SPECTRAL APPROXIMATION ORDERS OF LOCAL INTERPOLATION BY USING RADIAL BASIS FUNCTIONS

YEON JU LEE † AND JUNGHO YOON ‡

Abstract. The local radial basis function (RBF) interpolation method enables very large-scale data sets to be handled efficiently, overcoming the drawbacks of global interpolation which produces highly ill-conditioned linear systems. Whereas there have been intensive studies on the accuracy of global RBF interpolation, the error analysis of local RBF interpolation is much less investigated. In this regard, this study explores the approximation order of local RBF interpolation. We are particularly interested in using smooth RBFs, including Gaussian functions, because they can provide spectral approximation orders. In fact, most of the previous studies of smooth RBF interpolation have estimated errors for functions in a certain reproducing kernel Hilbert space $\mathcal{F}_\phi$. However, since the space $\mathcal{F}_\phi$ is very small when the function $\phi$ is smooth (e.g., a Gaussian function), this study focuses on proving approximation orders of local RBF interpolation for functions in the space $C^n$.

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1. Introduction

The problem of approximation from a set of scattered data can be found in a wide range of scientific and engineering applications, such as neural networks, topography, image processing, and computer graphics. Given a set $X$ of scattered points in $\mathbb{R}^d$ with $d \geq 1$ and values $f\vert_X$ sampled from an underlying function $f : \mathbb{R}^d \to \mathbb{R}$, we need to construct a function $s : \mathbb{R}^d \to \mathbb{R}$ such that $s$ approximates $f$ in some sense. These days, such problems also arise when reconstructing a three dimensional model from scattered data points sampled from the surface of the physical three dimensional object, as well as when solving partial differential equations. The strengths of the radial basis function (RBF) method are as follows: (i) it facilitates the evaluation of the approximant without using a mesh; (ii) it gives very accurate results both for interpolation problems and for solving elliptic partial differential equations; and (iii) there is enough flexibility in the choice of basis functions. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radial in the sense that $\phi(x) = \Phi(|x|)$ where $|x| := (x_1^2 + \cdots + x_d^2)^{1/2}$ indicates the usual Euclidean norm.

RBF interpolation to a multivariate function sampled at scattered points are constructed from the corresponding scattered shifts of an RBF $\phi$ with a possible addition of a polynomial term. The basis function $\phi$ is assumed to be either positive definite, or conditionally positive definite on $\mathbb{R}^d$. There are several reviews addressing the state-of-the-art of the investigation of RBF methods. The reader is referred to the survey papers [4], [6] and [22]. Another important source is the works

‡ Department of Mathematical sciences, KAIST, 373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701, S. Korea (lee08@kaist.ac.kr). † Department of Mathematics, Ewha W. University, Seoul, 120-750, S. Korea (yoon@ewha.ac.kr; corresponding author).
of Schaback and Wendland [25] and of Madych and Nelson [16, 17], who developed a theory of interpolation based on reproducing kernel Hilbert spaces. Local error bounds of RBF interpolation have been studied by Wu and Schaback [28]. Moreover, recent books have addressed the core of the underlying general mathematics for kernels and RBFs [5, 10, 27]. In addition, approximation orders of interpolation on the Sobolev space were observed by Yoon [29, 30, 31]. Also, error estimates for rough functions are studied by Brownlee and Light [3]. Recently, Narcowich, Ward and Wendland [20, 21] obtained Sobolev-type error estimates focusing on RBFs whose Fourier transforms decay algebraically around $\infty$.

The following notations are used in this paper. Let $\mathbb{Z}^d_+ := \{ (\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d : \gamma_k \geq 0 \}$. For $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}^d_+$, we set $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\alpha! := \alpha_1! \cdots \alpha_d!$, and $|\alpha|_1 := \sum_{k=1}^{d} \alpha_k$. The polynomial ordering of a sequence of multi-indices is determined in the way that the index $\alpha$ comes before $\beta$ (hereafter, denoted by $\alpha < \beta$) if $|\alpha|_1 < |\beta|_1$, or if $|\alpha|_1 = |\beta|_1$, $\alpha_j = \beta_j$, $j = 1, \ldots, p$ and $\alpha_{p+1} < \beta_{p+1}$. We use two notations for the polynomial space. First, $\Pi_k$ stands for the space consisting of all $d$-variate algebraic polynomials of degree $\leq k$. Second, for a given multi-index $\gamma \in \mathbb{Z}^d_+$, let $\Pi_\gamma := \text{span}\{x^\alpha : \alpha \leq \gamma\}$.

For a given integer $n \in \mathbb{N}$, put $n_d := \text{dim} \Pi_n$. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t) \exp(-i\theta \cdot t) \, dt.$$ 

This Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$.

The density of the center set $X$ in $\Omega$ is defined by

$$h_{X,\Omega} := \sup_{y \in \Omega} \min_{x_\gamma \in X} |y - x_\gamma|$$

which is called the fill-distance of $X$ in $\Omega$. The separation distance within $X$ is defined by

$$q_X := \min_{x_\ell \neq x_\gamma} |x_\ell - x_\gamma|/2.$$ 

The interpolation method using RBFs is certainly an important and well-established approach to solving the scattered data problem. However, in such fields, we often encounter the typical drawback of global methods, that is, the interpolation matrix is highly ill-conditioned. This leads to unstable solutions or unacceptable computational costs. Even worse, when the number of samples is huge (e.g. “point clouds”), this drawback is more serious. Recently, the methods of local RBF interpolation and approximation have been studied for the purpose of overcoming drawbacks of the global interpolation [8, 9, 12, 13, 14, 15, 26]. The local interpolation method enables very large data sets to be processed efficiently because the computation is limited to suitable small areas, thus avoiding the critical efficiency problem involved in global interpolation. Its main idea can be explained as follows. The domain $\Omega$ is divided into smaller (possibly overlapping) regions and then the RBF interpolation problem is solved in each region independently. As a post-process, non-negative weight functions constituting a partition of unity may be applied to blend local solutions [18, 26].

Whereas there have been intensive studies on the accuracy of global RBF interpolation methods, the error analysis of local RBF interpolation is much less investigated. Davydov et al. [13]
considered the local approximation algorithm based on interpolation or least squares, where the approximation order was studied for the case of thin plate splines. In addition, in [8], Iske provided error analysis of the local interpolation by polyharmonic splines, where instead of a set becoming dense in a small neighborhood, only a finite fixed set of centers is chosen and scaled by a parameter $h > 0$. Hence, this analysis can be applied to a very special type of data such as uniform grid. In [26], Wendland introduced a method combining the local RBF interpolation with a partition of unity and estimated the error bound for the functions in the so-called native space. All in all, there is an overwhelming need to study the accuracy of the local RBF interpolation. In this regard, the main objective of this paper is to provide the approximation order of local RBF interpolation, focusing on smooth RBFs because they can provide spectral approximation orders. The basis functions $\phi$ considered in this paper are assumed to be tempered distributions and have generalized Fourier transforms of the form

$$|\cdot|^{2s} \hat{\phi} = F \in L_\infty(\mathbb{R}^d), \quad s < d/2,$$

where the function $F$ decays exponentially around infinity, nonnegative on $\mathbb{R}^d$ and positive at least on an open subset of $\mathbb{R}^d$. (Specific examples will be given in section 4.) Indeed, most of the previous studies of smooth RBF interpolation have estimated errors for a class of functions $f$ whose Fourier transforms are dominated by the Fourier transform $\hat{\phi}$ in the sense that

$$\int_{\mathbb{R}^d} |\hat{f}(\theta)|^2 \hat{\phi}^{-1}(\theta) d\theta < \infty.$$

In this case, an approximand $f$ is required to have a certain smoothness associated with the expression (1.5). For instance, if $\phi$ is a Gaussian function, $\hat{\phi}^{-1}$ increases exponentially at infinity. Thus, the approximands $f$ need to be extremely smooth for an effective error analysis. Unfortunately, no convergence order for functions that are less smooth has been provided yet. In view of the above discussion, our major concern is to study the approximation order of local RBF interpolation for functions in $C^n$. Specifically, the prototype of our main result can be described as follows:

**Theorem 1.1.** Let $C$ be an open cell of a partition of a bounded domain $\Omega \subset \mathbb{R}^d$, and let $S$ be a set of scattered points in $C$. Assume that $k_d \leq \#S < (k+1)_d$ for some $k \in \mathbb{N}$. Let $L_f$ be the RBF interpolation on $S$. Then, if $f \in C^{n+1}(C)$ with $n \leq k$, we have the estimate

$$\|f - L_f\|_{L_\infty(C)} \leq \frac{c h^{n+1}_S}{h^{n+1}_{S,C}} \sum_{|\alpha| \leq n} \|f^{(\alpha)}\|_{L_\infty(C)}.$$  

under a suitable condition of $S$ and $C$, where $h_{S,C}$ is the density of $S$ in $C$ (see (1.2)).

Noteworthy is that to obtain the asymptotic convergence order $k+1$ in (1.6), we use only local $k_d$ values around the evaluation point, which is usually the minimal number of data points for the convergence order $k+1$. In this sense, therefore, our estimate is optimal.

**References**