Classification of Real Hypersurfaces in Complex Hyperbolic Space in Terms of Constant $\phi$-holomorphic Sectional Curvatures

Dedicated to Professor U-Hang Ki on his sixtieth birthday

DONG JOO SOHN  
Department of mathematics, Andong University, Andong, Kyungpook 760-749, Korea

YOUNG JIN SUH  
Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702-701, Korea

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In this paper we give a complete classification of real hypersurfaces in complex hyperbolic space $H_n(C)$ on which the sectional curvature of $\phi$-holomorphic planes is constant.

0. Introduction

A complex $n$-dimensional Kaeahlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space $P_nC$, a complex Euclidean space $C^n$ or a complex hyperbolic space $H_nC$, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface $M$ of $M_n(c)$ is denoted by $(\phi, \xi, \eta, g)$. Several authors have dealt with real hypersurfaces of a complex projective space $P_n(C)$ from some different point of view ([4],[8],[11]). In particular, Cecil-Ryan [4] showed also that they are realized as the tubes of constant radius over Kaeahler submanifolds when the structure vector field $\xi$ is principal.

In accordance with the development of the study of these real hypersurfaces in $P_n(C)$, many valuable interesting results of real hypersurfaces in a complex

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hyperbolic space $H_n(C)$ have been now investigated by several authors ([2],[9], [10]).

On the other hand, Kimura [7] have studied the classification of real hypersurfaces in $P_n(C)$ on which the sectional curvature of $\phi$-holomorphic plane is constant. In section 1 we also define the notion of constant $\phi$-holomorphic sectional curvature for real hypersurfaces in a complex hyperbolic space $H_n(C)$ and want to investigate the classification problem of these real hypersurfaces.

Hereafter, unless otherwise stated we treat an $n$-dimensional complex hyperbolic space $H_n(C)$ with the Bergman metric of constant holomorphic sectional curvature $-4$.

Now let us denote by $H$ the sectional curvature of a holomorphic 2-planes on a real hypersurface $M$ in $H_n(C)$, $N$ a unit normal vector at a point $x \in M$, and let $J$ the canonical complex structure on $H_n(C)$. Then $\xi = -JX$ is a tangent vector at $x \in M$.

A real hypersurface $M$ in a complex hyperbolic space $H_n(C)$ is said to be ruled if there exists a foliation of $M$ by totally geodesic complex hyperplanes $H_{n-1}(C)$ of $H_n(C)$.

Now in this paper we want to investigate the classification problem of real hypersurfaces in $H_n(C)$ with the constant $\phi$-holomorphic sectional curvature as the following

**Theorem.** Let $M$ be a real hypersurface of $H_n(C), n \geq 3$, on which the $\phi$ holomorphic sectional curvature $H$ is constant. Then $M$ is congruent to one of the following:

i) a horosphere or an open set of geodesic hypersphere, a tube over hyperplane $H_{n-1}(C)$, $(H > -4)$,

ii) a ruled real hypersurface $(H = -4)$. More precisely, let $T_0$ be the distribution defined by $T_0(x) = \{X \in T_x(M) : X \perp \xi\}$ for $x \in M$, then $T_0$ is integrable and its integral manifolds are totally geodesic $H_{n-1}(C)$,

iii) a real hypersurface foliated by the leaf of codimension 2 which is contained in $H_{n-1}(C)$ as a ruled hypersurface $(H = -4)$.

In section 3, we give some examples of ruled real hypersurfaces in $H_n(C)$. In this paper, all manifolds are to be $C^\infty$ and connected.

1. Preliminaries

Let $M$ be an orientable real hypersurface of $H_n(C)$ and let $N$ be a unit normal vector field on $M$. The Riemannian connections $\nabla$ in $H_n(C)$ and $\nabla$ in $M$ are related by the following formulas for any vector fields $X$ and $Y$ on $M$:

$$\nabla_X Y = \nabla_X Y + g(AX, Y)N,$$

(1.1)
\[
\tilde{\nabla}_X N = -AX,
\]

where \( g \) denotes the Riemannian metric of \( M \) induced from the Bergman metric \( G \) of \( H_n(C) \) and \( A \) is the shape operator of \( M \) in \( H_n(C) \). An eigenvector \( X \) of the shape operator \( A \) is called a principal curvature vector. Also an eigenvalue \( \lambda \) of \( A \) is called a principal curvature. In what follows, we denote by \( V_\lambda \) the eigenspace of \( A \) associated with eigenvalue \( \lambda \). It is known that \( M \) has an almost contact metric structure induced from the complex structure \( J \) on \( H_n(C) \), that is, we define a tensor field \( \phi \) of type \((1,1)\), a vector field \( \xi \) and a 1-form \( \eta \) on \( M \) by \( g(\phi X, Y) = G(JX, Y) \) and \( g(\xi, X) = \eta(X) = G(JX, N) \). Then we have

\[
\phi^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \phi \xi = 0.
\]

It follows from (1.1) that

\[
(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y) \xi,
\]

\[
\nabla_X \xi = \phi AX.
\]

Let \( \tilde{R} \) and \( R \) be the curvature tensors of \( H_n(C) \) and \( M \), respectively. Since the curvature tensor \( \tilde{R} \) has a nice form, we have the following Gauss and Codazzi equations:

\[
g(R(X, Y)Z, W) = -g(Y, Z)g(X, W) + g(X, Z)g(Y, W) - g(\phi Y, Z)g(\phi X, W) + g(\phi X, Z)g(\phi Y, W) + 2g(\phi X, Y)g(\phi Z, W) + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W),
\]

\[
(\nabla_X A) Y - (\nabla_Y A) X = -\eta(X)\phi Y + \eta(Y)\phi X + 2g(\phi X, Y)\xi.
\]

Now let \( H(X) \) be the sectional curvature of the holomorphic 2-plane spanned by a unit tangent vector \( X \) which is orthogonal to \( \xi \), that is, \( H(X) = \text{The sectional curvature of span}\{X, \phi X\} \), which is said to be \( \phi \)-holomorphic sectional curvature. Then from the equation of Gauss (1.6) we have

\[
H(X) = -4 + g(AX, X)g(A\phi X, \phi X) - g(AX, \phi X)^2.
\]

2. Proof of the Theorem

In this section, let us give a classification of real hypersurfaces in \( H_n(C) \) in terms of constant \( \phi \)-holomorphic sectional curvature \( H \). Now let us denote

\[
SH_x(M) = \{X \in T_x(M) \cap JT_x(M) : \|X\| = 1\}
\]

for \( x \in M \).
Now we suppose $H(X) = H$ (constant on $M$) for any $X \in SH_2(M)$. If $X(t), t \in (-\delta, \delta)$ is a part of a great circle in $X(0) = X, \ |X'(t)| = 1$ and $X'(0) = Y$, then, using (1.2), we have

\[
0 = \left. \frac{d}{dt} \right|_{t=0} H(X(t)) \tag{2.1}
= 2g(AX, Y)g(A\phi X, \phi X) + 2g(AX, X)g(A\phi X, \phi Y) - 2g(AX, \phi X)g((A\phi - \phi A)X, Y)
\]

and

\[
0 = \left. \frac{d^2}{dt^2} \right|_{t=0} H(X(t)) \tag{2.2}
= 2\{g(AY, Y) - g(AX, X)\}g(A\phi X, \phi X) + 8g(AX, Y)g(A\phi X, \phi Y) + 2g(AX, X)\{g(A\phi Y, \phi Y) - g(A\phi X, X)\}
+ 2g(AY, \phi X)\{g((A\phi - \phi A)Y, Y) - g((A\phi - \phi A)X, X)\}
= 2g(AY, Y)g(A\phi X, \phi X) + 8g(AX, Y)g(A\phi X, \phi Y) + 2g(AX, X)g(A\phi Y, \phi Y) - 2g((A\phi - \phi A)X, Y)^2 - 4g(AX, \phi X)g(AY, \phi Y) - 4(H + 4)
\]

Now we take an orthonormal basis \(\{e_1, ..., e_{2n-2}, \xi(= -JN)\}\) of the tangent space \(T_x(M)\) such that

\[
A e_j = \lambda_j e_j + \beta_j \xi, \quad 1 \leq j \leq 2n - 2. \tag{2.3}
\]

In (2.1), we put $X = e_j$ and $Y = e_k$ ($j \neq k$).

\[
0 = 2g(Ae_j, e_k)g(\phi e_j, \phi e_j) - 2g(Ae_j, e_j)g(A\phi e_j, \phi e_k) - 2g(Ae_j, \phi e_k)g((A\phi - \phi A)e_j, e_k)
= -2\lambda_j g(A\phi e_j, \phi e_k).
\]

Then we have

\[
\lambda_j g(A\phi e_j, \phi e_k) = 0. \tag{2.4}
\]

So if \(\lambda_j \neq 0\), then \(g(A\phi e_j, \phi e_k) = 0\) for any \(k \neq j\), and \(A\phi e_j \in \text{Span}\{\phi e_j, \xi\}\). The above argument shows that one can always choose an orthonormal basis

\[
\{e_1, \phi e_1, e_2, \phi e_2, ..., e_{n-1}, \xi\}
\]
such that $Ae_j = \lambda_j e_j + \beta_j \xi$ and $A\phi e_j = \bar{\lambda}_j \phi e_j + \bar{\beta}_j \xi$ $(1 \leq j \leq n - 1)$. In (2.2), we put $X = e_j$ and $Y = e_k$ $(j \neq k)$.

$$0 = 2g(A e_k, e_k)g(A\phi e_j, e_j) + 8g(A e_j, e_k)g(A\phi e_j, e_k) + 2g(A e_j, e_j)g(A\phi e_k, e_k) - 2g((\phi A - A\phi)e_j, e_k) - 4g(A e_j, \phi e_j)g(A e_k, \phi e_k) - 4(H + 4)$$

$$= 2\lambda_k \bar{\lambda}_j + 2\lambda_j \bar{\lambda}_k - 4(H + 4).$$

Then we get

$$\lambda_k \bar{\lambda}_j + \lambda_j \bar{\lambda}_k = 2(H + 4). \quad (2.5)$$

On the other hand, (1.2) implies that

$$-4 + \lambda_j \bar{\lambda}_j = -4 + \lambda_k \bar{\lambda}_k = H, \quad (2.6)$$

and we get

$$\lambda_j \bar{\lambda}_j + \lambda_k \bar{\lambda}_k = 2(H + 4). \quad (2.7)$$

From these formulas we can consider the following two cases:

(A) $\lambda_j = \bar{\lambda}_j = \lambda_k = \bar{\lambda}_k$ $(H \geq -4),$

(B) one of the $\lambda_j$ is non-zero and all the other $\lambda_k$, $\bar{\lambda}_k$ are zero $(H = -4)$. 

For example, $\bar{\lambda}_1 \neq 0$, $\lambda_2 = \cdots = \bar{\lambda}_1 = \cdots = \bar{\lambda}_{n-1} = 0$.

First, let us consider the case (A). We put $\lambda = \lambda_j$. Then $\lambda$ is constant and satisfies $-4 + \lambda^2 = H$, because of (1.2). We put $\beta U = A\xi - g(A\xi, \xi)\xi$, where $U$ is a unit tangent vector in $T_x(M)$. Then using (2.7), we have

$$A\xi = \alpha \xi + \beta U, \quad AU = \beta \xi + \lambda U \quad \text{and} \quad A\phi U = \lambda \phi U, \quad (2.8)$$

where $\alpha = g(A\xi, \xi)$. We shall show that either $\lambda = 0$ or $\beta = 0$. Let $X$ be a unit tangent vector at $x \in M$, which is orthogonal to $U, \phi U$ and $\xi$. Then (2.7) and (2.8) imply that $AX = \lambda X$, $A\phi X = \lambda \phi X$. Making use of (1.3), we obtain

$$(\nabla_X A)\phi X - (\nabla_{\phi X} A) X$$

$$= -\eta(X)\phi^2 X + \eta(\phi X)\phi X + 2g(\phi X, \phi X)\xi$$

$$= 2\xi, \quad (2.9)$$

and

$$(\nabla_X A)\phi X - (\nabla_{\phi X} A) X$$

$$= \nabla_X (A\phi X) - A\nabla_X (\phi X) - \nabla_{\phi X} (AX) + A\nabla_{\phi X} X$$

$$= (\lambda - A)(\nabla_X (\phi X) - \nabla_{\phi X} X),$$
where we have used the constancy of \( \lambda \). Then by virtue of (1.1), (2.8) and (2.9) we get

\[
0 = g((\nabla_X A)\phi X - (\nabla_{\phi X} A)X, U) \\
= g(\nabla_X (\phi X) - \nabla_{\phi X} X, (\lambda - A)U) \\
= g(\nabla_X (\phi X) - \nabla_{\phi X} X, -\beta \xi) \\
= \beta\{g(\phi X, \nabla_X \xi) - g(X, \nabla_{\phi X} \xi)\} \\
= \beta\{g(\phi X, \phi AX) - g(X, \phi \phi X)\} \\
= 2\lambda \beta.
\]

Since \( \lambda \) is constant, we have \( \lambda = 0 \) or \( \beta = 0 \) on \( M \). If \( \beta = 0 \) in \( M \), then \( \xi \) is principal and \( M \) has at most two distinct principal curvatures.

Suppose \( \lambda = 0 \) on \( M \). Then (2.8) is reduced to

\[
(2.10) \quad A\xi = \alpha \xi + \beta U, \quad AU = \beta \xi \text{ and } A\phi U = 0.
\]

Now let us show that the distribution \( T_0 \) defined by \( T_0(x) = \{X \in T_x(M) : \eta(X) = 0\} \) is integrable and totally geodesic in \( M \). If \( \beta = 0 \) on \( M \), then by a theorem of Ki and Suh [5] that \( \alpha \) is constant. Thus \( M \) has at most two distinct constant principal curvatures.

Now let us introduce a theorem of Montiel [9]

**Theorem A.** If \( M \) is a complete real hypersurface of \( H_n(C), n \geq 3 \), with at most two principal curvatures at each point, then \( M \) is congruent to one of the following spaces:

- (A) A “self-tube” \( M_n^* \),
- (A) A geodesic hypersphere,
- (A) A tube of arbitrary radius over a complex hyperbolic hyperplane,
- (B) A tube of radius \( \log^{1+\sqrt{3}} \) over a totally real hyperbolic hyperplane.

A tube of type (A), that is a geodesic hypersphere, has two constant principal curvatures: \( \lambda = \coth r \) of multiplicity \( 2n-2 \) and \( \alpha = 2\coth 2r \) of multiplicity 1 at each point for some \( r(r > 0) \), a tube of type (B) has two constant principal curvatures: \( \lambda = \tanh r \) of multiplicity \( n-1 \) and \( \alpha = \coth r \) of multiplicity \( n \), and a tube of type (A), which is said to be Montiel tube, has two constant principal curvatures: \( \lambda = 1 \) of multiplicity \( 2n-2 \) and \( \alpha = 2 \) of multiplicity 1. But these are all impossible. These types can not occur for the case A). So we can consider the open set \( M_0 \) of \( M \) defined by \( \beta \neq 0 \). Let \( T_0 \) be a distribution defined by

\[
T_0 = \{X \in T_x(M) : \eta(X) = g(X, U) = g(X, \phi U) = 0\}.
\]

Let \( X \in T_0 \), then using (1.1), (2.7) and (2.10), we have \( AX = 0, \nabla_X \xi = 0 \). Using (1.3) and (2.10), we
have
\[(\nabla_X A)\xi - (\nabla_\xi A)X = \phi X\]
and
\[(\nabla_X A)\xi - (\nabla_\xi A)X\]
\[= \nabla_X (A\xi) - A\nabla_X \xi - \nabla_\xi (AX) + A\nabla_\xi X\]
\[= \nabla_X (\beta U + \alpha \xi) + A\nabla_\xi X\]
\[= (X\beta)U + \beta \nabla_X U + (X\alpha)\xi + A\nabla_\xi X.\]

Hence we have
\[-\phi X + (X\beta)U + \beta \nabla_X U + (X\alpha)\xi + A\nabla_\xi X = 0.\]

This equation yields \(-\phi X + \beta \nabla_X U = 0\) since all other terms are linear combinations of \(\xi\) and \(U\). Thus we get

\[(2.11) \quad \nabla_X U = \frac{1}{\beta} \phi X \text{ on } M_0.\]

This shows that \(M = M_0\), because if \(\{x_j\}\) is a sequence of points in \(M_0\) such that it converges to some boundary point of \(M_0\) [hence \(\beta(x_j) \to 0\) as \(j \to \infty\)], then the sequence \(\{\|\nabla_X U\|(x_j)\}\) diverges. Moreover, using (1.1) and (2.10), we have

\[(2.12) \quad \begin{align*}
0 &= g(-\phi X + (X\beta)U + \beta \nabla_X U + A\nabla_\xi X, U) \\
&= X\beta + g(\nabla_\xi X, AU) \\
&= X\beta - \beta g(X, \nabla_\xi \xi) \\
&= X\beta - \beta g(X, \phi A\xi) \\
&= X\beta.
\end{align*}\]

Next, (1.3) also implies that
\[(\nabla_X A)U - (\nabla_U A)X\]
\[= -\eta(X)\phi U + \eta(U)\phi X + 2g(\phi X, U)\xi\]
\[= 0,
\]
and
\[(\nabla_X A)U - (\nabla_U A)X\]
\[= -\eta(X)\phi^2 U + \eta(\phi U)\phi X + 2g(\phi X, \phi U)\xi\]
\[= 0.
\]
By using (1.1), (2.10) and (2.12) we have

\[
(\nabla_X A)U - (\nabla_U A)X \\
= \nabla_X (AU) - A\nabla_X U - \nabla_U (AX) + A\nabla_U X \\
= \nabla_X (\beta \xi) - \frac{1}{\beta} A\phi X + A\nabla_U X \\
= \beta \nabla_X \xi + A\nabla_U X \\
= A\nabla_U X,
\]

and

\[
(\nabla_X A)\phi U - (\nabla_{\phi U} A)X \\
= \nabla_X (A\phi U) - A\nabla_X \phi U - \nabla_{\phi U} (AX) + A\nabla_{\phi U} X \\
= -A((\nabla_X \phi)U + \phi \nabla_X U) + A\nabla_{\phi U} X \\
= -A(\eta(U)AX - g(AX, U)\xi - \frac{1}{\beta} X) + A\nabla_{\phi U} X \\
= A\nabla_{\phi U} X.
\]

Hence \(\nabla_U X\) and \(\nabla_{\phi U} X\) are orthogonal to \(\xi\) and \(U\).

On the other hand, the equation of Codazzi (1.3) implies that

\[
(\nabla_{\xi} A)U - (\nabla_U A)\xi = -\phi U,
\]

and

\[
(\nabla_{\xi} A)U - (\nabla_U A)\xi \\
= \nabla_{\xi} (AU) - A\nabla_{\xi} U - \nabla_U (A\xi) + A\nabla_U \xi \\
= \nabla_{\xi} (\beta \xi) - A\nabla_{\xi} U - \nabla_U (\beta U + \alpha \xi) \\
= (\xi \beta)\xi + \beta \phi A\xi - A\nabla_{\xi} U - (U \beta)U - \beta \nabla_U U - (U \alpha)\xi \\
= (\xi \beta - U \alpha)\xi + \beta^2 \phi U - A\nabla_{\xi} U - (U \beta)U - \beta \nabla_U U,
\]

by using (2.10), (2.11) and \(\nabla_U \xi = 0\). This implies that

\[
g(\nabla_U U, \phi U) = \beta + \frac{1}{\beta}.
\]

Since \(g(\nabla_U X, U) = 0\) for \(X \in T_1\), using (1.1) and (2.10), we have

\[
(2.13) \quad \nabla_U U = (\beta + \frac{1}{\beta}) \phi U
\]
and
\[ \nabla_U (\phi U) = - (\beta + \frac{1}{\beta}) U. \]
Similarly, we obtain
\[
(\nabla_\xi A) \phi U - (\nabla_\phi U A) \xi \\
= - \eta(\xi) \phi^2 U + \eta(\phi U) \phi \xi + 2 g(\phi \xi, \phi U) \xi \\
= U
\]
and
\[
(\nabla_\xi A) \phi U - (\nabla_\phi U A) \xi \\
= \nabla_\xi (A \phi U) - A \nabla_\xi (\phi U) - \nabla_\phi U (A \xi) + A \nabla_\phi U \xi \\
= - A((\nabla_\xi \phi) U + \phi \nabla_\xi U) - \nabla_\phi U (\beta U + \alpha \xi) \\
= - A(\eta(U) A \xi - g(A \xi, U) \xi + \phi \nabla_\xi U) - (\phi \beta U) U \\
- \beta \nabla_\phi U U - (\phi U \alpha) \xi \\
= \beta A \xi - A \phi \nabla_\xi U - (\phi U \beta) U - \beta \nabla_\phi U U - (\phi U \alpha) \xi.
\]
These equations yield
\[(2.14) \quad \nabla_\phi U U = 0,
\]
because \(\nabla_\phi U U\) is orthogonal to \(\xi\) and \(U\), and
\[(2.15) \quad \phi U \beta = \beta^2 - 1.\]
Then using (1.1) and (2.14), we have
\[(2.16) \quad \nabla_\phi U (\phi U) = 0.\]
From these equations we can easily show that if \(X\) and \(Y\) are contained in \(T_0\), then \(\nabla_X Y \in T_0\) and \([X, Y] = \nabla_X Y - \nabla_Y X \in T_0\). Hence \(T_0\) is integrable and totally geodesic in \(M\). Moreover, (2.10) means that the integral manifold of \(T_0\) is totally geodesic in \(H_{n}(C)\). Since \(T_0\) is \(J\)-invariant, its integral manifold is a complex hypersurface \(H_{n-1}(C)\).

Next, we study the case (B). We put \(\lambda = \lambda_1\). Let \(M_1\) be an open subset of \(M\), defined by \(\lambda \neq 0\). Then we can choose an orthogonal frame field
\[\{e_1, \phi e_1, e_2, \phi e_2, \ldots, e_{n-1}, \phi e_{n-1}, \xi\}\]
on \(M_1\) such that
\[
A \xi = \alpha \xi + \beta_1 e_1 + \beta_1 \phi e_1 + \beta_2 e_2, \\
A e_1 = \lambda e_1 + \beta_1 \xi, \quad A \phi e_1 = \beta_1 \xi, \quad A e_2 = \beta_2 \xi, \\
A \phi e_j = A e_k = 0 \quad (2 \leq j \leq n - 1, \ 3 \leq k \leq n - 1).
\]
Firstly we show that the distribution $T_2$ defined by $T_2(x) = \{ X \in T_x(M) : \eta(X) = g(X, e_1) = 0 \}$ is integrable. Let $X, Y \in T_2$. Then by means of (1.1), (1.3) and (2.15) we obtain

$$(\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A) e_1$$

$$= -\eta(e_1) \phi^2 e_1 + \eta(\phi e_1) \phi e_1 + 2g(\phi e_1, \phi e_1) \xi$$

$$= 2\xi,$$

and

$$(\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A) e_1$$

$$= \nabla_{e_1} (A \phi e_1) - A \nabla_{e_1} (\phi e_1) - \nabla_{\phi e_1} (A e_1) + A \nabla_{\phi e_1} e_1$$

$$= \nabla_{e_1} (\beta_1 \xi) - A \nabla_{e_1} (\phi e_1) - \nabla_{\phi e_1} (\lambda e_1 + \beta_1 \xi) + A \nabla_{\phi e_1} e_1$$

$$= (\phi e_1 \beta_1) \xi - \beta_1 \nabla_{\phi e_1} \xi + A \nabla_{\phi e_1} e_1$$

$$= (e_1 \beta_1) \xi + \lambda \beta_1 \phi e_1 - A \nabla_{e_1} (\phi e_1) - (\phi e_1 \lambda) e_1 - \lambda \nabla_{\phi e_1} e_1$$

$$= (\phi e_1 \beta_1) \xi + A \nabla_{\phi e_1} e_1.$$

By taking the inner product with $\phi e_1$ we get

$$0 = g((\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A) e_1, \phi e_1)$$

$$= g((e_1 \beta_1) \xi + \lambda \beta_1 \phi e_1 - A \nabla_{e_1} (\phi e_1) - (\phi e_1 \lambda) e_1 - \lambda \nabla_{\phi e_1} e_1$$

$$= (\phi e_1 \beta_1) \xi + A \nabla_{\phi e_1} e_1, \phi e_1)$$

$$+ g(\nabla_{\phi e_1} e_1, A \phi e_1)$$

$$= \lambda \beta_1 - g(\nabla_{e_1} (\phi e_1), A \phi e_1) - \lambda g(\nabla_{\phi e_1} e_1, \phi e_1)$$

$$= \lambda \beta_1 - \beta_1 g(\phi e_1, \phi A e_1) - \lambda g(\nabla_{\phi e_1} e_1, \phi e_1)$$

$$= 2\lambda \beta_1 - \lambda g(\nabla_{\phi e_1} e_1, \phi e_1).$$

Hence, we obtain $g(\nabla_{\phi e_1} e_1, \phi e_1) = 2\beta_1$ on $M_1$. Similarly, we have the followings by taking the inner product with $e_2$ and $Y$ ($Y$ is orthogonal to $e_1, \phi e_1, e_2$ and $\xi$):

$$0 = g((\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A) e_1, e_2)$$

$$= -g(\nabla_{e_1} (\phi e_1), A e_2) - \lambda g(\nabla_{\phi e_1} e_1, e_2) + g(\nabla_{\phi e_1} e_1, A e_2)$$

$$= \beta_2 g(\phi e_1, \nabla_{e_1} \xi) - \lambda g(\nabla_{\phi e_1} e_1, e_2) - \beta_2 g(e_1, \nabla_{e_1} \xi)$$

$$= \beta_2 g(\phi e_1, \phi A e_1) - \lambda g(\nabla_{\phi e_1} e_1, e_2) - \beta_2 g(e_1, \phi A e_1)$$

$$= \lambda \beta_2 - \lambda g(\nabla_{\phi e_1} e_1, e_2).$$
and
\[
0 = g((\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A)e_1, Y)
\]
\[
= -g(\nabla_{e_1} (\phi e_1), Ae_2) - \lambda g(\nabla_{\phi e_1} e_1, e_2) + g(\nabla_{\phi e_1} e_1, Ae_2)
\]
\[
= \beta_2 g(\phi e_1, \nabla_{e_1} \xi) - \lambda g(\nabla_{\phi e_1} e_1, e_2) - \beta_2 g(e_1, \nabla_{\phi e_1} \xi)
\]
\[
= \beta_2 g(\phi e_1, \phi Ae_1) - \lambda g(\nabla_{\phi e_1} e_1, e_2) - \beta_2 g(e_1, \phi A\phi e_1)
\]
\[
= \lambda \beta_2 - \lambda g(\nabla_{\phi e_1} e_1, e_2),
\]
and
\[
0 = g((\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A)e_1, Y)
\]
\[
= -g(\nabla_{e_1} (\phi e_1), AY) - \lambda g(\nabla_{\phi e_1} e_1, Y) + g(\nabla_{\phi e_1} e_1, AY)
\]
\[
= -\lambda g(\nabla_{\phi e_1} e_1, Y).
\]
Hence, we have \(g(\nabla_{\phi e_1} e_1, e_2) = \beta_2\) and \(g(\nabla_{\phi e_1} e_1, Y) = 0\) on \(M_1\). Since \(\nabla_{\phi e_1} \xi = \phi A\phi e_1 = \phi(\bar{\beta}_1 \xi) = \bar{\beta}_1 (\phi \xi) = 0\), we get
\[
(2.18) \quad \nabla_{\phi e_1} e_1 = 2\bar{\beta}_1 \phi e_1 + \beta_2 e_2.
\]
Moreover, we obtain
\[
0 = g((\nabla_{e_1} A) \phi e_1 - (\nabla_{\phi e_1} A)e_1, e_1)
\]
\[
= -g(\nabla_{e_1} (\phi e_1), Ae_1) - (\phi e_1 \lambda) + g(\nabla_{\phi e_1} e_1, e_1 + \beta_1 \xi)
\]
\[
= -g(\nabla_{e_1} (\phi e_1), \nabla_{e_1} e_1) + \beta_1 g(\phi e_1, \phi Ae_1) - (\phi e_1 \lambda) - \beta_1 g(e_1, \phi A\phi e_1)
\]
\[
= \lambda g(\phi e_1, \nabla_{e_1} e_1) + \beta_1 g(\phi e_1, \phi Ae_1) - (\phi e_1 \lambda) - \beta_1 g(e_1, \phi A\phi e_1)
\]
\[
= \lambda g(\phi e_1, \nabla_{e_1} e_1) + \lambda \beta_1 - (\phi e_1 \lambda).
\]
Hence we get
\[
(2.19) \quad g(\nabla_{e_1} e_1, \phi e_1) = \phi e_1 |\log |\lambda| - \beta_1
\]
on \(M_1\). Now from the equation of Codazzi (1.3) it follows
\[
0 = g((\nabla_{e_1} A)e_2 - (\nabla_{e_2} A)e_1, e_1)
\]
\[
= g(\nabla_{e_1} (Ae_2) - A\nabla_{e_1} e_2 - \nabla_{e_2} (Ae_1) + A\nabla_{e_2} e_1, e_1)
\]
\[
= g(\beta_2 \phi Ae_1, e_1) - g(\nabla_{e_1} e_2, Ae_1) - e_2 \lambda
\]
\[
+ \beta_1 g(\phi Ae_2, e_1) + g(\nabla_{e_2} e_1, Ae_1)
\]
\[
= -\lambda g(\nabla_{e_1} e_2, e_1) - e_2 \lambda,
\]
so that
\[
(2.20) \quad g(\nabla_{e_1} e_2, e_1) = -\frac{1}{\lambda} e_2 \lambda = -e_2 |\log |\lambda|}
\]
Also from the equation of Codazzi (1.3) we have

\[ 0 = g((\nabla e_1 A)Y - (\nabla Y A)e_1, e_1) \]
\[ = g(\nabla e_1 (AY) - A\nabla e_1 Y - \nabla Y (Ae_1) + A\nabla Y e_1, e_1) \]
\[ = -g(\nabla e_1 Y, Ae_1) - g(\nabla Y (Ae_1), e_1) + g(\nabla Y e_1, Ae_1) \]
\[ = -g(\nabla e_1 Y, \lambda e_1 + \beta_1 \xi) - g(\beta_1 \phi AY, e_1) - \beta_1 g(e_1, \phi AY) - Y\lambda \]
\[ = -\lambda g(\nabla e_1 Y, e_1) + \beta_1 g(Y, \nabla e_1 \xi) - Y\lambda, \]

from this it follows

\[ (2.21) \quad g(\nabla e_1 Y, e_1) = -\frac{1}{\lambda} Y\lambda = -Y\log|\lambda|. \]

where \( Y \) is orthogonal to \( e_1, \phi e_1, e_2 \) and \( \xi \). Now if we put

\[ \nabla e_2 e_1 = \gamma \phi e_1 + \delta e_2 + \epsilon Y, \]

then we can prove \( \gamma = \beta_2 \), and the functions \( \delta, \epsilon \) are all vanishing. In fact

\[ 0 = g((\nabla e_1 A)e_2 - (\nabla e_2 A)e_1, \phi e_1) \]
\[ = g(\nabla e_1 (Ae_2) - A\nabla e_1 e_2 - \nabla e_2 (Ae_1) + A\nabla e_2 e_1, \phi e_1) \]
\[ = \beta_2 g(\phi Ae_1, \phi e_1) - g(\nabla e_1 e_2, \beta_1 \xi) - \lambda g(\nabla e_2 e_1, \phi e_1) \]
\[ -\beta_1 g(\phi Ae_2, \phi e_1) + g(\nabla e_2 e_1, A\phi e_1) \]
\[ = \lambda \beta_2 - \lambda g(\nabla e_2 e_1, \phi e_1). \]

This means \( \gamma = \beta_2 \). Moreover, we get \( \delta = 0 \) and \( \epsilon = 0 \), because

\[ 0 = g((\nabla e_1 A)e_2 - (\nabla e_2 A)e_1, e_2) \]
\[ = g(\nabla e_1 (Ae_2) - A\nabla e_1 e_2 - \nabla e_2 (Ae_1) + A\nabla e_2 e_1, e_2) \]
\[ + g(\nabla e_2 e_1, Ae_2) \]
\[ = -\lambda g(\nabla e_2 e_1, e_2), \]

and

\[ 0 = g((\nabla e_1 A)e_2 - (\nabla e_2 A)e_1, Y) \]
\[ = g(\nabla e_1 (Ae_2) - A\nabla e_1 e_2 - \nabla e_2 (Ae_1) + A\nabla e_2 e_1, Y) \]
\[ = -\lambda g(\nabla e_2 e_1, Y). \]

From these formulars we can write

\[ (2.22) \quad \nabla e_2 e_1 = \beta_2 \phi e_1. \]
Accordingly, the formulars above (2.19), (2.20) and (2.21), we have the following
\[
\nabla_{\varepsilon_1} e_1 = (e_2 \log |\lambda|) e_2 + (X \log |\lambda|) X \\
+ (\phi e_1 \log \lambda - \beta_1) \phi e_1 \\
= \text{grad } \log |\lambda| - \beta_1 \phi e_1.
\]
This shows that \( M = M_1 \). In fact, if \( M \neq M_1 \), then by the definition of \( M_1 \), there is a sequence of points \( \{y_j\} \) in \( M_1 \), so that \( \lim_{j \to \infty} \lambda(y_j) = 0 \). Then the above equation implies that the sequence
\[
\{||\nabla_{\varepsilon_1} e_1 + \beta_1 \phi e_1||(y_j)\}\]
diverges.

Next let us show that the distribution \( T_2 \) is integrable. For this by making use of (2.17), (2.20), (2.21) and (2.22) we have
\[
g([\phi e_1, e_2], \xi) = g(\nabla_{\phi e_1} e_2, \xi) - g(\nabla_{e_2} \phi e_1, \xi) \\
= -g(e_2, \phi A \phi e_1) + g(\phi e_1, \phi A e_2) \\
= 0,
\]
\[
g([\phi e_1, Y], \xi) = g(\nabla_{\phi e_1} Y, \xi) - g(\nabla_{Y} \phi e_1, \xi) \\
= 0 \\
= 0 \\
g([e_2, Y], \xi_1) = g(\nabla_{e_2} Y, \xi) - g(\nabla_{Y} e_2, \xi_1) \\
= 0 \\
g([Y, Z], \xi) = -g(Z, \phi A Y) + g(Y, \phi A Z) \\
= 0,
\]
where \( Y \) and \( Z \) are orthogonal to \( e_1, \phi e_1, e_2 \) and \( \xi \). Accordingly, the distribution \( T_2 \) is integrable.

Secondly, we show that each integral manifold of the distribution \( T_2 \) is contained in some complex hyperplane \( H_{n-1}(C) \). Now let us denote by \( L \) be a leaf of \( T_2 \). Then the normal space of the leaf \( L \) in \( H_n(C) \) is spanned by \( N, \xi \) and \( e_1 \). We denote \( \bar{A} \) (respectively \( \nabla^\perp \)) the shape operator (respectively the normal connection) of \( L \) in \( H_n(C) \). Then the equations of Weingarten and Gauss are given by
\[
\nabla_X \xi = -\bar{A}_\xi X + \nabla^\perp_X \xi, \quad \nabla_X e_1 = \sigma(X, \xi) \\
\nabla_X e_1 = -\bar{A}_{e_1} X + \nabla^\perp_X e_1, \quad \nabla_X e_1 = \nabla_X e_1 + \sigma(X, e_1) \\
\nabla_X N = -\bar{A}_N X + \nabla^\perp_X N, \quad \nabla_X N = -J \sigma(X, \xi),
\]
\[(2.23)\]
where \( X \in T_2 \), \( \nabla \) denotes the induced connection defined on \( M \) and \( \sigma \) denotes the second fundamental form of \( M \) in \( H_n(C) \). Now, making use of (1.1), (2.17), (2.18), (2.22) and (2.23), we have

\[
\bar{A}_\xi = 0, \quad \bar{A}_N = 0,
\]
\[
\bar{A}_{e_1}(\phi e_1) = -(\nabla_{\phi e_1} e_1)^T = -(2\bar{\beta}_1 \phi e_1 + \beta_2 e_2)^T
\]
\[
= -2\bar{\beta}_1 \phi e_1 - \beta_2 e_2,
\]
\[
\bar{A}_{e_2} e_2 = -(\nabla_{e_2} e_1)^T = -\beta_2 \phi e_1,
\]
\[
\bar{A}_{e_1} Y = -(\nabla_Y e_1)^T = 0,
\]

where \((X)^T\) denotes a \( T_2 \)-component of the vector field \( X \). Moreover, from the equations of Gauss and Weingarten we have

\[
\bar{\nabla}_{\phi e_1} \xi = \nabla_{\phi e_1} \xi + h(\phi e_1, \xi)N
\]
\[
= -\bar{A}_\xi \phi e_1 + \nabla^\perp_{\phi e_1} \xi.
\]

From this it follows

\[
\nabla^\perp_{\phi e_1} \xi = \bar{\beta}_1 N,
\]

where \( h(\phi e_1, \xi) \) is given by \( \bar{\beta}_1 \). Similar method by (2.20) we have

\[
\nabla^\perp_X e_1 = 0, \quad \nabla^\perp_Y N = 0
\]
\[
\nabla^\perp_{e_1} N = -\bar{\beta}_1 \xi, \quad \nabla^\perp_Y N = 0,
\]

where \( X, Y \in T_2 \) and \( Y_{\perp} \phi e_1, e_2 \). Hence the orthogonal complement of the first normal space is spanned by \( \xi \) and \( N \), and it is invariant under the complex structure \( J \) of \( H_n(C) \). Moreover, this subspace is invariant under the parallel displacement with respect to the normal connection \( \nabla^\perp \).

Now we define the following

Define. For \( x \in M^n \), the first normal space, \( N_1(x) \), is the orthogonal complement in \( T^\perp_x M \) of the set

\[
N_0(x) = \{ \xi \in T^\perp_x (M^n) | A_\xi = 0 \}.
\]

Now let us introduce the following

Proposition. Let \( f : M^n \rightarrow M^{n+p}(C) \) be a holomorphic and isometric immersion of a connected, complete, complex n-dimensional Kaehler manifold \( M^n \) into a complex space form \( M^{n+p}(C) \). Suppose the first normal space \( N_1(x) \) has constant dimension \( k \), and is parallel with respect to normal connection. Then
there is a totally geodesic \((n + k)\)-dimensional submanifold \(M^{n+k}(c)\) such that \(f(M^n) \subset M^{n+k}(c)\).

Cecil [3] proved the proposition above for submanifolds in a complex projective space \(P_{n+p}(C)\). With the minor change of the method used by Cecil [3] this proposition can be easily proved even for submanifolds in a complex Euclidean space \(C^{n+p}\) or in a complex hyperbolic space \(H_{n+p}(C)\). Though Cecil [3] proved this proposition for \(K\)epleher submanifolds, but he only used the invariance of the orthogonal complement of the first normal space by the complex structure \(J\). From this point of view and the formula above (2.24),(2.25) it follows the leaf \(L\) is contained in a totally geodesic \(H_{n-1}(C)\) in \(H_n(C)\). Finally, we can conclude that the leaf \(L\) in \(H_{n-1}(C)\) is a ruled real hypersurface, because of (2.23) and the consideration of the case (A). Thus we complete the proof of our assertion.

3. Examples of ruled real hypersurfaces in a complex hyperbolic space \(H_n(C)\)

In this section we give some examples of ruled real hypersurfaces of \(H_n(C)\). First of all, we recall about the fibration

\[
\pi : \ H_1^{2n+1} \longrightarrow H_n(C).
\]

In a complex Euclidean space \(C^{n+1}\) with the standard basis, let \(F\) be a Hermitian form defined by

\[
F(z, w) = -z_0 \bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k,
\]

where \(z = (z_0, ..., z_n)\) and \(w = (w_0, ..., w_n)\) are in \(C^{n+1}\). Then \((C^{n+1}, F)\) is a complex Minkowski space, which is simply denoted by \(C_1^{n+1}\). The scalar product given by \(\Re F(z, w)\) is an indefinite metric of index 2 in \(C_1^{n+1}\), where \(\Re z\) denotes the real part of the complex number \(z\). Let \(H_1^{2n+1}\) be a real Lorentzian hypersurface of \(C_1^{n+1}\) defined by

\[
H_1^{2n+1} = \{ z \in C_1^{n+1} : F(z, z) = -1 \},
\]

and let \(G\) be a Lorentzian metric of \(H_1^{2n+1}\) induced from the Lorentzian metric \(\Re F\). Then \((H_1^{2n+1}, G)\) is the Lorentzian manifold of constant sectional curvature -1, which is called an anti-De Sitter space.

For the anti-De Sitter space \(H_1^{2n+1}\) the tangent space \(T_z(H_1^{2n+1})\) at each point \(z\) can be identified (through the parallel displacement in \(C_1^{n+1}\)) with

\[
\{ w \in C_1^{n+1} | \Re F(z, w) = 0 \}.
\]
Let us denote by $T^\prime_z$ the orthogonal complement of the vector $iz$ in $T_zH_1^{2n+1}$, that is,

$$T^\prime_z = \{w \in C^{n+1}_1 : \Re F(z, w) = 0, \Re F(i z, w) = 0\}.$$

Let $S^1$ be the multivariate group of complex numbers of absolute value 1. Then $H_1^{2n+1}$ can be regarded as a principal fiber bundle over a complex hyperbolic space $H_n(C)$ with the group $S^1$ and the projection $\pi$. Furthermore, there is a connection such that $T^\prime_z$ is the horizontal subspace at $z$ which is invariant under the $S^1$-action. The natural projection $\pi$ of $H_1^{2n+1}$ onto $H_n(C)$ induces a linear isomorphism of $T^\prime_z$ onto $T_p(H_n(C))$, where $p = \pi(z)$. The metric $g$ of constant holomorphic sectional curvature $-4$ is given by $g_p(X, Y) = \Re F_z(X^\ast, Y^\ast)$ for any tangent vectors $X$ and $Y$ in $T_p(H_n(C))$, where $z$ is any point in the fiber $\pi^{-1}(p)$ and $X^\ast$ and $Y^\ast$ are vectors in $T^\prime_z$ such that $d\pi(X^\ast) = X$ and $d\pi(Y^\ast) = Y$, where $d\pi$ denotes the differential of the projection $\pi$.

On the other hand, a complex structure $J : w \to iw$ in the subspace $T^\prime_z$ is compatible with the action of $S^1$ and induces an almost complex structure $J$ on $H_n(C)$ such that $d\pi \circ J = J \circ d\pi$. Thus $H_n(C)$ is a complex hyperbolic space of constant holomorphic sectional curvature $-4$ and it is seen that the principal $S^1$-bundle $H_1^{2n+1}$ over $H_nC$ with the projection $\pi$ is a semi-Riemannian submersion with the fundamental tensor $J$ and totally geodesic time-like fibers.

Given a real hypersurface of $H_n(C)$, one can construct a Lorentzian hypersurface $N$ of $H_1^{2n+1}$ which is a principal $S^1$-bundle over $M$ with compatible totally geodesic time-like fibers and the projection $\pi : N \to M$ in such a way that the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{i'} & H_1^{2n+1} \\
\downarrow & & \downarrow \pi \\
M & \xrightarrow{i} & H_n(C)
\end{array}
\]

is commutative ($i, i'$ being the isometric immersions).

**Example 3.1** Now, a ruled real hypersurface $M$ of $H_n(C)$, can be defined as follows: Let $\gamma : I \to H_n(C)$ be any regular curve. Then for any $t(\in I)$ let $H^{(t)}_{n-1}(C)$ be a totally geodesic complex hypersurface of $H_n(C)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in H^{(t)}_{n-1}(C) : t \in I\}$. Then, by the construction, $M$ becomes a real hypersurface of $H_n(C)$, which is called a ruled real hypersurface. This means that there are many ruled real hypersurfaces of $H_n(C)$. Let $T_0$ be a distribution defined by $T_0(x) = \{u \in T_xM : u \perp \xi(x)\}$ in the tangent space $T_xM$ of $M$ at any point $x$ in $M$. Then it can be easily verified that the shape operator $A$ of a ruled real
hypersurface $M$ of $H_n(C)$ satisfies

$$A\xi = \alpha \xi + \beta U \quad (\beta \neq 0), \quad AU = \beta \xi, \quad AX = 0$$

for any vector $X$ orthogonal to $\xi$ and $U$, where $U$ is a unit vector orthogonal to $\xi$, and $\alpha$ and $\beta$ are smooth functions on $M$.

**Example 3.2** ([1]) In this example we construct a minimal ruled real hypersurfaces in $H_n(C)$. Now, let us denote by $N$ a real Lorentzian hypersurface of $H_1^{2n+1}$ given by

$$N = \{ z = (re^{i\phi}\cosh \theta, re^{i\phi}\sinh \theta, (r^2 - 1)^{\frac{1}{2}} z_2, ..., (r^2 - 1)^{\frac{1}{2}} z_n) \in C_1^{2n+1} = C_1^2 \times C_{n-1} : \Sigma_{j=2}^n |z_j|^2 = 1, r > 1, 0 \leq \phi < 2\pi, \quad \theta \in \mathbb{R} \}.$$ 

Let $T_z''$ be the subspace consisting of vectors $w = (0, 0, w_2, ..., w_n)$ orthogonal to vectors $z$ and $u$ through the parallel transformation in $C_1^{2n+1}$. These vectors span the tangent space $T_z N$ at the point $z$. A unit space-like normal vector $\overline{C}_z$ at $z$ is given by

$$\overline{C}_z = (ie^{i\phi}\sinh \theta, ie^{i\phi}\cosh \theta, 0, ..., 0).$$

On the other hand, for the vertical vector $iz$ with respect to the submersion $\pi$ at $z$ we have $iz = (\frac{\partial}{\partial \phi})_z + u$. We put

$$\overline{U}_z = (r^2 - 1)^{\frac{1}{2}} r^{-1} (\frac{\partial}{\partial \phi})_z + r(r^2 - 1)^{-\frac{1}{2}} u.$$ 

Then $\overline{U}_z$ is a unit space-like horizontal vector in $T_z N$ and again by (3.1) and (3.2) we get

$$\overline{J} \overline{U}_z = i\overline{U}_z = -(r^2 - 1)^{\frac{1}{2}} (\frac{\partial}{\partial r})_z.$$ 

Moreover, if we put $\overline{\xi}_z = -i\overline{C}_z$, then we get $\overline{\xi}_z = \frac{1}{r} (\frac{\partial}{\partial \theta})_z$.

Given for the Lorentzian hypersurface $N$ of a $(2n+1)$-dimensional anti-de Sitter space $H_1^{2n+1}$ in $C_1^{2n+1}$, a real hypersurface $M$ of a complex hyperbolic space $H_n(C)$ is given as follows: $N$ is a principal $S^1$- bundle over $M$ with time-like totally geodesic fibers and the projection $\pi : N \rightarrow M$. Since $N$ is $S^1$-invariant, $C_{\pi(z)} = d\pi(\overline{C}_z)$ provides a unit vector normal to $M$. The tangent space $T_x M$ of $M$ at $x = \pi(z)$ is spanned by the vectors $\xi_x = d\pi(\xi_z)$, $U_x = d\pi(\overline{U}_z)$, $\phi \overline{U}_x = J\overline{U}_x = d\pi(J\overline{U}_z)$ and $X_x = d\pi(X_z)$ where any vector $X_z \in T''_z$. In particular, $\xi$ is the structure vector field on $M$. It is seen by Montiel and Romero [10] that the shape operator $A$ of $M$ satisfies $AX = d\pi(\overline{AX})$, where $\overline{X}$ is the horizontal lift of the vector field $X$ on $M$. 
Now, because of \( u = -\left(\frac{\partial}{\partial z}\right)z + iz \), we get
\[
\tilde{U}_z = -r^{-1}(r^2 - 1)^{-\frac{1}{2}}(\frac{\partial}{\partial \phi})z + r(r^2 - 1)^{-\frac{1}{2}}iz
\]
by means of the definition of \( u \) and \( \tilde{U}_z \). Accordingly, by (3.4) and the above equation we have
\[
A\xi_x = d\pi(\frac{1}{r} \frac{\partial}{\partial \theta})z) = -\frac{1}{r^2} d\pi((\frac{\partial}{\partial \phi})z) = \frac{1}{r^2} d\pi(r(r^2 - 1)^{\frac{1}{2}} \tilde{U}_z)
\]
and therefore we get
\[
A\xi = (r^2 - 1)^{\frac{1}{2}} r^{-1} U_x,
\]
because \( iz \) is vertical. By the similar calculation to that developed as above we have also the following relations:
\[
A\xi = (r^2 - 1)^{\frac{1}{2}} r^{-1} U, \quad AU = (r^2 - 1)^{\frac{1}{2}} r^{-1} \xi, \quad AX = 0
\]
for any vector field \( X \) orthogonal to the structure vector field \( \xi \) and \( U \). By the similar discussion to that of the proof of the theorem the equations (3.7) mean that the real hypersurface \( M \) is minimal and the distribution \( T_0 \) is defined by \( \{ X(x) \in T_x M : X \perp \xi \} \) is integrable. Moreover the integral manifold is totally geodesic in \( H_{n-1}(C) \). Since \( T_0 \) is \( J \)-invariant, its integral manifold is a complex hypersurface \( H_{n-1}(C) \) and \( M \) is the ruled real hypersurface.

References


