LIE DERIVATIVES ON HOMOGENEOUS REAL HYPERSURFACES OF TYPE A IN COMPLEX SPACE FORMS

JUNG-HWAN KWON AND YOUNG JIN SUH

ABSTRACT. The purpose of this paper is to give some characterizations of homogeneous real hypersurfaces of type A in complex space forms $M_n(c)$, $c \neq 0$, in terms of Lie derivatives.

1. Introduction

A complex $n$-dimensional Kaehler manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. The complete and simply connected complex space form is isometric to a complex projective space $P_n(C)$, a complex Euclidean space $C^n$, or a complex hyperbolic space $H_n(C)$ according as $c > 0$, $c = 0$ or $c < 0$ respectively. The induced almost contact metric structure of a real hypersurface $M$ of $M_n(c)$ is denoted by $(\phi, \xi, \eta, g)$.

Now, there exist many studies about real hypersurfaces of $M_n(c)$, $c \neq 0$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space $P_n(C)$ by Takagi [13], who showed that these hypersurfaces of $P_n(C)$ could be divided into six types which are said to be of type $A_1, A_2, B, C, D$ and $E$, and in [3] Cecil-Ryan and in [8] Kimura proved that they are realized as the tubes of constant radius over Hermitian symmetric spaces of compact type of rank 1 or rank 2. Also Berndt [1], [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex
hyperbolic space $H_n(C)$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field $\xi$ is principal.

On the other hand, Okumura [12] and Montiel and Romero [11] proved the followings respectively.

**Theorem A.** Let $M$ be a real hypersurface of $P_n(C)$, $n \geq 2$. If it satisfies

\[
A\phi - \phi A = 0,
\]

then $M$ is locally congruent to a tube of radius $r$ over one of the following Kaehler submanifolds:

- $(A_1)$ a hyperplane $P_{n-1}(C)$, where $0 < r < \frac{\pi}{2}$,
- $(A_2)$ a totally geodesic $P_k(C)$ ($1 \leq k \leq n-2$), where $0 < r < \frac{\pi}{2}$.

**Theorem B.** Let $M$ be a real hypersurface of $H_n(C)$, $n \geq 2$. If it satisfies (1.1), then $M$ is locally congruent to one of the following hypersurfaces:

- $(A_0)$ a horosphere in $H_n(C)$, i.e., a Montiel tube,
- $(A_1)$ a tube of a totally geodesic hyperplane $H_k(C)$ ($k = 0$ or $n-1$),
- $(A_2)$ a tube of a totally geodesic $H_k(C)$ ($1 \leq k \leq n-2$).

Now hereafter, unless otherwise stated, the above kind of real hypersurfaces in Theorem A or in Theorem B are said to be of *real hypersurfaces of type A*.

From two decades ago there have been so many investigations for real hypersurfaces of type $A$ in $M_n(c)$, $c \neq 0$ and several characterizations of this type have been obtained by many differential geometers (See [1], [3], [7], [11] and [12]). But until now in terms of Lie derivatives only a few characterizations are known to us. From this point of view we have paid our attention to the works of Okumura [12] and Montiel and Romero [11] as in Theorem A and in Theorem B respectively. They showed that a real hypersurface $M$ in $P_n(C)$ or in $H_n(C)$ is locally congruent to a real hypersurface of type $A$ if and only if the structure vector $\xi$ is an infinitesimal isometry, that is $\mathcal{L}_{\xi} g = 0$, which is equivalent to (1.1), where $\mathcal{L}_{\xi}$ denotes the Lie derivative along the structure vector $\xi$. 

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Being motivated by these results Ki, Kim and Lee [4] proved that the Lie derivatives $\mathcal{L}_\xi g = 0$, $\mathcal{L}_\xi \phi = 0$ or $\mathcal{L}_\xi A = 0$ are equivalent to each other, where $A$ denotes the second fundamental tensor of $M$ in $M_n(c)$.

In this paper we want to generalize these results and to investigate further properties of real hypersurfaces of type $A$ in terms of the tensorial formulas concerned with the Lie derivatives along the structure vector field $\xi$ as follows:

**Theorem.** Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. Assume that the structure vector $\xi$ of $M$ satisfies one of the followings:

1. $\mathcal{L}_\xi g = fg$ for the induced Riemannian metric $g$,
2. $\mathcal{L}_\xi \phi = f\phi$ for the structure tensor $\phi$,
3. $\mathcal{L}_\xi \phi = fA$ for the second fundamental tensor $A$,
4. $\mathcal{L}_\xi \phi = fA\phi$ for the certain tensor $A\phi$ of type $(1,1)$ or,
5. $\mathcal{L}_\xi \phi = f\phi A$ for the certain tensor $\phi A$ of type $(1,1)$,

where $f$ denotes any differentiable function defined on $M$. Then $M$ is locally congruent to a real hypersurface of type $A$.

In section 2 the theory of real hypersurfaces in complex space forms is recalled and in section 3 we will prove the first part of the Theorem when $\xi$ becomes an infinitesimal conformal transformation. In section 4 we will give the complete proof of the latter parts of the Theorem in above. Namely, some characterizations of real hypersurfaces in $M_n(c)$ will be given in terms of the tensorial formulas concerned with the Lie derivatives $\mathcal{L}_\xi \phi$.

**2. Preliminaries**

Let $M$ be a real hypersurface of a complex $n$-dimensional complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ and let $C$ be a unit normal vector field on a neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood $x$ in $M$, the transformation of $X$ and $C$ under $J$ can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$
where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover it is seen that $g(\xi, X) = \eta(X)$, where $g$ denotes the induced Riemannian metric on $M$. By properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies

$$
(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,
$$

where $I$ denotes the identity transformation and $X$ denotes any vector field tangent to $M$. Accordingly, this set $(\phi, \xi, \eta, g)$ defines the almost contact metric structure on $M$. Furthermore the covariant derivative of the structure tensors are given by

$$
(2.2) \quad (\nabla_X \phi) Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,
$$

where $\nabla$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to the unit normal $C$ on $M$. Since the ambient space is of constant holomorphic sectional curvature $c$, the equations of Gauss and Codazzi are respectively given as follows:

$$
(2.3) \quad R(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\
- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,
$$

$$
(2.4) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},
$$

where $R$ denotes the Riemannian curvature tensor of $M$ and $\nabla_X A$ denotes the covariant derivative of the shape operator $A$ with respect to $X$.

Now, in order to get our result, we introduce a lemma which was proved by Ki and Suh [6] as follows:

**Lemma 2.1.** Let $M$ be a real hypersurface of a complex space form $M_n(c)$. If $A\phi + \phi A = 0$, then $c = 0$.  

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3. The infinitesimal conformal transformations

Before going to prove our assertion in Case (1), let us introduce a slight weaker condition than an infinitesimal isometry.

A vector field $X$ on a Riemannian manifold is said to be an infinitesimal conformal transformation if the metric tensor $g$ satisfies $\mathcal{L}_X g = fg$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field $X$ and $f$ denotes a differentiable function defined on $M$.

Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$, whose structure vector $\xi$ is an infinitesimal conformal transformation. Then the metric tensor $g$ on $M$ satisfies

$$(\mathcal{L}_\xi g)(X, Y) = g((\phi A - A\phi)X, Y) = fg(X, Y),$$

where $X$ and $Y$ are any vector fields tangent to $M$. It yields that

$$(\phi A - A\phi)X = fX$$

for any differentiable function $f$ on $M$. From this, putting $X = \xi$, we have

$$(3.1) \quad \phi A\xi = f\xi.$$ 

So, from applying the operator $\phi$ we have

$$(3.2) \quad A\xi = \alpha\xi,$$

where $\alpha$ denotes $g(A\xi, \xi)$. By virtue of the latter two formulas (3.1) and (3.2) we know that $f$ identically vanishes. This means the structure vector $\xi$ becomes an infinitesimal isometric transformation. Thus by Theorems A and B in the introduction, we have completed the proof of our Theorem in Case (1).

4. Some characterizations of real hypersurfaces in terms of $\mathcal{L}_\xi \phi$

In this section let us prove the latter part of our main Theorem. Namely, we will give some characterizations of real hypersurfaces of
type $A$ in terms of the Lie derivatives of the structure tensor $\phi$ along the structure vector $\xi$.

Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$ whose structure vector $\xi$ on $M$ satisfies

$$\mathcal{L}_\xi \phi = fT,$$

where $f$ is a differentiable function and $T$ is a tensor field of type $(1,1)$ defined on $M$. By the definition of the Lie derivative and (2.2) we have

$$(4.1) \quad \mathcal{L}_\xi \phi = \phi^2 A - \phi A \phi + A\xi \otimes \eta - \xi \otimes \eta(A) = fT,$$

from which together with (2.1), it follows that

$$(4.2) \quad A - A\xi \otimes \eta + \phi A \phi = -fT.$$

Operating the linear transformation (4.2) to the structure vector $\xi$ and taking account of (2.1), we have

$$(4.3) \quad fT\xi = 0.$$

Next, operating $\phi$ to (4.2) to the left and using (2.1), we have

$$(4.4) \quad A\phi - \phi A + \phi A\xi \otimes \eta - \xi \otimes \eta(A \phi) = f\phi T.$$

Operating $\phi$ to (4.2) to the right and making use of (2.1), we have

$$(4.5) \quad \phi A - A\phi - \phi A\xi \otimes \eta = fT\phi.$$

Taking the inner product of (4.2) with the structure vector $\xi$, we have for any $X$ in $TM$

$$g(AX, \xi) - \alpha\eta(X) + fg(TX, \xi) = 0.$$

Then from (4.4) and (4.5) we have
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**Lemma 4.1.** Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. Assume that the structure vector $\xi$ satisfies $L_\xi \phi = fT$, where $f$ is a differentiable function and $T$ is a tensor field of type $(1,1)$. If the structure vector $\xi$ is principal, then it satisfies

\begin{equation}
(4.7) \quad f\phi T + fT\phi = 0, \quad 2(A\phi - \phi A) = f(\phi T - T\phi).
\end{equation}

**Case (2):** $T = \phi$

Assume that $T = \phi$. In this case (2) the formula (4.6) yields the structure vector $\xi$ is principal. Then, by Lemma 4.1 we have $A\phi - \phi A = 0$. So by virtue of Theorems A and B, we have our assertion under this case.

**Case (3):** $T = A$.

We assume that $T = A$. By (4.4) and (4.5), we have

\begin{align}
(4.8) \quad &A\phi - (1 + f)\phi A + \phi A\xi \otimes \eta - \xi \otimes \eta(\phi A) = 0, \\
(4.9) \quad &\phi A - (1 + f)A\phi - \phi A\xi \otimes \eta = 0.
\end{align}

Acting the structure vector $\xi$ to the linear transformation (4.8), we get

\begin{equation}
(4.10) \quad f\phi A\xi = 0.
\end{equation}

Taking an inner product (4.9) with the structure vector $\xi$, we have

\begin{equation}
(4.11) \quad (1 + f)\phi A\xi = 0.
\end{equation}

From (4.10) and (4.11) we have

$$
\phi A\xi = 0,
$$

that is, $\xi$ is the principal curvature vector with principal curvature $\alpha$. Then by Lemma 4.1 we have

\begin{align}
(4.12) \quad &f(A\phi + \phi A) = 0, \\
(4.13) \quad &(2 + f)(A\phi - \phi A) = 0.
\end{align}
Let us denote by $M_1$ a subset of $M$ consisting of points at which $f(x) \neq 0$. Now let us assume $M_1$ is not empty. Then, by (4.12), we see that $A\phi + \phi A = 0$ on $M_1$, and hence $c = 0$ on $M_1$ by Lemma 2.1. This makes a contradiction. So $M_1$ is empty. Therefore the function $f$ vanishes identically on $M$. Then (4.13) together with Theorems A and B we have our assertion in Case (3).

Case (4): $T = A\phi$

Next, we assume that $T = A\phi$. Then, by (4.6), we have

\begin{equation}
A\xi - \alpha \xi = -f\phi A\xi.
\end{equation}

Applying $\phi$ to (4.14) and using (2.1) and (4.14), we have $(1 + f^2)\phi A\xi = 0$, that is, $\xi$ is the principal curvature vector with principal curvature $\alpha$. From this and (4.5) we have

\begin{equation}
\phi A - A\phi + f(A - \alpha \eta \otimes \xi) = 0.
\end{equation}

Operating $\phi$ to (4.15) to the right and using (2.1) and the fact $\xi$ is principal, we get

\begin{equation}
\phi A\phi + fA\phi + (A - \alpha \eta \otimes \xi) = 0,
\end{equation}

from which together with (4.15), it follows

\begin{equation}
\phi A - (1 + f^2)A\phi - f\phi A\phi = 0.
\end{equation}

Next, operating $\phi$ to (4.16) to the left and using (2.1), we get

\begin{equation}
\phi A - A\phi + f\phi A\phi = 0.
\end{equation}

From (4.15) and (4.18), we find

\begin{equation}
f\phi A\phi - f(A - \alpha \eta \otimes \xi) = 0.
\end{equation}

From this, operating $\phi$ to the left and using (2.1) and the fact $\xi$ is principal, we have

\begin{equation}
f(A\phi + \phi A) = 0.
\end{equation}
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Let $M_1$ be an open set consisting of points $x$ in $M$ such that $f(x) \neq 0$. If $M_1$ is not empty, then, by (4.20), we see that $A\phi + \phi A = 0$ on $M_1$, and hence $c = 0$ on $M_1$ by Lemma 2.1. This makes a contradiction. Hence $M_1$ is empty. Therefore the function $f$ vanishes identically on $M$. From this, together with (4.15), we have $\phi A = A\phi$. So by Theorems A and B, we have our assertion in this case.

Case (5): $T = \phi A$

Finally, we assume that $T = \phi A$. Then, by (4.6), the structure vector $\xi$ is principal curvature vector with principal curvature $\alpha$. From this together with (4.5) we have

(4.21) $\phi A - A\phi = f\phi A\phi$.

From this, applying $\phi$ to the left and using (2.1) and $\xi$ is principal, we get

(4.22) $\phi A\phi + (A - \alpha \eta \otimes \xi) = f A\phi$.

Next, operating $\phi$ to (4.22) to the right and using (2.1), we find

(4.23) $A\phi - \phi A + f(A - \alpha \eta \otimes \xi) = 0$,

from which together with (4.21) and (4.22) it follows

(4.24) $2(A\phi - \phi A) + f^2 A\phi = 0$.

Operating $\phi$ to (4.23) to the left and using (2.1) and also the fact $\xi$ is principal, we have

$\phi A\phi + f\phi A + (A - \alpha \eta \otimes \xi) = 0$,

from which together with (4.23) it follows

(4.25) $A\phi - \phi A = f\phi A\phi + f^2 A\phi$.

From (4.21) and (4.25) we have

$2(A\phi - \phi A) = f^2 A\phi$,

from which together with (4.24) it follows

$f^2(A\phi + \phi A) = 0$.

Let us also denote by $M_1$ an open set consisting of points $x$ in $M$ such that $f(x) \neq 0$. Then by the same argument as in above, we know that such an open subset $M_1$ do not exist. So $f$ vanishes identically on $M$. Thus we also have our assertion in Case (5).
References


Jung-Hwan Kwon
Department of Mathematics Education, Taegu University, Taegu 705-714, Korea

Young Jin Suh
Department of Mathematics, Kyungpook National University, Taegu 701-701, Korea
E-mail: yjsuh@bh.kyungpook.ac.kr