Numerical Verification of Solutions for Some Unilateral Boundary Value Problems

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Abstract—In this paper, we consider a numerical technique which enables us to verify the existence of solutions for some unilateral boundary value problems. Using the finite element approximations and explicit a priori error estimates, we construct, in a computer, a set of solutions which satisfies the hypothesis of Schauder's fixed-point theorem for a compact map on a certain Sobolev space. Further, the conditions of verifiability by this method are considered and some numerical examples are presented. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Recently, several methods for the numerical proof of existence of solutions for various variational inequalities have been developed ([1-3], etc.). These methods are known as new numerical approaches for the problems that are difficult to analytically prove the existence of solutions for variational inequalities. We propose a numerical method to verify the existence of solutions to some unilateral boundary value problems for the two-dimensional case; that is, we construct a computing algorithm which automatically encloses the solutions with guaranteed error bounds. In the following section, we describe the unilateral boundary value problem, and the main tool of the verification method is given a brief explanation at an abstract level. In order to verify solutions numerically, it is necessary to calculate the explicit a priori error estimates for approximate problems. These constants play an important role in the numerical verification method. In

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Section 3, we determine these constants. In particular, the explicit a priori error bound is presented for approximation by piecewise linear elements over triangles. In Section 4, we describe a computer algorithm to construct the set satisfying the verification condition. Finally, some numerical examples are illustrated.

2. PROBLEM AND OUTLINE OF THE VERIFICATION METHOD

Let $\Omega$ be a convex polygonal domain of $\mathbb{R}^2$. We define

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} uv \, dx,$$

where

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2}.$$

We now suppose the following conditions for the map $f$.

**Assumption 1.** $f$ is a continuous map from $H^1(\Omega)$ to $L^2(\Omega)$.

**Assumption 2.** For each bounded subset $U \in H^1(\Omega)$, $f(U)$ is also a bounded set in $L^2(\Omega)$.

Next, we define $K = \{ v \in H^1(\Omega) : v \geq 0 \text{ a.e. on } \Gamma \}$.

Now, let us consider the following unilateral boundary value problem:

$$\text{find } u \in K \text{ such that } a(u, v - u) \geq (f(u), v - u), \quad \forall v \in K. \quad (2.1)$$

We adopt $a(\phi, \psi) = (\nabla \phi, \nabla \psi) + (\phi, \psi)$ as the scalar product on $H^1(\Omega)$, where $(\cdot, \cdot)$ denotes the $L^2$-inner product on $\Omega$. Hence, the associated norm is defined by $\| \phi \|_{H^1(\Omega)} = a(\phi, \phi)$. We will discuss the existence and inclusion method for problem (2.1). This is the method providing the existence of a solution of problem (2.1) within explicitly computable bounds.

We consider the following auxiliary problem associated with (2.1), considering $g \in L^2(\Omega)$:

$$a(u, v - u) \geq (g, v - u), \quad \forall v \in K. \quad (2.2)$$

To verify the existence of a solution of (2.1) in a computer, we use the fixed-point formulation of a compact operator. In [3], problem (2.1) is equivalent to that of finding $u \in H^1(\Omega)$ such that

$$a(u, v - u) \geq a(F(u), v - u), \quad \forall v \in K,$$

where $F : H^1(\Omega) \to H^1(\Omega)$ is a compact operator. We have the following fixed-point problem for the compact operator $PKF$:

$$\text{find } u \in H^1(\Omega) \text{ such that } u = PKF(u). \quad (2.3)$$

Here, $PK$ denotes the projection operator from $H^1(\Omega)$ to $K$.

First, we describe the basic verification technique in the present paper. We now approximate the solution of (2.2) by means of finite element approximations. We denote by $I_\Omega = \{1, 2, 3, \ldots, m_0\}$ the set of all indices $i$ associated with the internal nodes $x_i$ of the domain $\Omega$, and we shall denote by $I_\Gamma = \{m_0 + 1, m_0 + 2, \ldots, m\}$ the set of all nodes indices $i$ associated with the boundary nodes $x_i$ of the domain $\Omega$ and let $I = I_\Omega \cup I_\Gamma$.

In what follows, we shall consider only the regular system of triangulations. In other words, when refining the partition of $\Omega$, the triangles of the given triangulation do not reduce to segments. Let $\{T_h\}$ be a regular system of triangulations of $\tilde{\Omega}$. The nodes of a triangulation lying on $I_\Gamma$ will be denoted by $p_{m_0 + 1}, p_{m_0 + 2}, \ldots, p_m$. We then approximate $H^1(\Omega)$ by

$$S_h = \{ v_h \in C^0(\tilde{\Omega}) : v_h|_T \in P_1(T), \forall T \in T_h \},$$
where $P_k(T)$ denotes the set of all polynomials of degree at most $k$ on the definition domain $T$. We then define $K_h$, an approximate subset of $K$, by

$$K_h = \{ v_h \in S_h, v_h(p_i) \geq 0, \forall i = m_0 + 1, m_0 + 2, \ldots, m \}$$

and the dual cone of $K_h$ by $K_h^* = \{ w \in H^1(\Omega) : a(w, v) \leq 0, \forall v \in K_h \}$. Notice that $K_h$ is a closed convex nonempty subset of $S_h$.

We then define the approximate problem corresponding to (2.2) as

$$a(u_h, v_h - u_h) \geq (g, v_h - u_h), \quad \forall v_h \in K_h.$$  \hspace{1cm} (2.4)

Let $u$ be the solution of (2.2) and $u_h \in K_h$ be the approximate solution of (2.4).

**Assumption 3.** For each $u$ and $u_h$, there exists a positive constant $C(h)$ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq C(h)\|g\|_{L^2(\Omega)}.$$  \hspace{1cm} (2.5)

Here, $C(h)$ has to be numerically determined. We describe the two-dimensional case in Section 3.

For a bounded, closed, and convex subset $U$ of $H^1(\Omega)$, define a set $V \subset H^1(\Omega)$ by

$$V = \{ v \in H^1(\Omega) : v = P_KF(u), \ u \in U \}.$$  

Our goal is to find a set $U$ which includes $V$. Once $V \subset U$ is obtained, noting that $P_KF$ is a compact operator on $H^1(\Omega)$, Schauder's fixed-point theorem gives the proof of existence of solutions to unilateral boundary value problem (2.1) in the set $V$, and in $U$. Next, let us introduce the procedure for finding such a set $U$ using computers. We describe how to obtain such a set of $H^1(\Omega)$ on a computer. For any $u \in H^1(\Omega)$, we define the rounding $R(P_KF(u)) \in K_h$ as the solution of the following problem:

$$a(R(P_KF(u)), v_h - R(P_KF(u))) \geq (f(u), v_h - R(P_KF(u))), \quad \forall v_h \in K_h.$$  

Then, for the set $V \subset H^1(\Omega)$, we define the rounding $R(V) \subset K_h$ as

$$R(V) = \{ v_h \in K_h : v_h = R(P_KF(u)), \ u \in U \}.$$  

Also, we define for $V \subset H^1(\Omega)$ the rounding error $RE(V) \subset K_h^*$ as

$$RE(V) = \left\{ v \in K_h^* : \|v\|_{H^1(\Omega)} \leq C(h) \sup_{u \in U} \|f(u)\|_{L^2(\Omega)} \right\}.$$  

We can denote $V \subset R(V) \oplus RE(V)$. In practice, $R(V)$ is represented as the linear combination of bases functions of $S_h$ with interval coefficients. On the other hand, $RE(V)$ is taken as a ball in $K_h^*$ obtained by its radius which is evaluated using the error estimation (2.5). Therefore, using $R(V) \oplus RE(V)$ instead of $V$, the verification condition becomes

$$R(V) \oplus RE(V) \subset U.$$  \hspace{1cm} (2.6)

### 3. COMPUTATION OF THE CONSTANTS

In this section, we give a bound of the constant $C(h)$ of (2.5). In the numerical verification methods, it is very important because the actual value for $C$ much influences the possibility and the accuracy of verification.

The smaller the constant $C$ is, the higher the possibility is attained in verifications with the procedure described in Section 4, as well as the higher accuracy is obtained. Thus, it is necessary that $C$ tends to be smaller as the mesh size $h$ of $K_h$ becomes smaller. Many researchers described
a numerical method to get a bound of the optimal constant in the error estimates of the finite element method. Considering partial differential equations, there are several approaches to the problem of computable error bounds. For variational inequalities, however, to this point, there have been very few such investigations. Up to now, determinations of constants appearing in \textit{a priori} error estimates have been discussed only for the one-dimensional problem [1-3]. In particular, we consider the unilateral boundary value problem for the two-dimensional case.

Now we describe how to estimate \( C \) in (2.5). We divide the domain into a small triangle with uniform mesh size \( h \). Let \( \Omega \) be a square with side length 1 and let \( T_h \) be the uniform triangulation of \( \Omega \) according to Figure 1.

First, notice that for any \( g \in L^2(\Omega) \), the basic model problem (2.2) has a unique solution \( u \in H^2(\Omega) \). As is well known, we can interpret the solution of (2.2) as follows: problem (2.2) has been formulated as a problem of finding \( u \) satisfying two subsets \( \Gamma_0 \) and \( \Gamma_+ \) such that

\[
\Gamma_0 \cup \Gamma_+ = \Gamma, \quad \Gamma_0 \cap \Gamma_+ = \emptyset, \quad -\Delta u + u = g, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \Gamma_0, \quad \frac{\partial u}{\partial n} \geq 0, \quad \text{on } \Gamma_0, \\
u \geq 0, \quad \text{on } \Gamma_+, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma_+, 
\]

where \( \frac{\partial}{\partial n} \) is the outer normal derivative on \( \Gamma \). We look at problem (2.2) as a free boundary problem because we do not know \( \Gamma_+ \) or \( \Gamma_0 \).

**Lemma 3.1.** Let \( u \) be a solution of problem (2.2) and if \( g \in L^2(\Omega) \), then we have

\[
\|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \leq \|g\|_{L^2(\Omega)}^2. \tag{3.2}
\]
PROOF. First, we consider the following integration:
\[ \int_\Omega (-\Delta u + u)^2 \, dx = \int_\Omega (\Delta u)^2 - 2 \int_\Omega u \cdot \Delta u \, dx + \int_\Omega u^2 \, dx. \]
Applying partial integration to the second term on the right-hand side and using the boundary condition for \( u \), we derive
\[ \int_\Omega (-\Delta u + u)^2 \, dx = \int_\Omega (\Delta u)^2 + 2 \int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, dx. \]
Hence, by using (3.1), we obtain the assertion.

**Lemma 3.2.** Let \( u \) be a solution of problem (2.2) and if \( g \in L^2(\Omega) \), then we have
\[ |u|_{H^2(\Omega)} \leq \|g\|_{L^2(\Omega)}, \tag{3.3} \]
where \( |w|_{H^2} \) implies the seminorm of \( w \) on \( H^2(\Omega) \) defined by
\[ |w|_{H^2(\Omega)}^2 \equiv \sum_{i,j=1}^2 \left\| \frac{\partial^2 w}{\partial x_i \partial x_j} \right\|_{L^2(\Omega)}^2. \]
To prove this lemma, we will need to make use of the following result which can be found in [4].
Let \( L \) denote the linear operator given by
\[ L[u] = -\Delta u + c(x) u, \quad u \in H^2(\Omega), \]
with a given function \( c \in L^\infty(\Omega) \). We will now assume that the constants \( K_0, c, \) and \( \bar{c} \) exist which satisfy
\[ \|u\|_{L^2(\Omega)} \leq K_0 \|L[u]\|_{L^2(\Omega)} \]
and
\[ c \leq c(x) \leq \bar{c}, \quad x \in \Omega. \]
Since \( \Omega \) is a convex polygonal domain, then we have the following estimate:
\[ |u|_{H^2(\Omega)} \leq \left( 1 + K_0 \max\left\{ \frac{1}{2} (\bar{c} - c), -c \right\} \right) \|L[u]\|_{L^2(\Omega)}. \]
Applying the above result with \( c = 1 \), we derive that \( |u|_{H^2(\Omega)} \leq \| -\Delta u + u \|_{L^2(\Omega)} \). By using (3.2), we obtain \( |u|_{H^2(\Omega)} \leq \|g\|_{L^2(\Omega)} \). Hence, the assertion of Lemma 3.2 holds true.

We have the following well-known estimation for the Courant’s triangles (see [5,6]).

**Lemma 3.3.** For any \( \varphi \in H^2(\Omega) \), let \( \tilde{\varphi} \) be the unique interpolating polynomial of degree \( \leq 1 \) of \( \varphi \). Then we have
\[ \| \nabla \tilde{\varphi} - \nabla \varphi \|_{L^2(\Omega)} \leq 0.81 h |\varphi|_{H^2(\Omega)} \]
and
\[ \| \tilde{\varphi} - \varphi \|_{L^2(\Omega)} \leq 3 \ell^2 |\varphi|_{H^2(\Omega)}, \]
where \( \ell \) is the greatest length of the sides of \( T \).

**Lemma 3.4.** Let \( u \in H^1(\Omega) \) and \( \Delta u \in L^2(\Omega) \). Then \( \partial u / \partial n \in H^{-1/2}(\Gamma) \) and
\[ (\nabla u, \nabla v) + (\Delta u, v) = \left( \frac{\partial u}{\partial n}, v \right)_\Gamma, \quad \forall v \in H^1(\Omega). \]
Furthermore, we have
\[ \left| \left( \frac{\partial u}{\partial n}, v \right)_\Gamma \right| \leq \left( \|u\|_{H^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 \right)^{1/2} \|v\|_{H^1(\Omega)}. \]
Regarding the approximation error \( \|u_h - u\|_{H^1(\Omega)} \), we then have the following theorem.
THEOREM 3.5. Let $T_h$ be the uniform triangulation of $\Omega$ and let $u$ and $u_h$ be solutions of problems (2.2) and (2.4), respectively. If $g \in L^2(\Omega)$, then we have

$$
\|u_h - u\|_{H^1(\Omega)} \leq C(h)\|g\|_{L^2(\Omega)}.
$$

Hence, we may take $C(h) = ((0.81h)^2 + (3\ell^2)^2 + 2((0.81h)^2 + (3\ell^2)^2)^{1/2})^{1/2}$ in (2.5).

PROOF. Equations (2.2) and (2.4) imply that

$$
a(u, u) \leq a(u, v) + (g, u - v), \quad \forall v \in K,
$$

$$
a(u_h, u_h) \leq a(u_h, v_h) + (g, u_h - v_h), \quad \forall v_h \in K_h.
$$

Hence,

$$
a(u - u_h, u - u_h) = a(u, u) + a(u_h, u_h) - a(u_h, u) - a(u, u_h)
\leq (g, u - v) + a(u, v) + (g, u_h - v_h) + a(u_h, v_h) - a(u_h, u) - a(u, u_h)
= (g, u - v_h) + (g, u_h - v) + a(u_h - u, v_h - u) + a(u, v - u_h).
$$

We deduce, by setting $v = u_h$, that $\forall v_h \in K_h \subset K$,

$$
\|u - u_h\|_{H^1(\Omega)}^2 \leq a(u - u_h, u - u_h) \leq (g, u - v_h) + a(u_h - u, v_h - u) + a(u, v - u_h).
$$

Let us set $v_h = r_h u$, where $r_h u$ denotes the piecewise linear Lagrange interpolation of the function $u$. As

$$
r_h u(p_i) = u(p_i) \geq 0, \quad \forall i = m_0 + 1, m_0 + 2, \ldots, m,
$$

we have $r_h u \in K_h$. Using (3.1), Lemma 3.4, and Green’s formula,

$$
a(u, r_h u - u) = (\nabla u, \nabla (r_h u - u)) + (u, r_h u - u)
= (-\Delta u, r_h u - u) + \left(\frac{\partial u}{\partial n}, r_h u - u\right)_\Gamma
= (g, r_h u - u) + \left(\frac{\partial u}{\partial n}, r_h u - u\right)_\Gamma.
$$

Substituting into the right-hand side of (3.4) and Lemma 3.4, we have

$$
\|u - u_h\|_{H^1(\Omega)}^2 \leq a(u - u_h, u - u_h)
\leq a(u_h - u, r_h u - u) + \left(\frac{\partial u}{\partial n}, r_h u - u\right)_\Gamma
\leq \frac{1}{2} \|u - u_h\|_{H^1(\Omega)}^2 + \frac{1}{2} \|u - r_h u\|_{H^1(\Omega)}^2
+ \left(\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)^{1/2} \|u - r_h u\|_{H^1(\Omega)}.
$$

Therefore, by (3.3) and standard results of approximation theory, we have

$$
\|u - u_h\|_{H^1(\Omega)}^2 \leq \|r_h u - u\|_{H^1(\Omega)}^2 + 2\|g\|_{L^2(\Omega)} \|r_h u - u\|_{H^1(\Omega)}
\leq \|\nabla (r_h u - u)\|_{L^2(\Omega)}^2 + \|r_h u - u\|_{L^2(\Omega)}^2
+ 2\|g\|_{L^2(\Omega)} \left(\|\nabla (r_h u - u)\|_{L^2(\Omega)}^2 + \|r_h u - u\|_{L^2(\Omega)}^2\right)^{1/2}
\leq (0.81h)^2 |u|_{H^2(\Omega)}^2 + (3\ell^2)^2 |u|_{H^2(\Omega)}^2
+ 2\|g\|_{L^2(\Omega)} \left((0.81h)^2 |u|_{H^2(\Omega)}^2 + (3\ell^2)^2 |u|_{H^2(\Omega)}^2\right)^{1/2}.
$$
Using Lemma 3.1, we conclude that
\[ \|u - u_h\|_{H^1(\Omega)} \leq \left( (0.81h)^2 + (3\ell^2)^2 + 2(0.81h)^2 + (3\ell^2)^2 \right)^{1/2} \|g\|_{L^2(\Omega)}, \]
where \( \ell \) is the greatest length of the sides of \( T \).

4. COMPUTING PROCEDURES FOR VERIFICATION

In this section, we propose a computer algorithm to obtain a set \( U(t) \) which satisfies the verification condition. As parameters to describe a function \( v_h \in S_h \), we choose the values \( v_h(p_i) \) of \( v_h \) at nodes \( p_i, i = 1, \ldots, m, \) of \( T_h \). The corresponding basis functions \( \phi_j \in S_h, j = 1, \ldots, m, \) are defined by \( \phi_j(p_i) = \delta_{ij} \) (Kronecker’s symbol). A function \( v_h \in S_h \) now has the representation
\[ v_h(t) = \sum_{j=1}^{m} Z_j \phi_j(t), \quad Z_j = v_h(p_j), \quad \text{for } t \in \bar{\Omega}. \]

Since the bilinear form \( a(\cdot, \cdot) \) is symmetric, (2.4) is equivalent to the quadratic programming problem
\[ \min_{v_h \in K_h} \left[ \frac{1}{2} a(v_h, v_h) - (g, v_h) \right]. \quad (4.1) \]

Then we can represent the above quadratic programming problem (4.1) in the form
\[ \min_{Z_j \geq 0} \left[ \frac{1}{2} Z_j^T D_{II} Z_j - P_j^T Z_j \right], \quad j \in I_\Gamma. \quad (4.2) \]

Here, \( D_{II} \equiv (a_{ij})_{i,j \in I} \), with \( a_{ij} = a(\phi_i, \phi_j) \) and \( Z_I \equiv (Z_j)_{j \in I} \) is the coefficient vector for \( \{\phi_j\} \) corresponding to the function \( v_h \) in (4.1). Further, \( P_I \equiv ((g, \phi_j))_{j \in I} \) is an \( m \)-dimensional vector.

By the Kuhn-Tucker theorem [3,7], a vector \( Z_I \) with \( Z_I \geq 0 \) is an optimal solution to (4.2) if and only if there exists \( Y_I \) such that
\[ D_{II} Z_I - P_I = 0, \]
\[ \langle Z_I, Y_I \rangle = 0, \]
\[ Y_I \geq 0, \]
\[ Z_I = 0. \quad (4.3) \]

Following Rump [7], condition (4.3) means, because of \( Z_I, Y_I \geq 0 \), that for \( I_\Gamma \) either \( Z_I = 0 \) or \( Y_I = 0 \). Now, consider the following system of nonlinear equations:
\[ D_{II} Z_I - P_I = 0, \]
\[ Y_j Z_j = 0, \quad j \in I_\Gamma. \quad (4.4) \]

Let \( (\hat{Z}_I, \hat{Y}_I) \) be an approximate solution of (4.4). Delete in (4.4) every variable \( Z_j, Y_j \) for which the corresponding component of \( \hat{Z}_I, \hat{Y}_I \) is approximately zero. Thus, it can be reduced to the following linear system:
\[ \hat{D}_{II} \hat{Z}_I - \hat{P}_I = 0, \]
where \( \hat{D}_{II} \) is an \( m \times m \) matrix and \( \hat{P}_I \) is an \( m \)-dimensional vector. Let \( \mathbb{R}^+ \) denote the set of all nonnegative real numbers. For \( \alpha \in \mathbb{R}^+ \), we associate
\[ \lfloor \alpha \rfloor \equiv \{ \psi \in K_h^* : \|\psi\|_{H^1(\Omega)} \leq \alpha \}. \]
Let $A_j$ ($1 \leq j \leq m$) be intervals on $\mathbb{R}^1$ and let $\sum_{j=1}^{m} A_j \phi_j$ be a linear combination of $\{\phi_j\}$, i.e., an element of the power set $2^S$ in the following sense:

$$\sum_{j=1}^{m} A_j \phi_j = \left\{ \sum_{j=1}^{m} a_j \phi_j : a_j \in A_j, \ 1 \leq j \leq m \right\}.$$

Then, setting $U = \sum_{j=1}^{m} A_j \phi_j \oplus [\alpha]$ and $g = f(U)$ in (4.1), we consider the following linear system:

$$D I \bar{Z}_I - \bar{P}_I = 0. \quad (4.5)$$

Here $P_I \equiv ((f(U), \phi_j))_{j \in I}$. In order to find a set $U$ satisfying the above verification condition (2.6), we use a simple iterative method. The simple iteration method is as follows.

First, we obtain an approximate solution $u_h^{(0)} \in S_h$ to (2.1) by some appropriate method. Set $U_h^{(0)} = \{u_h^{(0)}\}$ and $\alpha_0 = 0$. Next, we will define $R(V^{(i)})$ and $RE(V^{(i)})$ for $i \geq 0$, where $V^{(i)}$ is the set defined as follows:

$$V^{(i)} = \left\{ v^{(i)} \in K : v^{(i)} = P_K F \left( u^{(i)} \right), \ u^{(i)} \in U^{(i)} \right\}.$$

We define $R(V^{(i)}) \subset K_h$ according to

$$D I \bar{Z}_I = \left( f \left( u^{(i)} \right), \phi_j \right)_{j \in I}. \quad (4.6)$$

Here, $R(V^{(i)})$ is determined as the solution set of (4.6), as described above. In order to solve (4.6) with guaranteed accuracy, following [7], we have the following theorem.

**THEOREM 4.1.** Let $\bar{Z}_I$ be interval solutions of the linear system (4.6) containing the actual solutions. Then the following is true: If $\inf(\bar{Z}_I) \geq 0$, the quadratic programming problem (4.1) has an optimal solution $Z_I$. The nonzero components of $Z_I$ are included in $\bar{Z}_I$, and the others are zero.

Note that $R(V^{(i)})$ can be enclosed by $R(V^{(i)}) \subset \sum_{j=1}^{m} A_j \phi_j$, where $A_j = [\bar{A}_j, \bar{A}_j]$ are intervals. Next, $RE(V^{(i)})$ is defined by

$$RE \left( V^{(i)} \right) = \left\{ v \in K_h^* : \|v\|_{H^1(\Omega)} \leq C(h) \sup_{u^{(i)} \in U^{(i)}} \|f(u^{(i)})\|_{L^2(\Omega)} \right\}.$$

Using (2.6), $V^{(i)} \subset R(V^{(i)}) \oplus RE(V^{(i)})$ holds. Check the verification condition

$$R \left( V^{(i)} \right) \oplus RE \left( V^{(i)} \right) \subset U^{(i)}.$$

If the condition is satisfied, then $U^{(i)}$ is the desired set, and a solution to (2.1) exists in $V^{(i)}$, and hence, in $U^{(i)}$. If the condition is not satisfied, we continue the simple iteration by using $\delta$-inflation, i.e., let $\delta$ be a certain positive constant given beforehand, and take

$$\alpha_{i+1} = C(h) \sup_{u^{(i)} \in U^{(i)}} \|f(u^{(i)})\|_{L^2(\Omega)} + \delta,$$

$$[\alpha_{i+1}] = \left\{ v \in K_h^* : \|v\|_{H^1(\Omega)} \leq \alpha_{i+1} \right\},$$

$$U_h^{(i+1)} = \sum_{j=1}^{M} [A_j - \delta, \bar{A}_j + \delta] \phi_j,$$

$$U^{(i+1)} = U_h^{(i+1)} + [\alpha_{i+1}].$$

With the above, we can carry on the verification process by numerically checking the verification condition and the conditions of Theorem 4.1.
5. EXAMPLE OF NUMERICAL VERIFICATION

With the condition in Section 3, we provide some numerical examples of verification in the two-dimensional case according to the procedure described in the previous section. We divide the domain into a small triangle with uniform mesh size $h$, and choose the basis of $S_h$ as the pyramid functions.

**Example 1.** We consider the case $f(u) = u + (\pi^2/16)\sin(\pi/4x)y$.

Conditions:

- $\dim S_h = 64$.
- Initial value: $u_h^{(0)} = \text{Galerkin approximation}, a_0 = 0$.
- Extension parameter: $\delta = 10^{-5}$.

Results:

- Iteration numbers: 5.
- $H^1(\Omega)$-error bound: 0.927.
- Maximum width of coefficient intervals in $\{A_j\} = 0.00450766$.

Figure 2 shows the outline of the shape for the solution for $A_j$ of $A_j = [A_j, \bar{A}_j]$ in Section 4. Figure 3 shows the guaranteed intervals for an exact solution at each level. That is, it is verified that there exists a solution between these two curves.

**Example 2.** We consider the case $f(u) = Ku + \sin \pi x \cos 2\pi y$, where $K$ is a positive constant.

The execution conditions are as follows.

- $\dim S_h = 100$.
- $K = 1$.
- Initial value: $u_h^{(0)} = \text{Galerkin approximation}, a_0 = 0$, the outline of $u_h^{(0)}$ is shown in Figure 4.
- Extension parameters: $\delta = 10^{-5}$. 
Figure 3. Range of the nontrivial solution.

Figure 4. Approximate solution $u_h^{(0)}$. (a) Approximate solution. (b) Contour of the approximate solution.
Results are as follows:

Iteration numbers: 4.

$H^1(\Omega)$-error bound: 0.032.

Maximum width of coefficient intervals in $\{A_j\} = 0.000164284$.

Using computer arithmetic with double precision instead of strict interval computations (e.g., ACRITH, C-XSC, PROFIL, INTLAB, etc.), the numerical example is computed. So, the round-off errors in this example are neglected. However, it should be sufficient for our present purposes.

REFERENCES