1 Systems of Distinct Representatives (SDR)

Definition

A system of distinct representatives (SDR) for $\mathcal{A}$ is a set of $n$ elements $\{e_1, e_2, \ldots, e_n\}$ such that $e_i \in A_i$ for all $i = 1, 2, \ldots, n$.

Another name for an SDR is transversal.

Example. If $A_1 = \{1, 2, 3\}, A_2 = \{1, 4, 5\}, A_3 = \{3, 5\}$, then $\{1, 4, 5\}$ is an SDR, and $\{1, 2, 5\}$ is another SDR, but $\{1, 3\}$ is not (it is too small).

If $A_1 = \{2, 3, 4, 5\}, A_2 = \{4, 5\}, A_3 = \{4, 5\}, A_4 = \{5\}$, then no SDR exists!
Lemma 9.2.1 (MC)
Let \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \) be a multiset of sets.

If there exists an SDR for \( \mathcal{A} \), then it follows for every subset \( S \subseteq \{1, 2, \ldots, n\} \), that the union of the sets \( A_i \) with \( i \) belonging to \( S \) has at least as many elements as \( S \):

\[
| \bigcup_{i \in S} A_i | \geq |S| \quad \text{(Marriage Condition, MC).}
\]

Proof of MC
Assume that \( T = \{e_1, e_2, \ldots, e_n\} \) is an SDR for \( \mathcal{A} \), with \( e_i \in A_i \) for every \( i = 1, 2, \ldots, n \).

Let \( S = \{i_1, i_2, \ldots, i_k\} \) be a \( k \)-subset of \( \{1, 2, \ldots, n\} \).

Let \( E = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \). Then \( e_j \in E \) for all \( j = 1, 2, \ldots, k \).

We now have:

\[
| \bigcup_{i \in S} A_i | = |E| \geq |\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}| = k = |S|.  
\]

The Marriage Theorem

Theorem 9.2.2 (Philip Hall, 1935)
A multiset \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \) has an SDR if and only if it satisfies the Marriage Condition.

Proof of Hall’s Theorem
We have already proved that if \( \mathcal{A} \) has an SDR, then MC is satisfied.

Conversely, we now assume that \( \mathcal{A} \) satisfies MC, and we show that \( \mathcal{A} \) has an SDR.

The proof is by induction on \( n \).

Start of the induction
For \( n = 0 \) the multiset is empty: \( \mathcal{A} = \emptyset \).

The empty set \( \emptyset \) satisfies the conditions for being an SDR of \( \mathcal{A} \).

So the statement is true for \( n = 0 \).
The induction step

We assume \( n > 0 \), and the theorem is true for every multiset of fewer than \( n \) sets. If \( S \) is a subset of \( \{1, 2, \ldots, n\} \), we can write

\[
\mathcal{A}(S) = \bigcup_{i \in S} A_i.
\]

The MC condition is the same as

\[ |\mathcal{A}(S)| \geq |S| \quad \text{for all} \quad S \subseteq \{1, 2, \ldots, n\}. \]

The two cases of the proof

There are two possibilities:

- **Case 1**: \( |\mathcal{A}(S)| > |S| \) for every \( S \neq \emptyset \neq \{1, 2, \ldots, n\} \).
- **Case 2**: \( |\mathcal{A}(S)| = |S| \) for some \( S \neq \emptyset \neq \{1, 2, \ldots, n\} \).

**Case 1**: \( |\mathcal{A}(S)| > |S| \) for every \( S \neq \emptyset \neq \{1, 2, \ldots, n\} \).

If \( S = \{n\} \), the Marriage Condition implies \( |\mathcal{A}(S)| = |A_n| \geq 1 \).

Therefore \( A_n \) is not empty, so we can choose \( x \in A_n \).

Let \( \mathcal{B} = \{A_1 \setminus \{x\}, A_2 \setminus \{x\}, \ldots, A_{n-1} \setminus \{x\}\} \).

If \( S \) is a non-empty subset of \( \{1, 2, \ldots, n-1\} \), then

\[
|\mathcal{B}(S)| = \left|\left(\bigcup_{i \in S} A_i \setminus \{x\}\right)\right| = \left|\left(\bigcup_{i \in S} A_i\right) \setminus \{x\}\right|
\]

\[
\geq \left|\bigcup_{i \in S} A_i \setminus \{x\}\right| = |\mathcal{A}(S)| - 1 \geq |S|.
\]

If \( S = \emptyset \), then \( |\mathcal{B}(S)| = 0 = |S| \).

By the induction hypothesis, we can find an SDR of \( \mathcal{B} \).

Let \( \{e_1, e_2, \ldots, e_{n-1}\} \) be such an SDR. Then \( e_i \in A_i \setminus \{x\} \) for every \( i = 1, 2, \ldots, n-1 \).

Let \( e_n = x \). Then \( e_i \in A_i \) for every \( i = 1, 2, \ldots, n \).

It follows that \( T = \{e_1, e_2, \ldots, e_n\} \) is an SDR for \( \mathcal{A} \).

This proves Case 1.

**Case 2**: \( |\mathcal{A}(S)| = |S| \) for some \( S \neq \emptyset \neq \{1, 2, \ldots, n\} \).

Let \( S = \{i_1, i_2, \ldots, i_k\} \), where \( 0 < k < n \), and \( |\mathcal{A}(S)| = |S| \).

Let \( \mathcal{C} = \{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\} \). By the induction hypothesis, there is an SDR \( \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \) of \( \mathcal{C} \).

Let \( \{i_{k+1}, i_{k+2}, \ldots, i_n\} = \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\} \), and \( \mathcal{D} = \{A_{i_{k+1}} \setminus \mathcal{A}(S), A_{i_{k+2}} \setminus \mathcal{A}(S), \ldots, A_{i_n} \setminus \mathcal{A}(S)\} \).

If \( \mathcal{D} \) satisfies the MC, then there exists an SDR \( \{e_{i_{k+1}}, e_{i_{k+2}}, \ldots, e_{i_n}\} \) of \( \mathcal{D} \), by the induction hypothesis.

Then \( T = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}, e_{i_{k+1}}, \ldots, e_{i_n}\} \) will be an SDR of \( \mathcal{A} \) and the proof is finished.

So we prove that \( \mathcal{D} \) satisfies the marriage condition: if \( R \) is a subset of \( \{i_{k+1}, i_{k+2}, \ldots, i_n\} \), then \( |\mathcal{D}(R)| \geq |R| \).

We get:

\[
|R| + |S| = |R \cup S| \leq |\mathcal{A}(R \cup S)| = |\mathcal{A}(R) \cup \mathcal{A}(S)|
\]

\[
= |\mathcal{D}(R) \cup \mathcal{A}(S)| = |\mathcal{D}(R)| + |\mathcal{A}(S)| = |\mathcal{D}(R)| + |S|.
\]

We have shown that \( |\mathcal{D}(R)| \geq |R| \) is true for every \( R \subseteq \{i_{k+1}, i_{k+2}, \ldots, i_n\} \). So \( \mathcal{D} \) satisfies MC, which proves the theorem.
2 Largest Subset with an SDR

Finding the largest subset with an SDR

Theorem 9.2.3
Let \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \).

There exists a subfamily \( \mathcal{T} \) of \( t \) sets such that \( \mathcal{T} \) has an SDR if and only if
\[
|\mathcal{A}(S)| \geq |S| - (n - t)
\]
is true for every subset \( S \) of \( \{1, 2, \ldots, n\} \).

Proof of Theorem 9.2.3
Let \( F \) be a set of \( n - t \) new elements, \( F \cap \mathcal{A}(\{1, 2, \ldots, n\}) = \emptyset \).

Define \( \mathcal{B} = \{A_1 \cup F, A_2 \cup F, \ldots, A_n \cup F\} \).

If \( \mathcal{B} \) has an SDR \( \{e_1, e_2, \ldots, e_n\} \), then \( \{e_1, e_2, \ldots, e_n\} \setminus F \) is an SDR for \( n - (n - t) = t \) of the sets from \( \mathcal{A} \).

Conversely, if \( T = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\} \) is an SDR of \( \{A_{i_1}, A_{i_2}, \ldots, A_{i_t}\} \), then \( T \cup F \) is an SDR of all of \( \mathcal{B} \).

We deduce that \( \mathcal{B} \) has an SDR if and only if there exists an SDR of a subset of \( t \) of the sets from \( \mathcal{A} \).

Proof of Theorem 9.2.3 continued
The proof is finished if we show that the condition
\[
|\mathcal{A}(S)| \geq |S| - (n - t)
\]
for \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \) is equivalent to MC for \( \mathcal{B} = \{A_1 \cup F, A_2 \cup F, \ldots, A_n \cup F\} \).

The MC for \( \mathcal{B} \) states that for every \( k \)-subset \( S = \{i_1, i_2, \ldots, i_k\} \),
\[
|S| \leq |\mathcal{B}(S)| = |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \cup F| \\
= |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| + |F| \\
= |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| + (n - t).
\]
So the MC for \( \mathcal{B} \) is equivalent to the condition
\[
|\bigcup_{i \in S} A_i| \geq |S| - (n - t) \quad \text{for all } S \subseteq \{1, 2, \ldots, n\}.
\]

3 Conclusion

Conclusion

This ends the lecture!
Next time:
Combinatorial Designs