ON SECTIONAL AND RICCI CURVATURES OF SEMI-
RIEMANNIAN SUBMERSIONS*

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Abstract

O'Neill introduced a notion of Riemannian submersion [7]. In this paper we give a new notion of semi-Riemannian submersion and want to investigate some geometric properties concerned with sectional and Ricci curvatures of this submersion.

1. Introduction

The theory of Riemannian submersion was firstly introduced by O'Neill ([7]) and its geometric properties have been studied by many differential geometers (Besse [1], Escobales Jr. [2], [3], Gray [4], Magid [5], Nakagawa and Takagi [6], and Takagi and Yorozu [11]). In this paper we introduce a new notion of a semi-Riemannian submersion which is more general than the notion of Riemannian submersion and want to investigate its geometric properties.

The main purpose of section 2 is to give the notion of semi-Riemannian submersion which contains the concepts of both Riemannian and indefinite Riemannian (or said to be pseudo-Riemannian) submersions and to construct some fundamental formulas for this submersion.

In section 3 we will give a typical example of semi-Riemannian submersion of pseudo-hyperbolic space $H^{m+n}$.

Now in section 4 the sectional curvature of semi-Riemannian submersion will be defined and the sufficient conditions for the horizontal distribution $\mathcal{D}_H$ of the minimal semi-Riemannian submersion to be totally geodesic and integrable will be studied in terms of sectional curvature.

Finally, in section 5 we also define the notion of Ricci curvature of the semi-Riemannian submersion and want to investigate some geometric properties for the horizontal distribution $\mathcal{D}_H$ of the minimal semi-Riemannian submersion to be totally geodesic and integrable in terms of Ricci curvature. Moreover, we will give another example of minimal semi-Riemannian submersion which is not totally geodesic.

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2. Preliminaries

Let $M$ be an $(m + n)$-dimensional connected semi-Riemannian manifold of index $r + s (0 \leq r \leq m, 0 \leq s \leq n)$, which is denoted by $M_{r+s}^{m+n}$ and let $B$ be an $n$-dimensional connected semi-Riemannian manifold of index $s$, which is denoted by $B^n$. A semi-Riemannian submersion $\pi: M \to B$ is a submersion of semi-Riemannian manifolds $M$ and $B$ such that

1. The fiber $\pi^{-1}(b), b \in B$, are semi-Riemannian submanifolds of $M$.
2. The differential $d\pi$ of $\pi$ preserves scalar products of vectors normal to fibers.

For a semi-Riemannian submersion $\pi: M \to B$ vectors tangent to fibers are said to be vertical and those normal to fibers are said to be horizontal. Any vector field $X$ on $M$ can be decomposed as

$$X = X' + X''$$

where $X'$ (resp. $X''$) denotes a vertical (resp. horizontal) part of $X$. We define two tensors $T$ and $A$ of type $(1, 2)$ on $M$ by

\[
\begin{align*}
T(X, Y) &= (\nabla_x Y')' + (\nabla_x Y'')', \\
A(X, Y) &= (\nabla_x Y')' + (\nabla_x Y'')',
\end{align*}
\]

for any vector fields $X$ and $Y$ on $M$, where $\nabla$ denotes the Levi-Civita connection on $M$. They are called integrability tensors for the semi-Riemannian submersion $\pi: M \to B$. We choose a local field $e_1, \ldots, e_{m+n}$ of orthonormal frames adapted to the semi-Riemannian metric of $M$ in such a way that, restricted to the fiber $\pi^{-1}(b), b \in B, e_1, \ldots, e_m$ is a local field of orthonormal frames adapted to a semi-Riemannian metric of $\pi^{-1}(b)$ induced from that on the semi-Riemannian manifold $M$. The following convention on the range of indices will be used throughout this paper:

\[A, B, C, D, E, F, \ldots = 1, \ldots, m + n;\]
\[i, j, k, l, \ldots = 1, \ldots, m;\]
\[\alpha, \beta, \gamma, \delta, \ldots = m + 1, \ldots, m + n,\]

where $m$ denotes the dimension of fibers. The summation $\Sigma$ is taken over all repeated indices, unless otherwise stated. Then we have $< e_A, e_B > = e_A e_B$, where $<, >$ denotes the scalar product on $M$. The dual coframe field is denoted by $\{\omega_A\}$. The connection form $\omega_{AB}$ are characterized by the structure equations of $M$:

\[
\begin{align*}
d\omega_A + \sum e_B \omega_{AB} \wedge \omega_B &= 0, \\
\omega_{AB} + \omega_{BA} &= 0,
\end{align*}
\]

\[
\begin{align*}
d\omega_{AB} + \sum e_C \omega_{AC} \wedge \omega_{CB} &= \Omega_{AB}, \\
\Omega_{AB} &= -\frac{1}{2} \sum e_C e_D R_{ABCD} \omega_C \wedge \omega_D,
\end{align*}
\]

where $\Omega_{AB}$ denotes the curvature form of $M$ and $R_{ABCD}$ are components of the
Riemannian curvature tensor $R$ with respect to the semi-Riemannian metric. The Levi-Civita connection $\nabla$ on $M$ is given by
\begin{equation}
\nabla_e e_B = \sum \epsilon_C \omega_{CB}(e_A)e_C.
\end{equation}
We define two tensors $h$ and $A_0$ of type $(1, 2)$ on $M$ by
\begin{equation}
h(X, Y) = (\nabla_Y X')', \quad A_0(X, Y) = -(\nabla_Y X''),
\end{equation}
for any vector fields $X$ and $Y$ on $M$. They are also called integrability tensors for the semi-Riemannian submersion $\pi: M \to B$. The integrability tensor $h$ restricted to a fiber means the second fundamental form of the fiber. It follows from (2.2) and (2.4) that
\begin{equation}
h(e_i, e_j) = \sum \epsilon_a \omega_{ai}(e_j)e_a, \quad A_0(e_\alpha, e_\beta) = \sum \epsilon_j \omega_{aj}(e_\beta)e_j.
\end{equation}
In fact, by the definition we get
\begin{equation}
h(e_i, e_j) = (\sum \epsilon_C \omega_{Ci}(e_j)e_C)^{\prime\prime} = \sum \epsilon_a \omega_{ai}(e_j)e_a.
\end{equation}
On the other hand, it is seen by Gray [4] that the integrability tensor $A_0$ satisfies the following relation:
\begin{equation}
A_0(e_\alpha, e_\beta) = -A_0(e_\beta, e_\alpha) = -\frac{1}{2}[e_\alpha, e_\beta]',
\end{equation}
and hence we get
\begin{equation}
A_0(e_\alpha, e_\beta) = -(\sum \epsilon_C \omega_{Ca}(e_\beta)e_C)' = \sum \epsilon_j \omega_{aj}(e_\beta)e_j.
\end{equation}
Thus the only components $h^A_{\beta C}$ (resp. $A^B_{CD}$) of $h$ (resp. $A_0$) which may not vanish are
\begin{equation}
h^\alpha_{ij} = \omega_{ai}(e_j), \quad (\text{resp. } A^i_{\alpha \beta} = \omega_{ai}(e_\beta)).
\end{equation}
Accordingly the connection form $\omega_{ai}$ are given by
\begin{equation}
\omega_{ai} = \sum \epsilon_j h^\alpha_{ij}\omega_j + \sum \epsilon_B A^i_{\alpha \beta}\omega_B.
\end{equation}
We may choose a suitable semi-Riemannian metric on the tangent bundle $TM$ of $M$ and decompose $TM$ as a direct product of a vertical distribution $\mathcal{D}_V$ and a horizontal one $\mathcal{D}_H$, where the vertical (resp. horizontal) distribution is defined by an assignment of any point $x$ in $M$ with a tangent space (resp. the orthonormal subspace) to a fiber through $x$. A distribution $\mathcal{D}$ is said to be integrable if $[X, Y]$ belong to $\mathcal{D}$ whenever vector fields $X$ and $Y$ belong to $\mathcal{D}$. Since the vertical distribution $\mathcal{D}_V$ is defined by $\omega_\alpha = 0$ and it is integrable, by Cartan's lemma we have
\begin{equation}
h^\alpha_{ij} = h^\alpha_{ji}.
\end{equation}
Since the integrability tensor $A_0$ is also skew-symmetric, we get
\begin{equation}
A^i_{\alpha \beta} = A^i_{\beta \alpha}.
\end{equation}
The semi-Riemannian submersion $\pi: M \to B$ is said to be minimal if each fiber is minimal, i.e., if it satisfies $\sum \epsilon_j h^\alpha_{ij} = 0$. The semi-Riemannian submersion
$\pi: M \to B$ is said to be totally geodesic if each fiber is totally geodesic, i.e., if it satisfies $h^o_{ij} = 0$. By (2.5) the horizontal distribution $\mathcal{D}_h$ is integrable if and only if

$$A^e_{\alpha \beta} = 0.$$  

Now, for a tensor field $T = (T^A_{B_1 \cdots B_n})$ on $M$, we define the covariant derivative $T_{\alpha B_1 \cdots B_n}$ by

$$\sum e_c T^A_{B_1 \cdots B_n} \omega_C = dT^A_{B_1 \cdots B_n} - \sum e_c T^A_{B_1 \cdots B_n} \omega_{CA} - \sum e_c T^A_{B_1 \cdots B_n} \omega_{B_n CB}.$$  

Then, from the definition of $(h^A_{BCD})$, $(A^e_{BCD})$ and (2.6), it follows that

$$h^o_{ijk} = - \sum e_r h^o_{i jk}, \quad h^o_{ijr} = - \sum e_r h^o_{ijk},$$

$$h^o_{i jk} = \sum e_r h^o_{i jk}, \quad h^o_{ijr} = - \sum e_r h^o_{ijk},$$

$$A^e_{ijk} = - \sum e_r A^e_{ijk}, \quad A^e_{ijr} = - \sum e_r A^e_{ijk},$$

$$A^e_{ijk} = - \sum e_r A^e_{ijk}, \quad A^e_{ijr} = - \sum e_r A^e_{ijk}.$$  

Moreover, by the exterior derivatives of (2.6) and by means of (2.2), (2.3) and (2.10)–(2.14), we have

$$R^e_{ijk} = h^o_{ijk} - h^o_{kij} + A^e_{aje} - A^e_{aje},$$

$$R^e_{ijr} = h^o_{ijk} - h^o_{ijr} + A^e_{aje} - A^e_{aje},$$

$$R^e_{ijr} = h^o_{ijk} - h^o_{ijr} + A^e_{aje} - A^e_{aje}.$$  

Next, by virtue of (2.2), (2.3) and (2.10) we have the Ricci formulas for the second covariant derivatives of $h$ as the following

$$h^o_{BCDE} - h^o_{BCED} = \sum e_F (h^e_{BC} R_{AFDE} + h^e_{BC} R_{BFDE} + h^e_{BC} R_{CFDE}).$$

3. Examples

In this section typical examples of semi-Riemannian submersion of an $(m + n)$-dimensional pseudo-hyperbolic space $H^{n+1}_{m+n}$ are considered.

Let $C$ or $H$ be the field consisting of complex numbers or quaternion numbers. They are simply denoted by $K$. In $K^{n+1}$ with the standard basis, a
semi-Hermitian form $F$ is defined by

$$F(z, w) = -\sum_{i=1}^{r} z_i \overline{w}_i + \sum_{j=r+1}^{n+1} z_j \overline{w}_j,$$

where $z = (z_1, \ldots, z_{n+1})$ and $w = (w_1, \ldots, w_{n+1})$ are in $K^{n+1}$. The complex or quaternion semi-Euclidean space $(K^{n+1}, F)$ is simply denoted by $K^{n+1}_r$. The scalar product $g'(z, w)$ is given by $ReF(z, w)$ is a semi-Riemannian metric of index $dr$ in $K^{n+1}_r$, where $d = 2$ or $d = 4$ according as $K = C$ or $K = H$. Let $H^{dn+d-1}_{dr-1}$ be a real hypersurface of $K^{n+1}_r$, $r \geq 1$, defined by

$$H^{dn+d-1}_{dr-1} = \{ z \in K^{n+1}_r : F(z, z) = -1 \},$$

and let $g$ be a semi-Riemannian metric of $H^{dn+d-1}_{dr-1}$ induced from the semi-Riemann metric $g'$. Then $(H^{dn+d-1}_{dr-1}, g)$ is the semi-Riemann manifold of constant sectional curvature $-1$ and with index $dr - 1$, which is called a unit pseudo-hyperbolic space. For the unit pseudo-hyperbolic space $H^{dn+d-1}_{dr-1}$ with index $dr - 1$ the tangent space $T_z(H^{dn+d-1}_{dr-1})$ at each point $z$ can be identified (through the parallel displacement in $K^{n+1}_r$) with $\{ w \in K^{n+1}_r : ReF(z, w) = 0 \}$.

Let $T_z$ be the orthogonal complement of the vector $iz$ in $T_z(H^{dn+d-1}_{dr-1})$ or the vectors $iz$, $jz$ and $kz$ in $T_z(H^{3n+3}_2)$, where we denote by $i$ an imaginary unit in $C$ and by 1, $i$, $j$ and $k$ a basis for $H$ so that they satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Let $H^{r-1}_{dr-1}$ be the multiplicative group of these numbers of absolute value 1. Then $H^{dn+d-1}_{dr-1}$ can be considered a principal fiber bundle over a pseudo-hyperbolic $K$-space $H^{r-1}_d K$ with group $H^{r-1}_{dr-1}$ and the projection $\pi$. Furthermore there is a connection such that $T_z$ is the horizontal subspace at $z$ which is invariant under the $H^{r-1}_{dr-1}$-action. The metric $g_0$ of constant holomorphic sectional curvature $-4$ is given by $g_0(X, Y) = g_z(X^*, Y^*)$ for any tangent vectors $X$ and $Y$ in $T_b(H^{r-1}_d K)$, where $z$ is any point in the fiber $\pi^{-1}(b)$ and $X^*$ and $Y^*$ are vectors in $T_z$ such that $d\pi X^* = X$ and $d\pi Y^* = Y$.

On the other hand, complex structures $I: w \mapsto iw$, $J: w \mapsto jw$ and $K: w \mapsto kw$ in $T_z$ is compatible with the action of $H^{r-1}_{dr-1}$ and induce almost complex structures $I$, $J$ and $K$ on $H^{r-1}_d K$ such that $d\pi I = I o d\pi$, $d\pi J = J o d\pi$ and $d\pi K = K o d\pi$. Thus $H^{r-1}_d K$ is a pseudo-hyperbolic space over $K$ of constant holomorphic sectional curvature $-4$ and it is seen that the principal $H^{d-1}_{dr-1}$-bundle $H^{dn+d-1}_{dr-1}$ over $H^{r-1}_d K$ with projection $\pi$ is a semi-Riemannian submersion with the fundamental tensors $I$, $J$ and $K$. A distribution $\mathcal{D}$ determined by the subspace spanned by $iz$, $jz$ and $kz$ at any point $z$ is integrable. In fact, we have

$$\nabla_{iz}(jz) = j\nabla_{iz}(z) = jiz = -kz,$$

because $j$ is parallel and $H^{dn+d-1}_{dr-1}$ is totally umbilic in $K^{n+1}_r$. This shows that $[iz, jz] = -2kz$. Since the others hold similarly, it means that the distribution $\mathcal{D}$ is integrable. On the other hand, (3.1) implies that the maximal integral submanifold of $\mathcal{D}$ is totally geodesic. Thus the semi-Riemannian submersions have totally geodesic time-like fibers $H^{d-1}_{dr-1}$. 


\[ H_{d-1}^{d-1} \rightarrow H_{d-1}^{d} \]
\[ \downarrow \quad \pi \]
\[ H_{d-1}^{d} \]

In particular, we consider the case \( r = 1 \). Then there exist totally geodesic space-like submersions \( \pi: H_{1}^{n+1} \rightarrow H_{C}^{n} \) and \( \pi: H_{3}^{n+3} \rightarrow H_{H}^{n} \) whose basic manifold is Riemannian.

4. Sectional curvatures

Let \( M = M_{m+s}^{m+n} \) be an \( (m + n) \)-dimensional semi-Riemannian manifold of index \( r + s \) and \( B = B_{s}^{n} \) be an \( n \)-dimensional semi-Riemannian manifold of index \( s \). We denote by \( \mathcal{P}_{D} \) and \( \mathcal{P}_{I} \) the set of all definite plane sections and all non-degenerate plane sections, respectively. For any non-degenerate plane section \( P_{I} \) the sectional curvature is denoted by \( K(P_{I}) \). Let \( \pi: M \rightarrow B \) be a semi-Riemannian submersion. Then we have

\[ R_{\alpha\beta} = h_{ij}^{\alpha} - \sum \epsilon_{k}h_{ij}^{\alpha}h_{kj}^{\beta} + \sum \epsilon_{r}A_{\alpha r}A_{\beta r} - A_{\alpha\beta}^{I} \]

by means of (2.8), (2.11), (2.13) and (2.16). Assume that the semi-Riemannian submersion \( \pi: M \rightarrow B \) is minimal. Then it is easily seen that we have

\[ \sum \epsilon_{j}h_{ij}^{\beta} = 0, \]

from which the following

\[ \sum \epsilon_{j}R_{\alpha\beta} = -\sum \epsilon_{j}h_{ij}^{\alpha}h_{kj}^{\beta} + \sum \epsilon_{r}A_{\alpha r}A_{\beta r} - \sum \epsilon_{j}A_{\alpha\beta}^{I} \]

is derived. Since the left hand side and the first two terms of the right hand side are symmetric with respect to indices \( \alpha \) and \( \beta \) and the last one is skew-symmetric, we have

\[ \sum \epsilon_{j}R_{\alpha\beta} = \sum \epsilon_{j}h_{ij}^{\alpha}h_{kj}^{\beta} + \sum \epsilon_{r}A_{\alpha r}A_{\beta r}. \]

**Theorem 4.1.** Let \( \pi: M_{m+s}^{m+n} \rightarrow B_{s}^{n} \) be a semi-Riemannian submersion. If \( K(P_{I}) \geq 0 \) and if the submersion is minimal, then it is totally geodesic and horizontal distribution is integrable.

**Proof.** By (4.2) we get

\[ \sum \epsilon_{j}R_{\alpha\beta} = -\sum \epsilon_{j}h_{ij}^{\alpha}h_{kj}^{\beta} + \sum \epsilon_{r}A_{\alpha r}A_{\beta r} \leq 0, \]

because of \( \epsilon_{j} = -1 \) and \( \epsilon_{r} = 1 \). By the assumption \( K(P_{I}) \geq 0 \) we get \( \epsilon_{j}\epsilon_{r}R_{\alpha\beta} \geq 0 \). Thus we get \( h_{ij}^{\alpha} = 0 \) and \( A_{\alpha\beta}^{I} = 0 \) for any indices. \( \square \)

Similarly, using (4.2) one can prove the following:

**Corollary 4.2.** Let \( \pi: M_{m+s}^{m+n} \rightarrow B_{s}^{n} \) be a semi-Riemannian submersion. If \( K(P_{I}) \leq 0 \) and if the submersion is minimal, then it is totally geodesic and the horizontal distribution is integrable.
Certain semi-Riemannian submersions like those in Theorem 4.1 and Corollary 4.2 have simple geometric situation. The distribution $\mathcal{D}$ is said to be parallel if the vector field $\nabla_X Y$ belong to $\mathcal{D}$ whenever a vector field $Y$ belongs to $\mathcal{D}$. Let $\pi: M \to B$ be a semi-Riemannian submersion with totally geodesic fibers. We assume that the horizontal distribution $\mathcal{D}_H$ is integrable. Then by (2.4) and (2.6) we have

$$\nabla_{e_A} e_\beta = \sum \varepsilon_\iota \alpha_\iota \beta_\iota (e_A) e_\gamma,$$

which means that the horizontal distribution is parallel. Thus the vertical distribution orthogonal to $\mathcal{D}_H$ is also parallel and hence one finds the following:

**Theorem 4.3.** Let $\pi: M^{m+n}_{m+n} \to B^n$ be a semi-Riemannian submersion. If it is totally geodesic and if the horizontal distribution is integrable, then the total space $M$ is locally decomposed into the product manifold $F \times B$.

**Remark 1.** Let $\pi: M^{m+n}_{m+n} \to B^n$ be a semi-Riemannian submersion. If it is totally geodesic, the mixed sectional curvature $K(U, X)$ is always non-positive, where $U$ (resp. $X$) is a vertical vector (resp. a horizontal vector).

Now, an $m$-dimensional semi-Riemannian manifold of index $r$ and of constant curvature $c$ is called a semi-Riemannian space form, which is denoted by $M^{m+n}_{m+n}(c)$. Let $\pi: M^{m+n}_{m+n}(c) \to B^n$ be a minimal semi-Riemannian submersion. We denote by $S$ the square of the length of the second fundamental form of the fiber, that is, $S = \sum \varepsilon_i \varepsilon_j \varepsilon_\alpha \alpha_\alpha \alpha_\beta = \sum h^i_j h^i_j$, because of $\varepsilon_i = -1$ and $\varepsilon_\alpha = 1$. Then by (4.2) we obtain

$$S = \sum \varepsilon_\alpha \varepsilon_\beta \varepsilon_j A_{\alpha \beta} A^j_{\alpha \beta} - mnc.$$

From this equation we can conclude the following properties: For a minimal semi-Riemannian submersion $\pi: M^{m+n}_{m+n}(c) \to B^n$

1. $c \geq 0$ implies that $c = 0$, $h^i_j = 0$ and $A_{\alpha \beta} = 0$ for any indices $i$, $j$, $\alpha$ and $\beta$.
2. $c < 0$ implies $S \leq -mnc$, where the equality holds if and only if $A_{\alpha \beta} = 0$ for all indices $\alpha$, $\beta$ and $j$.

Therefore we can state the following lemmas:

**Lemma 4.4.** Let $\pi: M^{m+n}_{m+n}(c) \to B^n$ be a semi-Riemannian submersion. If the submersion is minimal and if $c \geq 0$, then $c = 0$. Moreover it is totally geodesic and the horizontal distribution is integrable.

**Lemma 4.5.** Let $\pi: M^{m+n}_{m+n}(c) \to B^n$ be a semi-Riemannian submersion. If the submersion is minimal and if $c < 0$, then $S \leq -mnc$, where the equality holds if and only if the horizontal distribution is integrable.

Thus we can prove the following:

**Theorem 4.6.** Let $M$ be an $(m+n)$-dimensional semi-Riemannian space form $M^{m+n}_{m+n}(c)$ of index $m$ and $B$ be an $n$-dimensional Riemannian manifold. If $c > 0$, then there are no minimal semi-Riemannian submersions $\pi: M \to B$. 
Similarly, the following properties can be verified:

**Lemma 4.7.** Let \( \pi: M^{\text{m+n}}_n(c) \to B^m_n \) be a semi-Riemannian submersion. If the submersion is minimal and if \( c \leq 0 \), then \( c = 0 \). Moreover it is totally geodesic and the horizontal distribution is integrable.

**Theorem 4.8.** Let \( M \) be an \((m+n)\)-dimensional semi-Riemannian space form \( M^{\text{m+n}}_n(c) \) of index \( n \) and \( B \) be an \( n \)-dimensional semi-Riemannian manifold of index \( n \). If \( c < 0 \), then there are no minimal semi-Riemannian submersions \( \pi: M \to B \).

5. Ricci curvatures

Let \( M \) be an \((m+n)\)-dimensional semi-Riemannian manifold of index \( r+s \) \((r, s \geq 0) \) and \( B \) be an \( n \)-dimensional semi-Riemannian manifold of index \( s \). Let \( \pi: M \to B \) be a semi-Riemannian submersion. We choose a local field \( \{e_A\} \) of orthonormal frames, restricted to the fiber \( \pi^{-1}(b), b \in B \), \( \{e_j\} \) is a local field of orthonormal frames \( \pi^{-1}(b) \). By means of the semi-Riemannian submersion, \( \{e^\alpha_a = \pi(e_a)\} \) is a local field of orthonormal frames on \( B \). The dual coframe field is denoted by \( \{\omega^\alpha_a\} \) on \( B \) with respect to \( \{e^\alpha_a\} \). It is easily seen that we get

\[
\omega^\alpha_a = \pi^* \omega^\alpha_a.
\]

The connection forms \( \{\omega^\alpha_{ab}\} \) are characterized by the structure equations of \( B \):

\[
\begin{cases}
\omega^\alpha_{ab} + \sum \varepsilon^\beta \omega^\beta_{ab} \wedge \omega^\beta_a = 0, \\
\omega^\alpha_{ba} = \omega^\alpha_{ab},
\end{cases}
\]

where \( \Omega^\alpha_{ab} \) denotes the curvature form of \( B \) and \( R^m_{abc} \) are component of the Ricci curvature tensor \( R^m \) of \( B \). Differentiating (5.1) exteriorly we get

\[d\omega^\alpha_a = d(\pi^* \omega^\alpha_a) = \pi^* (d\omega^\alpha_a)\]
and hence, using (2.2), (2.6), (2.7), (5.1) and (5.2) we get

\[
\sum \varepsilon^\beta (\omega^\beta_a - \pi^* \omega^\beta_{ab}) \wedge \omega^\beta - \sum \varepsilon^\beta \varepsilon^\beta \Lambda^\alpha_{ab} \omega^\beta \wedge \omega^\beta = 0.
\]

We put

\[
\omega_{ab} - \pi^* \omega^\alpha_{ab} = \sum \varepsilon_C W_{ab} \omega_C, \quad \omega_{ai} = \sum \varepsilon_C W_{ai} \omega_C.
\]

Then by (2.6), (5.4) and (5.5) we have

\[
W_{ab} = 0, \quad W_{ab} = A_{ab}, \quad W_{ai} = h^a_i, \quad W_{ai} = A_{ai}.
\]

where we have used the fact that \( W_{ab} \) is skew-symmetric with respect to \( a \) and \( b \). A tensor \( W \) whose components are given by \( W_{abc} \) is called the **structure**
tensor for the semi-Riemannian submersion \( \pi : M \to B \). We denote by \( W_{aBCD} \) components of the covariant derivative of the structure tensor \( W \) with respect to the connection form \( \pi^* \omega'_{\alpha\beta} \). Then it is defined by

\[
(5.7) \quad \sum \varepsilon_D W_{aBCD} \omega_D = dW_{aBC} - \sum \varepsilon_D W_{\delta BC} \pi^* \omega'_{\delta \alpha} - \sum \varepsilon_D (W_{aDC} \omega_{DB} + W_{aBD} \omega_{DC}).
\]

Taking account of (2.6), (5.6) and (5.7) we can easily obtain

\[
(5.8) \quad \begin{cases} 
W_{ap\gamma} = \sum \varepsilon_j (A_{ap} h_{ji}^\gamma + A_{a\gamma} h_{ji}^p), \\
W_{ap\gamma\delta} = \sum \varepsilon_j (A_{ap} A_{a\gamma} A_{a\delta} + A_{a\gamma} A_{a\delta}). 
\end{cases}
\]

Differentiating (5.5) exteriorly we get

\[
d(\omega_{\alpha\beta} - \pi^* \omega'_{\alpha\beta}) = \varepsilon_C d(W_{a\beta C} \omega_C).
\]

Accordingly, making use of the structure equations (2.2), (2.3) for \( M \) and the structure equations (5.1), (5.2) and (5.3) for \( B \) together with (5.5) and (5.7), we can directly obtained the following Ricci formula:

\[
(5.9) \quad W_{a\beta CD} - W_{a\beta DC} = R_{a\beta CD} - \delta_{C\gamma} \delta_{D\alpha} R'_{a\gamma\delta\beta}.
\]

Thus, from (5.8) and (5.9) we have for the semi-Riemannian submersion \( \pi : M \to B \)

\[
(5.10) \quad R_{a\gamma\delta} - R'_{a\gamma\delta} = \sum \varepsilon_j (2A_{a\gamma} A_{a\delta} + A_{a\gamma} A_{a\delta} - A_{a\delta} A_{a\delta}),
\]

\[
(5.11) \quad R_{a\beta \alpha} - R''_{a\beta \alpha} = -3 \sum \varepsilon_j A_{a\gamma} A_{a\beta}.
\]

Remark 1. The above equations (5.10) and (5.11) in the Riemannian submersion are already obtained by Besse [1], Escobales Jr. [3], Gray [4], and O'Neill [7].

For the non-degenerate plane spanned by vectors \( u \) and \( v \) at any point on the semi-Riemannian manifold \( B \) the sectional curvature of the plane section is denoted by \( K'(u, v) \).

Lemma 5.1. For a semi-Riemannian submersion \( \pi : M^{m+n} \to B^n (n \geq 2) \), we have

\[
K(e_\alpha, e_\beta) \geq K''(d\pi e_\alpha, d\pi e_\beta) * \pi,
\]

where the equality holds if and only if \( A_{a\beta} = 0 \) for any index \( j \).

Proof. By the assumption of the semi-Riemannian submersion we have \( \varepsilon_j = -1 \), which implies that (5.11) is equivalent to

\[
R_{a\beta \alpha} - R''_{a\beta \alpha} = -3 \sum \varepsilon_j A_{a\beta} A_{a\beta} \geq 0.
\]

Since the sectional curvature of the plane section spanned by \( e_\alpha \) and \( e_\beta \) (resp. \( d\pi e_\alpha \) and \( d\pi e_\beta \)) is given by
we get \( K(e_\alpha, e_\beta) \geq K'(d\pi e_\alpha, d\pi e_\beta) \circ \pi \), where the equality holds if and only if \( A_{\alpha\beta} = 0 \) for any index \( j \).

\[ \begin{align*}
K(e_\alpha, e_\beta) &= R_{\alpha\beta\alpha\beta}, \\
K'(d\pi e_\alpha, d\pi e_\beta) &= R'_{\alpha\beta\alpha\beta}.
\end{align*} \]

**Theorem 5.2.** For a semi-Riemannian submersion \( \pi : M^{m+n}_n \rightarrow B^n (n \geq 2) \) if \( K(P) \leq 0 \), then there exists at least one plane \( P' \) in \( T_bB \), \( b \in B \), such that \( K''(P') < 0 \) or \( B \) is locally flat and the horizontal distribution is integrable.

**Proof.** Suppose that there does not exist a plane section \( P' \) such that \( K''(P') < 0 \). Then, for any point \( b \in B \) and for any plane section \( P' \) in \( T_bB \) we have \( K''(P') \geq 0 \). Accordingly Lemma 5.1 means that

\[ K(e_\alpha, e_\beta) \geq K'(d\pi e_\alpha, d\pi e_\beta) \circ \pi \geq 0 \]

for any indices \( \alpha \) and \( \beta \). Since the plane section spanned by \( e_\alpha \) and \( e_\beta \) is definite, by the assumption we get \( K(e_\alpha, e_\beta) \leq 0 \) for any indices \( \alpha \) and \( \beta \), which means that \( B \) is locally flat and \( A_{\alpha\beta} = 0 \) for any indices.

From now on we assume that the semi-Riemannian submersion \( \pi : M \rightarrow B \) is minimal. Then we have the formula (4.2) given in section 4. By virtue of this formula we can prove

**Theorem 5.3.** Let \( M^{m+1}_n \) be a Lorentzian manifold satisfying the strongly energy condition and \( \pi : M^{m+1}_1 \rightarrow B^1_1 \) be a semi-Riemannian submersion of codimension one and with space-like fibers. If it is minimal, then it is totally geodesic.

**Proof.** By the assumption of codimension we have \( \dim B = 1 \) and each fiber is a space-like hypersurface, which implies that \( B \) is time-like. The assumption for the strongly energy condition means that the Ricci tensor \( \text{Ric}(e_\alpha) = \text{Ric}(e_\alpha, e_\alpha) \) in the direction of the time-like vector \( e_\alpha \) of \( M \) satisfies \( \text{Ric}(e_\alpha) = R_{\alpha\alpha} \geq 0 \), where \( \alpha = m + 1 \).

On the other hand, (4.2) is reformed as

\[ R_{\alpha\alpha} = -\sum h^{ij} h_{jk} - \sum A_{\alpha\gamma} A_{\beta\gamma} \leq 0, \]

because of \( \varepsilon_j = 1 \) and \( \varepsilon_\alpha = -1 \). Thus, by the strongly energy condition, the equality holds and hence we have \( h^{ij}_{ij} = 0 \) for any indices.

**Remark 2.** Let \( M \) be a compact Riemannian manifold whose Ricci curvature is positive semi-definite. It is proved by Oshikiri [9] that if a foliation \( (M, g, \mathcal{F}) \) of codimension one is minimal, then \( \mathcal{F} \) is totally geodesic and the metric \( g \) is bundle-like.

Next we study the Ricci curvature and the Einstein condition for the semi-Riemannian submersions. Since the fibers are submanifolds of the total space \( M \), the Riemannian curvature tensor \( R' \) satisfies the Gauss equation

\[ R'_{ijkl} = R_{ijkl} + \sum \varepsilon_\alpha (h^\alpha_{ij} h^\alpha_{jk} - h^\alpha_{ik} h^\alpha_{jk}), \]

\[ (5.12) \]
where $R'_{ijkl}$ are components of the Riemannian curvature tensor $R'$ of the fiber. We denote by $R_{AB}$ (resp. $R'_{ij}$ and $R''_{ij}$) the components of the Ricci curvature tensor of $M$ (resp. the fiber and $B$). Because of

$$R_{ij} = \sum \varepsilon_k R_{kij} + \sum \varepsilon_{\beta} R_{\beta ij},$$

we obtain by (2.8), (4.1) and the Gauss equation (5.12)

$$R_{ij} = R'_{ij} + \sum \varepsilon_{\beta} h_{ij}^{\beta} - \sum \varepsilon_k \varepsilon_{\beta} h_{kk}^{\beta} h_{ij}^{\beta} + \sum \varepsilon_{\beta} \varepsilon_{\gamma} A'_{\beta \gamma} A'_{\gamma},$$

(5.13)

Similarly we get

$$R_{\alpha \beta} = R''_{\alpha \beta} - 2 \sum \varepsilon_j \varepsilon_{\alpha} A'_{\alpha j} A'_{\gamma} - \sum \varepsilon_{\alpha} h_{ij}^{\alpha} h_{ij}^{\beta} + \sum \varepsilon_{\alpha} A'_{\alpha j},$$

(5.14)

where we have used (4.1) and (5.10). On the other hand, we have

$$R_{ij} = \sum \varepsilon_j (h_{ij}^{\alpha} - h_{ij}^{\beta}) + 2 \sum \varepsilon_k \varepsilon_{\alpha} A'_{ik} A'_{\alpha} + \sum \varepsilon_{\alpha} A'_{\alpha},$$

(5.15)

by (2.12), (2.14), (2.15) and (2.17). Now we denote by $\text{Ric}$ (resp. $\text{Ric}'$ and $\text{Ric}''$) the Ricci curvature tensor of $M$ (resp. the fiber and $B$). The Ricci curvature in the direction of $e_{j}$ of $M$ is denoted by $\text{Ric}(e_{j}) = \text{Ric}(e_{j}, e_{j})$. From (5.13) one finds the following:

**Theorem 5.4.** For a minimal semi-Riemannian submersion $\pi: M_{m+n} \rightarrow B$, or $\pi: M_{n+n} \rightarrow B_{n}$ if

$$\text{Ric}'(e_{j}) \geq \text{Ric}(e_{j})$$

for any index $j$, then the horizontal distribution is integrable.

**Proof.** By the assumption and (5.13) we have

$$0 \geq \text{Ric}(e_{j}) - \text{Ric}'(e_{j}) = \sum \varepsilon_{\beta} h_{ij}^{\beta} + \sum \varepsilon_{\beta} \varepsilon_{\gamma} A'_{\beta \gamma} A'_{\gamma} \geq \sum \varepsilon_{\beta} h_{ij}^{\beta},$$

where the equality holds if and only if $A'_{\beta \gamma} = 0$ for any indices.

On the other hand, we have $\sum \varepsilon_{j} \varepsilon_{\beta} h_{ij}^{\beta} = 0$, which implies

$$\sum \varepsilon_{j} \{\text{Ric}(e_{j}) - \text{Ric}'(e_{j})\} = 0.$$

Thus we get $\text{Ric}(e_{j}) - \text{Ric}'(e_{j}) = 0$ for any index $j$. •

We say the horizontal distribution $\mathcal{D}_{H}$ satisfies the Yang-Mills condition if it satisfies

$$\sum \varepsilon_{\beta} A'_{\alpha \beta} = \sum \varepsilon_{k} \varepsilon_{\beta} h_{ik}^{\beta} A'_{\alpha k},$$

for any indices $i$ and $\alpha$. It is important for Einstein Riemannian submersions (cf. pp. 243 Besse [1]). From the formula between Ricci curvatures we get the following:

**Proposition 5.5.** Let $\pi: M_{m+n} \rightarrow B$, (resp. $\pi: M_{n+n} \rightarrow B_{n}$) be a semi-
Riemannian submersion with time-like (resp. space-like) totally geodesic fibers. Then $M$ is Einstein if and only if the horizontal distribution $\mathcal{H}$ satisfies the Yang-Mills condition and the Ricci curvatures of the fibers and the base manifold $B$ satisfy

1. $R_{ij} = \lambda \delta_{ij} - \sum \varepsilon_{\alpha} \varepsilon_{\beta} A^{\alpha}_{ij} A_{\alpha}^{\beta}$
2. $R_{\alpha\beta} = \lambda \delta_{\alpha\beta} + 2 \sum \varepsilon_{\alpha} \varepsilon_{\gamma} A^{\gamma}_{\alpha\beta}$

for a constant $\lambda$.

Next we assume that the semi-Riemannian submersion $\pi: M \to B$ is minimal and we denote by $r$ (resp. $r'$ or $r''$) the scalar curvature of $M$ (resp. the fiber or $B$). Then by the definition we get

$$r - r'' = \sum \varepsilon_{\alpha} R_{\alpha\alpha} + \sum \varepsilon_{i} R_{ii} - \sum \varepsilon_{\alpha} R_{\alpha\alpha}.$$

Then by (5.13) and (5.14) it is reformed as

$$r - r' - r'' = -\sum \varepsilon_{i} \varepsilon_{\alpha} A^{\alpha}_{ij} A_{\alpha}^{i} - \sum \varepsilon_{\alpha} \varepsilon_{i} \varepsilon_{j} h_{ij}^{\alpha},$$

where $r' = \sum \varepsilon_{i} R_{ii}$ and $r'' = \sum \varepsilon_{\alpha} R_{\alpha\alpha}^{\prime \prime}$. Thus we have the followings:

**Theorem 5.6.** For a minimal Riemannian submersion $\pi: M^{m+n} \to B^n$, we have

$$r \leq r' + r'',$$

where the equality holds if and only if it is totally geodesic and the horizontal distribution is integrable.

**Corollary 5.7.** For a Riemannian submersion $\pi: M^{m+n} \to B^n$ if there is a point $x \in M$ such that $r(x) > r'(x) + r''(x)$, then it is not minimal.

**Remark 3.** The following results are proved by Watson [10]. Let $M$ be a compact Riemannian manifold whose Ricci tensor is positive semi-definite and $B$ be a Riemannian manifold whose Ricci tensor is negative semi-definite. If there is a point on $M$ at which the Ricci tensor is positive definite, then there are no minimal submersions $\pi: M \to B$. In particular, if $B$ is of negative curvature, there are no minimal submersions $\pi: M \to B$.

From (5.13) and (5.16) we have

$$\sum \varepsilon_{\alpha} \{\text{Ric}(e_{\alpha}) - \text{Ric}''(d\pi e_{\alpha})\} = 2(r - r' - r'') + \sum \varepsilon_{\alpha} \varepsilon_{i} \varepsilon_{j} h_{ij}^{\alpha} h_{ij}. $$

Thus we prove the following:

**Lemma 5.8.** Let $\pi: M^{m+n} \to B^n$ be a semi-Riemannian submersion. If it is minimal, then

$$\sum \varepsilon_{\alpha} \{\text{Ric}(e_{\alpha}) - \text{Ric}''(d\pi e_{\alpha})\} \geq 2(r - r' - r''),$$

where the equality holds if and only if it is totally geodesic.
As a direct consequence of (5.16) and Lemma 5.8 we get

**Theorem 5.9.** Let \( \pi: M_{m+n}^m \to B^n \) be a semi-Riemannian submersion. If it is minimal and if \( \text{Ric}(e_a) \leq \text{Ric}(dx e_a) \) and \( r - r' - r'' \geq 0 \), then it is totally geodesic and the horizontal distribution is integrable.

**Example.** An example of minimal semi-Riemannian submersion \( \pi: M_{m+n}^m \to B^n \) which is not totally geodesic is here constructed.

Let \( \{f_A\} \) be the set of smooth positive functions on \( \mathbb{R}^n \). Let \( M_{m+n}^m \) (resp. \( M_{n}^n \)) be an \((m + n)\)-dimensional semi-Riemannian manifold of index \( m \) (resp. index \( n \)) defined by

\[
M = M_{m+n}^m \quad \text{(resp. } M_{n}^n \text{)}
\]

\[
= \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : g = (g_{AB}), g_{AB}(x, y) = \varepsilon f_A^2(y) \delta_{AB}\},
\]

where \( \varepsilon_j = -1, \varepsilon_a = 1 \) (resp. \( \varepsilon_j = 1, \varepsilon_a = -1 \)). Also, let \( B = B^n \) (resp. \( B^n \)) be an \( n \)-dimensional Riemannian (resp. semi-Riemannian) manifold defined by

\[
B = \{y \in \mathbb{R}^n : g'' = (g'_{\alpha\beta}), g''_{\alpha\beta} = \varepsilon f^2(y) \delta_{\alpha\beta}\}.
\]

Then, for the natural projection \( \pi: M \to B \), it is a semi-Riemannian submersion whose fibers are defined by a fixed point \( y \in \mathbb{R}^n \). For the natural coordinate system \( \{x_A\} \) the natural basis \( \{\partial/\partial x_A\} \) satisfies

\[
\begin{align*}
g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= \varepsilon f^2 \delta_{ij}, \\
g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= 0, \\
g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}) &= \varepsilon f^2 \delta_{i\beta}.
\end{align*}
\]

Accordingly an orthonormal basis \( \{e_A\} \) is given by

\[
e_i = \frac{1}{f_i} \frac{\partial}{\partial x_i}, \quad e_\alpha = \frac{1}{f_\alpha} \frac{\partial}{\partial x_\alpha}.
\]

Thus, calculating \( \nabla e \varepsilon e_j \) we can get

\[
h_{ij}^\alpha = -\frac{\varepsilon_i}{f_i} \frac{\partial f_i}{\partial x_\alpha} \delta_{ij},
\]

Consequently we have

\[
\sum \varepsilon_i h_{ij}^\alpha = -\frac{1}{f_\alpha} \frac{\partial}{\partial x_\alpha} \log \prod f_j.
\]

This shows that if the functions \( f_1, \ldots, f_m \) satisfy the condition \( \prod f_j = \text{constant} \), then the submersion is minimal, but in general not totally geodesic.

**References**


