((5.16)\), and using \((5.11)\), we know that

\[
A\phi = -\phi A
\]

on the distribution \( \mathcal{D}_{2} \). From this and \((5.4)\) it follows that for any \( Y \in \mathcal{D}_{2} \),

\[
0 = \phi Y + \phi_{1}Y = A\phi AY = -A^{2}\phi Y.
\]

So \( \phi Y \in \mathcal{D}_{2} \) gives \( A^{2}Y = 0 \). Then from this and \((5.5)\), and using \( \phi Y = -\phi_{1}Y \) we have

\[
6\eta_{2}(AY)\xi_{2} + 6\eta_{3}(AY)\xi_{3} - 4\phi_{1}A\phi Y = 0,
\]

where we have used that \( \phi \phi_{1}A\phi = \phi_{1}A\phi Y + \eta_{1}(AY)\xi = \phi_{1}A\phi Y \). From this, taking an inner product with \( \xi_{2} \) and using the formulas in Section 2, we have

\[
\eta_{2}(AY) = 0.
\]

Similarly, we can assert that \( \eta_{3}(AY) = 0 \) for any \( Y \in \mathcal{D}_{2} \). So combining this with the fact that \( \eta_{\nu}(AY) = 0 \) for any \( \nu = 1, 2, 3 \), and \( Y \in \mathcal{D}_{1} \), we have proved that \( \eta_{\nu}(AY) = 0 \) for any \( Y \in \mathcal{D}, \nu = 1, 2, 3 \). Accordingly, we have \( g(\mathcal{A}\mathcal{D}, \mathcal{D}^{\perp}) = 0 \) for Hopf hypersurfaces \( M \) in \( G_{2}(C^{m+2}) \) with commuting Ricci tensor and its Reeb vector \( \xi \in \mathcal{D}^{\perp} \). Then, by virtue of Theorem A we know that \( M \) is locally congruent to real hypersurfaces of type (A), that is, a tube over a totally geodesic \( G_{2}(C^{m+1}) \) in \( G_{2}(C^{m+2}) \). □

We introduce in Theorem A, relating to this kind of hypersurface, another proposition due to Berndt and the present author [6] as follows:

**Proposition C.** Let \( M \) be a connected real hypersurface of \( G_{2}(C^{m+2}) \). Suppose that \( \mathcal{A}\mathcal{D} \subset \mathcal{D}, \mathcal{A}\xi = \alpha \xi, \) and \( \xi \) is tangent to \( \mathcal{D}^{\perp} \). Let \( J_{1} \in \mathcal{J} \) be the almost Hermitian structure such that \( JN = J_{1}N \). Then \( M \) has three (if \( r = \pi/2 \)) or four (otherwise) distinct constant principal curvatures

\[
\alpha = \alpha_{i} = \sqrt{8} \cot \left( \sqrt{8}r \right), \quad \alpha_{j} = \alpha_{k} = \sqrt{2} \cot \left( \sqrt{2}r \right), \quad \lambda = -\sqrt{2} \tan \left( \sqrt{2}r \right), \quad \mu = 0
\]
with some \( r \in (0, \pi / \sqrt{8}) \). The corresponding multiplicities are
\[
m(\alpha_i) = 1, \quad m(\alpha_j) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),
\]
and for the corresponding eigenspaces we have
\[
\begin{align*}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\
T_\beta &= C^+\xi = C^+N, \\
T_\lambda &= \{X|X \perp \mathbb{R}\xi, JX = J_1X\}, \\
T_\mu &= \{X|X \perp \mathbb{R}\xi, JX = -J_1X\}.
\end{align*}
\]

In the paper [7] due to Berndt and the present author we have given a characterization of real hypersurfaces of type (A) in Theorem A when the shape operator \( A \) of \( M \) in \( G_2(\mathbb{C}^{m+2}) \) commutes with the structure tensor \( \phi \), which is equivalent to the condition that the Reeb flow on \( M \) is isometric, that is, \( \mathcal{L}_\xi g = 0 \), where \( \mathcal{L} \) (resp. \( g \)) denotes the Lie derivative (resp. the induced Riemannian metric) of \( M \) in the direction of the Reeb vector field \( \xi \). Namely, Berndt and the present author [7] proved the following:

**Theorem D.** Let \( M \) be a connected orientable real hypersurface in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \). Then the Reeb flow on \( M \) is isometric if and only if \( M \) is an open part of a tube around some totally geodesic \( G_2(\mathbb{C}^{m+1}) \) in \( G_2(\mathbb{C}^{m+2}) \).

Now let us check for the real hypersurfaces of type (A) mentioned in Proposition C and Theorem D whether they satisfy a commuting Ricci tensor, that is, \( S\phi = \phi S \). Then by Theorem D for the commuting shape operator, that is, \( A\phi = \phi A \), the commuting Ricci tensor \( S\phi = \phi S \) implies
\[
\begin{align*}
&-3\eta_2(\phi Y)\xi_2 - 3\eta_3(\phi Y)\xi_3 + \phi_1\phi^2 Y - \eta(\phi_2\phi Y)\phi_2\xi - \eta(\phi_3\phi Y)\phi_3\xi \\
&= -3\sum_{\gamma=1}^3\eta_\gamma(Y)\phi\xi_\gamma + \phi\phi_1\phi Y - \eta(\phi_2 Y)\phi_2\xi - \eta(\phi_3 Y)\phi_3\xi. 
\end{align*}
\]

(5.18)

Now let us check case by case whether the two sides in (5.18) are equal to each other as follows:

Case 1. \( Y = \xi = \xi_1 \).
In this case it can be easily checked that the two sides are equal to each other.
Case 2. \( Y = \xi_2, \xi_3 \).
Then by putting \( X = \xi_2 \) in (5.18) we have
\[
\begin{align*}
-3\eta_2(\phi_3\xi_2)\xi_3 - \phi_1\xi_3 + (\eta_2(\xi_2)\phi_2\xi + \eta_3(\xi_2)\phi_3\xi) = -3\phi_3\xi_2 + \phi_1\phi_3\xi_2 - \eta(\phi_3\xi_2)\phi_3\xi,
\end{align*}
\]

which implies that both sides are equal to \( \xi_3 \).
Case 3. \( Y \in T_\xi \uplus T_\mu \).
In such a case we have immediately \( S\phi Y = \phi SY \).

**Remark 5.1.** In the paper due to Pérez and the author [13] we have proved that there do not exist any real hypersurfaces \( M \) in \( G_2(\mathbb{C}^{m+2}) \) with parallel and commuting Ricci tensor. Such a geometric condition is stronger than our commuting Ricci tensor in this paper. In the paper [12] we also have proved the non-existence property for real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) with commuting shape operator, that is, \( A\phi_i = \phi A, \quad i = 1, 2, 3 \).

**References**