A CHARACTERIZATION OF RULED REAL HYPERSURFACES IN $P_n(C)$

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Introduction

Let $P_n(C)$ denote an $n$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$. Real hypersurfaces in $P_n(C)$ have been studied by many differential geometers (See [2], [3], [6], [9] and [11]).

As for a problem concerned with the type number $t$ which is defined by the rank of the second fundamental tensor of real hypersurfaces $M$ in $P_n(C)$, Takagi [9], Yano and Kon [11] showed that there is a point $p$ in $M$ such that $t(p) \leq 2$.

On the other hand, Kimura and Maeda [4] found a non-trivial example of non-homogeneous real hypersurfaces in $P_n(C)$ which is called a ruled real hypersurface. Also it is known that this ruled real hypersurface is not complete and its type number is equal to 2 on the whole $M$ (See Kimura and Maeda [5]). Then it naturally rises to the question that "Is a ruled real hypersurface the only real hypersurface of $P_n(C)$ ($n \geq 3$) satisfying $t = 2$". The purpose of this paper is to answer this problem affirmatively. Thus as a characterization of a ruled real hypersurface we have the following

**THEOREM A.** Let $M$ be a real hypersurface in $P_n(C)$ ($n \geq 3$) satisfying $t(p) \leq 2$ for any point $p$ in $M$. Then $M$ is a ruled real hypersurface.

It is known that a ruled real hypersurface in $P_n(C)$ ($n \geq 3$) is not complete. Thus, as an application of Theorem A, we also have the following

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THEOREM B. Let \( M \) be a complete real hypersurface in \( P_n(\mathbb{C}) \) \((n \geq 3)\). Then there exists a point \( p \) on \( M \) such that \( t(p) \geq 3 \).

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1. Preliminaries

Let \( M \) be a real hypersurface in \( P_n(\mathbb{C}) \) \((n \geq 2)\). Let \( \{e_1, \ldots, e_{2n}\} \) be a local field of orthonormal frames in \( P_n(\mathbb{C}) \) such that, restricted to \( M \), \( e_1, \ldots, e_{2n-1} \) are tangent to \( M \). Denote its dual frame field by \( \theta_1, \ldots, \theta_{2n} \). We use the following convention on the range of indices unless otherwise stated; \( A, B, \ldots, = 1, \ldots, 2n \) and \( i, j, \ldots, = 1, \ldots, 2n-1 \).

The connection forms \( \theta_{AB} \) are defined as the 1-forms satisfying

\[
\begin{align*}
(1.1) \quad d\theta_A &= -\sum \theta_{AB} \wedge \theta_B, \quad \theta_{AB} + \theta_{BA} = 0.
\end{align*}
\]

Restrict the forms under consideration to \( M \). Then, we get \( \theta_{2n} = 0 \) and the forms \( \theta_{2n,i} \) can be written as

\[
(1.2) \quad \phi_i \equiv \theta_{2n,i} = \sum h_{ij} \theta_j, \quad h_{ij} = h_{ji}.
\]

The quadratic form \( \sum h_{ij} \theta_i \otimes \theta_j \) is called the second fundamental form of \( M \) with direction of \( e_{2n} \). The curvature forms \( \Theta_{ij} \) of \( M \) are defined by

\[
(1.3) \quad \begin{align*}
\Theta_{ij} &= d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj}, \\
\Theta_{ij} &= \frac{1}{2} \sum R_{ijk\ell} \theta_k \wedge \theta_\ell,
\end{align*}
\]

where \( R_{ijk\ell} \) denotes the component of the Riemannian curvature tensor of \( M \).

We denote by \( J \) the complex structure of \( P_n(\mathbb{C}) \), and put

\[
Je_i = \sum J_{ji} e_j + f_i e_{2n}.
\]

Then the almost contact structure \( (J_{ij}, f_k) \) satisfies

\[
(1.4) \quad \begin{align*}
\sum J_{ik} J_{kj} &= f_i f_j - \delta_{ij}, \quad \sum f_j J_{ji} = 0 \\
\sum f_i^2 &= 1, \quad J_{ij} + J_{ji} = 0.
\end{align*}
\]
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\begin{equation}
    dJ_{ij} = \sum (J_{ik}\theta_{kj} - J_{jk}\theta_{ki}) - f_i\phi_j + f_j\phi_i,
\end{equation}

\begin{equation}
    df_i = \sum (f_j\theta_{ji} - J_{ji}\phi_j).
\end{equation}

The equations of Gauss and Codazzi are given by

\begin{equation}
    \Theta_{ij} = \phi_i \wedge \phi_j + c\theta_i \wedge \theta_j
    + c \sum (J_{ik}J_{jt} + J_{ij}J_{kt})\theta_k \wedge \theta_t,
\end{equation}

\begin{equation}
    d\phi_i = -\sum \phi_j \wedge \theta_{ji}
    + c \sum (f_iJ_{jk} + f_jJ_{ik})\theta_j \wedge \theta_k,
\end{equation}

respectively. Then it follows from (1.3) and (1.6) that the components of the Riemannian curvature tensor are given by

\begin{equation}
    R_{ijkl} = c\{(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}) + J_{ik}J_{jt} - J_{it}J_{jk} + 2J_{ij}J_{kt}\}
    + h_{ik}h_{jt} - h_{it}h_{jk}.
\end{equation}

2. Lemmas

Let $M$ be a real hypersurface in $P_n(\mathbb{C})$. We choose an arbitrary point $p$ in $M$, and use the following convention on the range of indices; $a, b, \ldots, = 1, \ldots, t(p)$ and $r, s, \ldots, = t(p) + 1, \ldots, 2n - 1$. Then we can take a field $\{e_1, \ldots, e_{2n}\}$ of orthonormal frames on a neighborhood of $p$ in such a way that the 1-forms $\phi_i$ can be written as

\begin{equation}
    \phi_a = \sum h_{ba}\theta_b, \quad h_{ab} = h_{ba},
    \phi_r = 0,
\end{equation}

at $p$. We call such a field $\{e_1, \ldots, e_{2n}\}$ to be associated with a point $p$.

Under this notation we have

**Lemma 2.1.** Assume that $J_{rs}(p) = 0$ at a point $p$ on $M$. Then $t(p) \geq n - 1$. Furthermore, the equality holds if and only if $f_a = 0$ and $J_{ab} = 0$ at $p$.

**Proof.** By (1.4) we have

\begin{equation}
    \sum b J_{ab}^2 + \sum r J_{ar}^2 + f_a^2 = 1,
\end{equation}

\begin{equation}
    \sum a J_{ra}^2 + f_r^2 = 1,
\end{equation}

\begin{equation}
    \sum (\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}) + J_{ik}J_{jt} - J_{it}J_{jk} + 2J_{ij}J_{kt}\}
    + h_{ik}h_{jt} - h_{it}h_{jk}.
\end{equation}
Summing up (2.2) on $a$, and (2.3) on $r$, we have

$$\sum_{a,b} J_{ab}^2 + \sum_{a,r} J_{ar}^2 + \sum_{a} f_a^2 = t(p), \tag{2.4}$$

$$\sum_{a,r} J_{ar}^2 + \sum_{r} f_r^2 = 2n - 1 - t(p). \tag{2.5}$$

Substituting (2.5) into (2.4) and making use of $\sum_a f_a^2 + \sum_r f_r^2 = 1$, we have

$$\sum_{a,b} J_{ab}^2 + 2\sum_{a} f_a^2 = 2(t(p) - (n - 1)) \geq 0,$$

and so our assertion follows.

This concludes the proof.

Now we consider a point $p$ where the type number $t$ attains the maximal value, say $T$. Then there is a neighborhood $U$ of $p$, on which the function $t$ is constant and the equation (2.1) holds.

Put $\theta_{ar} = \sum A_{arb} \theta_b + \sum B_{ars} \theta_s$. Then, taking the exterior derivative of $\phi_r = 0$ and using (1.7), we have

$$\sum h_{ab} \theta_b \wedge \theta_{ar} - c \sum (f_r J_{ij} + f_i J_{rj}) \theta_i \wedge \theta_j = 0,$$

from which we have

$$\sum h_{ab} B_{brs} - cf_a J_{rs} + cf_s J_{ra} - 2cf_r J_{as} = 0, \tag{2.6}$$

$$f_s J_{rt} - f_t J_{rs} + 2f_r J_{st} = 0. \tag{2.7}$$

It is easy to see that (2.7) is reduced to

$$f_r J_{st} = 0. \tag{2.8}$$

Under such a situation we have

**Lemma 2.2.** If $J_{rs} = 0$ on $U$, then $T \geq n$ on $U$.

**Proof.** If $T < n$, then by Lemma 2.1 we have $T = n - 1$, and $f_a = 0$ on $U$. For a suitable choice of a field $\{e_r\}$ of orthonormal frames, if
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necessary, we may set $f_{2n-1} = 1$ and $f_r = 0$ for $r = n, \ldots, 2n - 2$. Then from (1.5) we have

$$0 = df_r = - \sum J_{ar} \phi_a.$$  

But, since rank $J = 2n - 2$, we have $\det(J_{ar}) \neq 0$ ($a = 1, \ldots, n - 1, r = n, \ldots, 2n - 2$). Thus the above equation implies $\phi_a = 0$, which contradicts the fact that $\det(h_{ab}) \neq 0$.

This concludes the proof.

In the remainder of this section we restrict the forms under consideration to the following open set $V_T$ defined by

$$V_T = \{p \in M \mid \text{J}_{rs}(p) \neq 0, \ t(p) = T\},$$

where $\text{J}_{rs}(p) \neq 0$ means "$\text{J}_{rs}(p) \neq 0$ for some $r, s = T + 1, \ldots, 2n - 1".

First from (2.8) we have $f_r = 0$. Thus we may set $f_1 = 1$, and $f_a = 0$ for $a \geq 2$. Hence from (1.4) we have

$$\text{(2.9)} \quad J_{1a} = 0, \quad J_{1r} = 0.$$  

Furthermore, $df_r = 0$ gives

$$\text{(2.10)} \quad A_{1ra} = \sum h_{ab} J_{br},$$  

$$\text{(2.11)} \quad B_{1rs} = 0.$$  

The equation (2.6) amounts to

$$\text{(2.12)} \quad \sum h_{ab} B_{brs} = c f_a J_{rs}.$$  

**Lemma 2.3.** $\det(h_{ab}) = 0$ ($a, b = 2, \ldots, T$) on $V_T$.

**Proof.** Here indices $a, b$ run from 2 to $T$. If $\det(h_{ab}) \neq 0$, then by (2.12) we have $B_{ars} = 0$, which together with (2.11) gives $J_{rs} = 0$. A contradiction to the fact $J_{rs}(p) \neq 0$ on $V_T$.

This concludes the proof.
3. The proofs of Theorem A and Theorem B

Let $M$ be a real hypersurface of $P_n(C)$ $(n \geq 3)$ with $t(p) \leq 2$ for any point $p$ in $M$. Let us now construct the following sets which will be used in the later.

$$V = \{ p \in M \mid J_{rs}(p) \neq 0 \}, \ (r, s, \ldots, = t(p) + 1, \ldots, 2n - 1),$$

(3.1) $$M_1 = \{ p \in M \mid t(p) \leq 1 \}, \text{ and}$$

$$M_2 = \{ p \in M \mid t(p) = 2 \}.$$ 

Then $M_2 = M - M_1$. Moreover we have $\text{Int}(M_1) = \phi$, because Takagi [9] showed that for any point $p$ in $M$ there exists an open neighborhood $U$ of $p$ in $M$ such that $t(p) \geq 2$, where “Int” means the interior of the given set.

From (3.1) we also construct the following sets

(3.2) $$V_1 = V \cap M_1 \text{ and } V_2 = V \cap M_2.$$ 

Then $V_2$ coincides with the open set $V_T$ which is defined in §2 for the case $T = 2$. Since we have assumed $T = 2$, let us restrict the forms under consideration to $V_2$ unless otherwise stated. Then (2.8) gives $f_r = 0$ for $r = 3, \ldots, 2n - 1$, because $J_{rs} \neq 0$. Thus we may set $f_1 = 1$ and $f_a = 0$, $a \geq 2$. From this fact we know that $e_1$ becomes an almost contact structure vector field.

On the other hand, by Lemma 2.3 we have $h_{22} = 0$ on $V_2$. From which together with the formula of (2.1) we have

(3.3) $$A = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that is, $Ae_1 = \alpha e_1 + \beta e_2$ and $Ae_2 = \beta e_1$, where $A$ is the second fundamental tensor of $M$ in $P_n(C)$.

Firstly, we now assert that the holomorphic sectional curvature $H = H(e_i)$ is constant on $V_2$. Here “the holomorphic section” means the section spanned by $\{e_i, Je_i\}$ for $i \neq 1$. Then the holomorphic sectional curvature is given by

$$H(e_i) = R_{ii} *_{ii} * = \sum_{i, \ell} R_{ijkl} J_{ji} J_{li},$$
where $e_i^*$ means $J e_i = \sum_j J_{ji} e_j$, ($i \neq 1$). From which together with (1.4) and (1.8) we have

$$H(e_i) = c\{1 - f_i^2 + 3(1 - f_i^2)^2\} + \sum_{j,t}(h_{ii}h_{tj}J_{ti}J_{ji} - h_{ji}h_{ti}J_{ti}J_{ji}).$$

(3.4)

Since $f_i = 0$ and $h_{ii} = 0$ on $V_2$ for $i = 2, \ldots, 2n - 1$, (3.4) reduces to

$$H(e_i) = 4c - \sum_{j,t} h_{ji}h_{ti}J_{ti}J_{ji}.$$

Thus for a case where $i \geq 3$, $H(e_i) = 4c$, because $h_{ii} = 0$ for $\ell = 1, \ldots, 2n - 1$. For a case where $i = 2$, (2.9) implies that $H(e_2) = 4c$. Thus we have our assertion.

Next we want to show that the holomorphic sectional curvature $H$ is constant on $M$. The set given in (3.1) becomes

$$V = V \cap M = V \cap (M_1 \cup M_2) = (V \cap M_1) \cup (V \cap M_2) = V_1 \cup V_2.$$

Since $\text{Int}(V_1) = \emptyset$, from the above formula we have that $H(e_i) = 4c$ on $V$. Then let us consider an orthogonal complement set of $V$ in $M$ such that $W = M - V$. Thus $J_{rs} = 0$ on $\text{Int}(W)$. From this fact Lemma 2.1 gives $t(p) \geq n - 1 \geq 2$ for any point $p$ in $\text{Int}(W)$. On the other hand, we have assumed $t(p) \leq 2$ for any point $p$ in $M$. Thus $t(p) = 2$ and $J_{rs}(p) = 0$ on $\text{Int}(W)$. But Lemma 2.2 means that “$J_{rs} = 0$ on $\text{Int}(W)$” implies $T \geq 3$. This makes a contradiction. Consequently, $\text{Int}(W) = \emptyset$. Thus we conclude that the constancy of the holomorphic sectional curvature $H$ can be extended to $M$ globally.

Now let us recall a theorem which is proved by Kimura [4].

**Theorem C.** Let $M$ be a real hypersurface in $P_n(C)$ ($n \geq 3$) on which $H$ is constant. Then $M$ is one of the following:

(a) an open subset of a geodesic hypersphere ($H > 4c$),
(b) a ruled hypersurface ($H = 4c$). More precisely, let $T_0$ be the distribution defined by $T_0(x) = \{X \in T_x(M) \mid X \perp \xi\}$ for $x \in M$, then $T_0$ is integrable, and its integral manifolds are a totally geodesic $P_{n-1}(C)$. 
(c) a real hypersurface on which there is a foliation of codimension two such that each leaf of the foliation is contained in some complex hyperplane \( P_{n-1}(\mathbb{C}) \) as a ruled hypersurface \( (H = 4c) \).

It is known that for the case (b) of Theorem C the second fundamental tensor \( A \) is given by \( A\xi = a\xi + \nu U, \ A U = \nu \xi \) and \( AX = 0 \) for any \( X \) orthogonal to \( \xi \) and \( U \), where \( \xi \) and \( U \) correspond to \( e_1 \) and \( e_2 \), respectively. Combining this fact with our above assertion we complete the proof of Theorem A. Theorem B immediately follows from Theorem A and the non-completeness of a ruled real hypersurface of the case (b) in Theorem C.

**Remark 1.** The above Theorem B is the main result of the paper [7]. In that paper we directly gave the proof of Theorem B by solving a differential equation which is derived from the exterior derivative of (2.1).

**Remark 2.** In the paper [8] the present author and Takagi obtained another new rigidity theorem for isometric immersions \( \iota \) and \( \hat{\iota} \) of a real hypersurface \( M \) into \( P_n(\mathbb{C}) \) under the additional condition such that the type number of \( (M, \iota) \) or \( (M, \hat{\iota}) \) is not equal to 2 at each point of \( M \). As an application of Theorem B to the homogeneous real hypersurface in \( P_n(\mathbb{C}) \) \((n \geq 3)\) we also obtained a rigidity Theorem without the above additional condition.

**References**

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