ON THE REPRESENTATION OF FINITE DISTRIBUTIVE LATTICES

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ABSTRACT. Classical results of Birkhoff and of Dilworth yield an extremely useful representation of a finite distributive lattice $L$ as the downset lattice $D(J_L)$ of poset $J_L$ of join-irreducible elements of $L$. Through this they get a correspondence between the chain decompositions of $J_L$ and the tight embeddings of $L$ into a product of chains.

We use a simple but elegant result of Rival which characterizes sublattices $L$ in terms of the possible make-up of the sets $L' - L$, to give an alternate interpretation of a chain decomposition of $J_L$ as a catalog of the irreducible intervals removed from a product of chains to get $L$. This interpretation allows us to extend the correspondence of Birkhoff and Dilworth to non-tight embeddings of $L$ into products of chains. In the non-tight case, $J_L$ is replaced with a digraph $D$, which is no longer a poset, and no longer canonical, and $L$ is represented as the lattice of terminal sets of $D$. Taking a quotient of this $D$ yields a one-to-one correspondence between the embeddings of $L$ into products of chains, and what we call loose chain covers of a a canonical supergraph $J_L^\infty$ of $J_L$.

There is another useful representation, from results of Dilworth and of Koh, of a finite distributive lattice $L$ as the lattices of maximal antichains $A(P)$ of various posets $P$. Through the correspondence of Birkhoff and Dilworth this relates chain decompositions of various posets $P$ to tight embeddings of $L$ into products of chains. But it is not a correspondence.

We similarly extend the representation theory of Dilworth and Koh to non-tight embeddings of $L$ into products of chains. This allows us to correspond any chain decomposition of any finite poset to an embedding of a finite distributive lattice into a product of chains; in a way, completing the correspondence of Dilworth and Koh.

The development of our two representations is essentially the same, and we move between them via a simple construction based on a construction of Koh.

1. Introduction

In this section we give a brief survey of several classical results about the representations of finite distributive lattices. We then explain how our results generalize them.

Though we say ‘generalize’, our ideas come very naturally from a result of Rival which yields a different interpretation than do the classical constructions. In many cases, this yields simplified proofs to the classical results, and we view their value as much in this natural interpretation as in the fact that they are generalizations. Indeed, this paper grew out of [4], in which we find a characterisation of the graphs

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admitting distributive lattice polymorphisms. In that paper, we found it was necessary to consider non-tight embedding of lattices into product of chains. It was then that we realized they had been largely overlooked in the literature. The point of view we develop here differs from the classical one in one point in particular. We start from an implicit description of the different sublattices of a given product of chains, whereas the classical view starts by describing the different embeddings of a given lattice into various products of chains.

A finite lattice $L$ is a partial ordering $\leq$ of a finite set $L$ such that for every pair of elements $x, y \in L$ there is a greatest lower bound or meet denoted $x \wedge y$ and a least upper bound or join denoted $x \vee y$. The lattice is distributive if the meet and join operations distribute over each other.

1.1. Representation as lattices of downsets. The set $2^X$ of subsets of a finite set $X$ ordered under inclusion is the canonical first example of a distributive lattice. Indeed it is easy to check that ‘union’ and ‘intersection’ act as join and meet operations, respectively, and it is an elementary exercise in set theory that these operations distribute with each other. More generally, any family of sets, closed under union and intersection, is a distributive lattice under inclusion.

In [1], Birkhoff proved a representation theorem for distributive lattices, showing that any distributive lattice $L$ can be represented as the inclusion lattice of a family of sets. An element $x$ of $L$ is join-irreducible if $x = a \vee b$ implies that $x = a$ or $x = b$. So the set $J_L$ of non-zero join-irreducible elements of $L$ is a poset under the partial order induced by $L$. A downset of a poset $P$ is a subset $S$ of $P$ such that if $x \in P$ and $y \leq x$ then $y \in P$. It is easy to see that the set $\mathcal{D}(P)$ of downsets of a poset $P$ is a sublattice of $2^P$; that is, it is closed under its meet and join operations. This immediately implies that $\mathcal{D}(P)$ is a distributive lattice. Birkhoff showed that $L \cong \mathcal{D}(J_L)$. The main step of Birkhoff’s representation theorem is the following theorem.

**Theorem 1.1.** [1] The map

$$S : x \mapsto S_x := \{ \alpha \in J_L | \alpha \leq x \}$$

is an isomorphism of $L$ to $\mathcal{D}(J_L)$.

The proof of this is now the simple exercise of showing that $S$ has inverse operation $S_x \mapsto \bigvee S_x$, and that it commutes with the join and the meet. (As our lattices are finite, the meet $\bigvee S$ is well defined for any subset $S$ of elements.)

Further one can show that $P \cong J_{\mathcal{D}(P)}$ by observing that a downset in $\mathcal{D}(P)$ is join-irreducible if and only if it has a unique maximal element. Thus Theorem 1.1 gives the following, which completes a one-to-one correspondence between finite posets and finite distributive lattices.

**Corollary 1.2.** If $\mathcal{D}(P) \cong \mathcal{D}(P')$ then $P \cong P'$.

A totally ordered lattice is called a chain. The $n$-chain denoted $Z_n$ is the chain on the set $[n]_0 = \{0\} \cup [n] = \{0, \ldots, n\}$ with the usual ordering $0 \leq \cdots \leq n$. A product of chains will be written as $\mathcal{P} = \prod_{i=1}^d C_i$ where $C_i$ is the chain isomorphic to $Z_{n_i}$, for the implicitly defined integer $n_i$. Where elements of $\mathcal{P}$ are the $d$-tuples $x = (x_1, \ldots, x_d)$ in the Cartesian product $\prod_{i=1}^d [n_i]_0$ of sets, the product ordering on $\mathcal{P}$ is as follows.
It is a simple task to show that $2^X$ is isomorphic to the product of $|X|$ copies of the chain $Z_1$, so Theorem 1.1 yields the following well known fact.

**Fact 1.3.** Any finite distributive lattice $L$ can be embedded into a product of chains.

Recall that an embedding of one lattice into another is an isomorphism of the first to a sublattice of the second.

For a chain $C_i \cong Z_{n_i}$, let $C_i^* \cong Z_{n_i-1}$ be the chain on the set $[n_i] = \{1, \ldots, n_i\}$ that we get from $C_i$ removing the element 0. A subchain of a poset is a subposet isomorphic to a chain. A decomposition of a poset $P$ into subchains is a partition of its elements into a set $\mathcal{C}^*$ of chains $C_1^*, \ldots, C_d^*$.

In [2], Dilworth proved the following embedding theorem whose proof is now a simple exercise.

**Theorem 1.4.** [2] For any decomposition $\mathcal{C}^* = \{C_1^*, \ldots, C_d^*\}$ of a poset $P$ into chains $C_1^*, \ldots, C_d^*$, the map

$$S_x \mapsto (|S_x \cap C_1^*|, \ldots, |S_x \cap C_d^*|)$$

is an embedding of $\mathcal{D}(P)$ into the product $\mathcal{P} = \prod_{i=1}^d C_i$.

With Theorem 1.1, this gives the following.

**Corollary 1.5.** For any decomposition $\mathcal{C}^* = \{C_1^*, \ldots, C_d^*\}$ of $J_L$ into $d$ chains, the map

$$E_C : L \to \mathcal{P} : x \mapsto (|S_x \cap C_1^*|, \ldots, |S_x \cap C_d^*|)$$

is an embedding of $L$ into $\mathcal{P}$.

We mention here that Dilworth proved Theorem 1.4 in showing that the minimum number $d$ of chains such that $L$ embeds into a product of $d$ chains, is equal to the size $w(L)$ of the largest antichain of $L$: the largest set of pairwise incomparable elements. This also required his famous Decomposition Theorem, which we will use at the end of Section 4.

**Theorem 1.6.** [2] For any poset $P$ having maximum antichain of size $d$, there is a decomposition of $P$ into $d$ chains.

In [3] Dilworth writes the following with regards Theorem 1.4 and its corollary.

Now it is easily shown that subdirect union representations of a finite distributive lattice in terms of chains correspond to the decompositions of the partially ordered set of join irreducibles into a set union of chains.

One might interpret this as the statement that a converse to Corollary 1.5 is easily shown. However without some quantification, which was perhaps implicit at the time, this cannot be the case: the lattice $Z_1$ embeds into, say $Z_1^2$, in several ways, but there is only one decomposition of the poset $J_{Z_1}$ of a single element into chains, this decomposition corresponds to the embedding of $Z_1$ into $Z_1$. In [6], Larson makes explicit a converse to Corollary 1.5.

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1Here 'subdirect union representation' is not defined but seems to refer to the result in [2] in which a lattice is represented as a sublattice of the direct union of chains. The use of the word 'subdirect' is less restrictive than the modern usage of the word which we assume in this paper.
The minimum or zero element of the lattice \( L \) is denoted \( 0 \) or \( 0_L \) and the maximum or one element is denoted \( 1 \) or \( 1_L \). A cover of a lattice \( L \) is a pair \( x \geq y \) of elements such that there is no element \( z \) different from \( x \) and \( y \) with \( x \geq z \geq y \). An embedding \( L \leq P \) of a lattice into a product of chains is tight if preserves \( 0 \) and \( 1 \) and maps covers to covers. Larson shows, essentially, the following.

**Theorem 1.7.** The embeddings \( E_C \) are tight, and for every tight embedding \( E \) there is some chain decomposition \( C^*_E \) of \( J_L \) such that \( E = E_{C^*_E} \).

It was not mentioned, but it is a trivial observation that different chain decompositions of \( J_L \) yield different embeddings of \( L \) into products of chains, so with Corollary 1.5, this yield the following correspondence.

**Corollary 1.8.** There is a one-to-one correspondence between the chain decompositions \( C^* \) of \( J_L \) and the tight embeddings \( E \) of \( L \) into products of chains.

It was not made explicit in the above papers, but from their proofs, one can see that the chain decomposition \( C^*_E \) of \( J_L \) corresponding to an embedding \( E \) of \( L \) into \( P \) is as follows. An element \( x \in J_L \) is identified with the unique irreducible element \( P \) in \( S_{E(x)} - S_{E(x')} \), where \( x' \) is the unique element covered by \( x \) in \( L \).

We now explain our results that are related to the above discussion. In Section 3 we generalize the poset \( D(P) \) of downsets of a poset \( P \) to the poset \( D(D) \) of terminal sets of a reflexive digraph \( D \). Theorem 3.5, our first main theorem, gives a correspondence between the sublattices of a product of chains \( P \) and the supergraphs \( D \) of a digraph \( C^\infty \) defined by \( P \). We see this as a generalization of Theorems 1.1 and 1.4, and Corollary 3.6 as an extension of Corollary 1.5 from tight embeddings to all embeddings. In our extension, we lose the ‘one-to-one’-ness of Corollary 1.8. In Section 6 we recover this by taking a quotient of our digraph \( D \), to get in Theorem 6.3 a one-to-one correspondence between embeddings of a lattice \( L \) into products of chains, and loose chain covers of a canonical extension \( J_L^\infty \) of \( J_L \). Finally, with Corollary 6.5 we restrict ourselves to full embeddings to make it very clear that we have an extension of Corollary 1.8. Section 5 further corrects some mistakes in [6], and extends the results of that same paper, to give a more comprehensive relation between the type of embedding and the corresponding properties of \( D \) (and so \( J_L \)).

1.2. **Representation as lattice of maximal antichains.** The correspondence taking a downset of a poset \( P \) to its subset of maximal elements gives one-to-one correspondence between the downsets \( D(P) \) of \( P \) and the antichains \( A(P) \) of \( P \). It is not hard to see that by defining the following ordering on \( A(P) \), the correspondence can be extended to a lattice isomorphism: for antichains \( I, I' \in A(P) \), set \( I \leq I' \) if for all \( a \in I \) there is some \( a' \in I' \) such that \( a \leq a' \).

In [3], Dilworth showed the following

**Theorem 1.9.** [3] The poset \( A_m(P) \leq A(P) \) of all maximum sized antichains of a poset \( P \) is a distributive lattice.

This proof takes some work. As does the complementing representation theorem, proved by Koh in [5].

**Theorem 1.10.** [5] For every finite distributive lattice \( L \), and every chain decomposition \( C^* \) of \( J_L \) there is a poset \( P_C \) such that \( L \cong A_m(P_C) \).
Through the correspondence to chain decompositions of $J_L$, each of the posets $P_C$ given by Koh, corresponds to a tight embedding of $L$ into a product of chains. However, there are other posets $P$ such that $A_m(P) = L$, that do not arise from Koh’s results.

Replacing the poset $P$ with a more general digraph $A$ and generalizing antichains to independent sets we generalize this by showing, in our second main theorem, Theorem 4.6, that the poset $A_m(A)$ of maximal independent sets of a digraph $A$ is a distributive lattice. Further, by a simple alteration of our construction of the digraph $D$, we associate to every sublattice of a product of chains $\mathcal{P}$ a digraph $A$ and show in Corollary 4.10 that for any digraph $A$ with $A_m(A) = L$, the digraph $A$ corresponds to some sublattice of a product of chains $\mathcal{P}$.

The proof of these results uses Theorem 4.5 which relates the digraphs $A$ and $D$ corresponding to a sublattice $L$ of $\mathcal{P}$, through a construction $K_C$, defined in Section 2. This construction arises very naturally from Rival’s result, and is essentially Koh’s construction $(J_L, C^*) \to P_C$. The only difference is that it removes loops from $P_C$. We view Theorem 4.5 as the ‘essence’ of Koh’s paper [5].

1.3. Representation through irreducible intervals. Let $J_L^+ = J_L \cup \{0\}$ denote the full set of join-irreducible elements of a lattice $L$ and let $M_L^+$ analogously denote the set of all meet-irreducible elements of $L$. An irreducible interval of $L$ is the set

$$[\alpha, \beta] = \{x \in L \mid \alpha \leq x \leq \beta\}$$

for some $\alpha \in J^+_L$ and $\beta \in M^+_L$. For a set $\mathcal{I}$ of irreducible intervals of $L$ we let $\cup \mathcal{I} = \cup_{I \in \mathcal{I}} I$. The set $\mathcal{I}$ is called closed if $I \subseteq \cup \mathcal{I}$ implies $I \in \mathcal{I}$.

In [7], Rival showed the following.

**Theorem 1.11.** If $L$ is a sublattice of a lattice $L'$ of finite length, then

$$L = L' - \cup \mathcal{I}$$

for some family $\mathcal{I}$ of irreducible intervals of $L$. Further, $\mathcal{I}$ may be assumed to be closed.

For a product $\mathcal{P}$ of chains, it is easy to see that $J^+_\mathcal{P}$ consists of those elements that are non-zero in at most one coordinate, and $M^+_\mathcal{P}$ consists of those elements $x = (x_1, \ldots, x_d)$ such that $x_i < n_i$ for at most one $i \in [d]$. Thus the irreducible intervals of $\mathcal{P}$ are sets of the form

$$i[\alpha, \beta] = \{x \in L \mid \alpha \leq x_i \text{ and } x_j \leq \beta\}$$

for $\alpha \in [n_i]_0$ and $\beta \in [n_j]_0$.

We depict a poset by its Hasse Diagram, the digraph on its vertices where $(a, b)$ is an arc if $a$ is a cover of $b$; that is, if $a \geq b$ but there is no $x$ different from $a$ and $b$ with $a \geq x \geq b$. In our figures, the arcs of a Hasse diagram will always go down, so we omit the arrows. Figure 1(i) shows a lattice $L$ represented as the sublattice $\mathcal{P} - 2[3, 2]_1$ of $\mathcal{P} = Z_4 \times Z_5$.

1.4. The idea of our approach. For a product of chains $\mathcal{P} = \prod_{i=1}^d C_i$ we let $\mathcal{C} = \cup_{i=1}^d C_i$ be the disjoint union of the chains $C_1, \ldots, C_d$ and $\mathcal{C}^* = \cup_{i=1}^d C_i^*$ be the disjoint union of the chains $C_i^*$. We let $\alpha e_i$ represent the element $\alpha$ of $C_i$, and let $x = (x_1, \ldots, x_d)$ represent the $d$-tuple $(x_1 e_1, \ldots, x_d e_d)$ of $\mathcal{P}$.

Though $\mathcal{C}$ is clearly isomorphic to $J^+_\mathcal{P}$, we view elements of $\mathcal{C}$ as ‘coordinates’ of $\mathcal{P}$, rather than join-irreducible elements of $\mathcal{P}$: an element $x = (x_1, \ldots, x_d)$ of $\mathcal{P}$ is
uniquely determined by its (antichain of) coordinates \( \{x_1 e_1, \ldots, x_d e_d\} \). Further, it is uniquely determined by the (downset of) coordinates below it. In fact, it is easy to show, and is shown up to change of notation, in Lemmas 3.4 and 4.4, that

(i) the map \( x \mapsto T_x := \{\alpha e_i \mid \alpha \leq x_i\} \) is an isomorphism of \( \mathcal{P} \) to \( D(\mathcal{C}^*) \), and

(ii) the map \( x \mapsto \{x_1 e_1, \ldots, x_d e_d\} \) is an isomorphism of \( \mathcal{P} \) to \( A_m(\mathcal{C}) \).

The main idea behind our results comes from the simple observation of what we must do to \( \mathcal{C}^* \) and \( \mathcal{C} \) so that these statements remain true when we remove an irreducible interval \( [\alpha, \beta] \) from \( \mathcal{P} \).

Using the example in Figure 1, we see that to keep statement (i) true when we remove \( 2[3,2] \), we should add the edge \( 3e_2 \to 3e_1 \) to \( \mathcal{C}^* \), as in Figure 1(ii). This ensures that, for example, the set \( T_{(2,3)} = \{2e_1, 1e_1, 3e_2, 2e_2, 1e_2\} \) is no longer a downset. In general it ‘kills’ any downset \( T \) containing \( 3e_2 \) but not \( 3e_1 \). These are

\[
\begin{align*}
&\mathcal{P} = \mathbb{Z}_4 \times \mathbb{Z}_5 \\
\vspace{0.5cm} &\mathcal{P} - 2[3,2]_3 \\
&\mathcal{P} = \mathbb{Z}_4 \times \mathbb{Z}_5 \\
&\mathcal{P} - 2[3,2]_3 \\
&\mathcal{P} = \mathbb{Z}_4 \times \mathbb{Z}_5 \\
&\mathcal{P} - 2[3,2]_3
\end{align*}
\]
exactly $T_x$ for $x \in _2[3,2]$. Our digraph $D$ will be defined from $\mathcal{C}^*$ by adding this arc for every irreducible interval that is removed.

Similarly, to keep statement (ii) true, we should add the edge $3e_2 \rightarrow 2e_1$ to $\mathcal{C}$, as in Figure 1(iii), so that $(2,3) = \{2e_1,3e_2\}$ is no longer an antichain. We ‘kill’ any antichain containing an element above $3e_2$ and an element below $2e_1$. These are exactly the antichains $\{x_1e_1, x_2e_2\}$ for $x \in _2[3,2]$. Our digraph $A$ will be defined from $\mathcal{C}$ by adding this arc for every irreducible interval that is removed.

The intuition holds even after $\mathcal{C}^*$ and $\mathcal{C}$ are no longer posets. For this though we will, in Sections 3 and 4, extend the posets $\mathcal{D}$ and $\mathcal{A}_m$ of downsets and antichains to posets $\mathcal{D}$ and $\mathcal{A}_m$ of terminal sets and independent sets.

For technical reasons we will replace the posets $\mathcal{C}$ and $\mathcal{C}^*$ of this outline with the digraph $\mathcal{T}$ and $\mathcal{C}^\infty$ respectively. We define these now.

2. Setup and the digraphs $A_{\mathcal{C}}(\mathcal{J})$ and $D_{\mathcal{C}}(\mathcal{J})$

2.1. Notation. For the rest of the paper, $\mathcal{P}$ will always denote a product of $d$ chains $C_1, \ldots, C_d$ and $\mathcal{C}$, $\mathcal{C}^\infty$ will be the disjoint unions of the $d$ corresponding chains $C_i$, and $C_i^\infty$, and $\mathcal{T} = \cup_{i=1}^d T_i$ will denote the disjoint union of transitive tournaments, where $T_i$ is the transitive tournament that we get from $C_i$ by removing loops.

Defining one will define the others with the same $d$ and $n_1, \ldots, n_d$.

For a poset $P$ let $P^\infty$ be the pointed poset we get by adding a new maximum element $\infty$ and a new minimum element $0$; we make one exception, letting $\mathcal{C}^\infty = (\mathcal{C}^\infty)^\infty$.

We retain the notation $x_i e_i$ from last section to refer to elements of $\mathcal{C}$, $\mathcal{C}^\infty$ and $\mathcal{T}$, and extend this notation to $\mathcal{C}^\infty$ by letting $0e_i$ and $(n_i + 1)e_i$ refer to $0$ and $\infty$ respectively, for all $i \in [d]$.

Partial order relations are now viewed as digraphs, a comparability $a \geq b$ being viewed as an arc $a \rightarrow b$. A graph is reflexive if every vertex $a$ has a loop $a \rightarrow a$, and is irreflexive if it has no loops. $A$ and $D$ will always refer to digraphs, $D$ will generally be reflexive. They will often contain $\mathcal{T}$ or $\mathcal{C}^\infty$, respectively, as spanning subgraphs. In this case we will refer to them a spanning supergraphs of $\mathcal{T}$ and $\mathcal{C}^\infty$ respectively.

2.2. Definitions of $A_{\mathcal{C}}(\mathcal{J})$ and $D_{\mathcal{C}}(\mathcal{J})$. By Theorem 1.11 any sublattice of a product $\mathcal{P}$ of chains can be represented as $\mathcal{P} - \cup \beta$ for some family $\mathcal{J}$ of irreducible intervals of $\mathcal{P}$.

**Definition 2.1** ($A_{\mathcal{C}}(\mathcal{J})$). For a family of irreducible intervals $\mathcal{J}$ of $\mathcal{P}$, let $A_{\mathcal{C}}(\mathcal{J})$ be the spanning supergraph of $\mathcal{T}$ we get by letting $\alpha e_i \geq \beta e_j$ for each $i[\alpha, \beta], j \in \mathcal{J}$.

**Definition 2.2** ($D_{\mathcal{C}}(\mathcal{J})$). For a family of irreducible intervals $\mathcal{J}$ of $\mathcal{P}$, let $D_{\mathcal{C}}(\mathcal{J})$ be the spanning supergraph of $\mathcal{C}^\infty$ defined by letting $\alpha e_i \geq (\beta + 1)e_j$ for each $i[\alpha, \beta], j \in \mathcal{J}$.

Observe that we get $\mathcal{C}^\infty$ from $\mathcal{T}$ by replacing every arc $\alpha e_i \rightarrow \beta e_j$ with the arc $\alpha e_i \rightarrow (\beta + 1)e_j$. More generally, it should be clear that $D_{\mathcal{C}}(\mathcal{J}) = K_{\mathcal{C}}(A_{\mathcal{C}}(\mathcal{J}))$ where $K_{\mathcal{C}}$ is the following construction, which as we mentioned in the introduction, is essentially a generalization of Koh’s construction from [5].

**Definition 2.3.** For a spanning supergraph $A$ of $\mathcal{C}$, let $D = K_{\mathcal{C}}(A)$ be graph on $\mathcal{C}^\infty$ with arcs

$$\{\alpha e_i \rightarrow (\beta + 1)e_j \mid \alpha e_i \rightarrow \beta e_j \text{ in } A\}.$$
Now the construction $J \mapsto A_c(J)$ has a well defined inverse construction $A \mapsto J_c(A)$. To say the same about the other two constructions requires some discussion of closure. Recall the observation in Theorem 1.11 that we may assume that $J$ is closed: $i[\alpha, \beta]_j \in \cup J$ implies $i[\alpha, \beta]_j \in J$. Noting that the interval $i[0, \beta]_j$ is independent of $i$ and the interval $i[\alpha, n_j]_j$ is independent of $j$, $J$ being closed clearly implies the following three conditions:

(i) $J$ contains the empty intervals of the form $i[\alpha, \beta]_i$ for all $i$ and $0 \leq \beta < \alpha \leq n_i$,
(ii) $i[0, \beta]_j \in J$ for some $i$ implies $i[0, \beta]_j \in J$ for all $i$.
(iii) $i[\alpha, n_j]_j \in J$ for some $j$ implies $i[\alpha, n_j]_j \in J$ for all $j$.

Firstly, we will always assume that a family $J$ satisfies (i). This corresponds to the fact that $D_c(J)$ contains $\mathcal{C}^\infty$ and that $A_c(J)$ contains $\mathcal{F}$. Secondly, it is clear that $J$ satisfying (ii) and (iii) translates to $A = A_c(J)$ satisfying the following.

(ii) If $0e_i \rightarrow \beta e_j$ is in $A$ for any $i \in [d]$, then $0e_i \rightarrow \beta e_j$ is in $A$ for all $i \in [d]$.
(iii) If $\alpha e_i \rightarrow (n_j + 1)e_j$ is in $A$ for any $j \in [d]$, then $\alpha e_i \rightarrow (n_j + 1)e_j$ is in $A$ for all $j \in [d]$.

These properties hold precisely when $K_{\mathcal{C}}^{-1}(K_{\mathcal{C}}(A)) = A$ for the obviously defined inverse $K_{\mathcal{C}}^{-1}$ of construction $K_{\mathcal{C}}$. So exactly when $D_{\mathcal{C}}^{-1} = K_{\mathcal{C}}^{-1} \circ A_{\mathcal{C}}^{-1}$ is the inverse construction of $D_{\mathcal{C}}$. The construction $D_{\mathcal{C}}(J)$ can be seen as assuming properties (ii) and (iii), so is slightly less general than the construction $A_c(J)$. We will not always assume these properties, as doing so would require more conditions on $A$ in the statement of such results as Corollary 4.8, but we will see, in Corollary 4.7, that apart from this, one may assume properties (ii) and (iii) without much harm.

2.3. Closure of $J$ and transitivity. We finish this section by looking at the full implications of assuming closure of $J$. We start with a technical lemma.

Lemma 2.4. Let $D = D_c(J)$ for some subgraph $L = \mathcal{P} - \cup J$ of $\mathcal{P}$. Then the following are equivalent for $\alpha \leq n_i$ and $\beta \leq n_j$.

(i) $i[\alpha, \beta]_j \subset \cup J$.
(ii) For all $x \in L$, $x_i \geq \alpha \Rightarrow x_j > \beta$.
(iii) There is an $(\alpha e_i, (\beta + 1) e_j)$-path in $D$.

Proof. The equivalence of (i) and (ii) is immediate as both are equivalent to the statement that there are no elements $x \in L$ with $x_i \leq \alpha$ and $\beta \leq x_j$. So we show the equivalence of (ii) and (iii).

On the one hand, assume that $\alpha i_j, e_i \rightarrow \alpha i_{j+1}e_{i+1}$ is a path in $D$ where $\beta + 1 = \alpha i_j$. By definition of $D$ this means that $i[\alpha i_j, \alpha i_{j+1} - 1]_i \in J$ for each $i$. Thus by the equivalence of (i) and (ii), we have for all $x \in L$ that $x_i \geq \alpha i_j$ implies $x_{i+1} > \alpha i_{j+1} - 1$ which implies $x_{i+2} > \alpha i_{j+2} - 1$, etc., until we get that $x_{i+\beta} > \alpha i_{j+\beta} - 1 = \beta$, as needed.

On the other hand, assume that there is no such path from $\alpha i_j e_i$ to $\alpha i_{j+1} e_i$. We will find $x$ in $L$ with $x_i \geq \alpha i_j$ and $x_{i+1} < \alpha i_{j+1}$. Indeed, for $j \in [d]$ let $x_j$ be the maximum $\alpha$ such that there is a path in $D$ from $\alpha i_j e_i$ to $\alpha i_{j+1} e_{j+1}$. If no such path exists, let $x_j$ be $0$. Clearly $x_i \geq \alpha i_j$, and by the assumption that there is no path from $\alpha i_j e_i$ to $\alpha i_{j+1} e_i$ we have that $x_{i+1} < \alpha i_{j+1}$. We have just to show that $x$ is in $L$.

If it is not in $L$, then it is in some $i[\alpha, \beta]_j \in J$. Thus we have that $\alpha \leq x_i$ and $\beta \geq x_j$. So $\alpha e_i \rightarrow (\beta + 1) e_j$ is in $A$. Now $\alpha \leq x_i$ implies that there is an path from
α_i e_i to α e_i, so the edge α e_i \rightarrow (\beta + 1) e_j gives us a path from α_i e_i to (\beta + 1) e_j. But x_j < \beta + 1, which contradicts the choice of x_j.

\[ \square \]

**Proposition 2.5.** Let \( D = D_\mathcal{P}(\mathcal{I}) \) for some sublattice \( \mathcal{P} \cap \mathcal{J} \) of \( \mathcal{P} \). Then \( \mathcal{I} \) is closed if and only if \( D \) is transitive.

**Proof.** On the one hand, assume that \( \mathcal{I} \) is closed. Then we can replace (i) of Lemma 2.4 with with \( \mathcal{I}[\alpha, \beta] \in \mathcal{J} \). Now \( \alpha e_i \rightarrow \beta e_j \) and \( \beta e_j \rightarrow \gamma e_k \) in \( D \) give by Lemma 2.4 that \( \mathcal{I}[\alpha, \gamma] \in \mathcal{J} \) which means \( \alpha e_i \rightarrow \gamma e_k \) in \( D \). So \( D \) is transitive.

On the other hand, assume that \( \mathcal{I} \) is not closed. Then there is some \( \alpha e_i \rightarrow \beta e_j \) in \( \mathcal{J} \) but not in \( \mathcal{I} \). So there is a \((\alpha e_i, \beta e_j)\)-path in \( D \), while \( \alpha e_i \rightarrow \beta e_j \) is not in \( D \). This gives the first statement of the lemma. \[ \square \]

Now to get a similar result for \( A = A_\mathcal{P}(\mathcal{I}) \), recall that if \( \mathcal{J} \) is closed we may assume that \( A = K^{-1}(D) \) where \( D = D_\mathcal{P}(\mathcal{I}) \) is transitive. It is not to hard to see from Definition 2.3, that for \( \alpha_i e_i \) and \( \alpha_{i+1} e_{i+1} \) in \( A \), there is a \((\alpha_i e_i, \alpha_{i+1} e_{i+1})\)-path in \( D \) if and only if there is an alternating \((\alpha_i e_i, \alpha_{i+1} e_{i+1})\)-path in \( A \):

\[ \alpha_i e_i \rightarrow (\alpha_{i+1} e_{i+1} \leftarrow \alpha_{i+2} e_{i+2} \rightarrow (\alpha_{i+3} e_{i+3} \leftarrow \alpha_{i+4} e_{i+4} \rightarrow \cdots \rightarrow (\alpha_{i+\ell} e_{i+\ell} \leftarrow \alpha_i e_i) \right) \]

One should pay attention here to the particular nature of the backwards edges. It follows from Proposition 2.5 that \( \mathcal{J} \) is closed if and only if \( A \) satisfies the conditions (ii) and (iii) of the last subsection, and is ‘alternating path transitive’, a concept we feel is clear enough in light of the fact that we will not make rigourous use of it latter. We make the following weaker observation formal.

**Proposition 2.6.** Let \( A = A_\mathcal{P}(\mathcal{J}) \) for a family \( \mathcal{J} \). If \( \mathcal{J} \) is closed then \( A = K^{-1}(D_\mathcal{P}(\mathcal{J})) \) and \( A \) is transitive.

**Proof.** We have discussed that \( \mathcal{J} \) being closed allows us to assume that \( A = K_\mathcal{P}(D) \) where \( D = D_\mathcal{P}(\mathcal{J}) \). To see that \( A \) is transitive, let \( \alpha e_i \rightarrow \beta e_j \rightarrow \gamma e_k \) in \( A \). Then \( \alpha e_i \rightarrow (\beta + 1) e_j \) and \( \beta e_j \rightarrow (\gamma + 1) e_k \) are in \( D \). As \((\beta + 1) e_j \rightarrow (\beta) e_j \) is always in \( D \) we have by transitivity that \( \alpha e_i \rightarrow (\gamma + 1) e_k \) is in \( D \). So \( \alpha e_i \rightarrow \gamma e_k \) is in \( D \), as needed for transitivity. \[ \square \]

### 3. Generalizing downsets to terminal sets

A directed path or an \( xy \)-path in a digraph is a sequence of arcs

\[ x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_d = y. \]

**Definition 3.1.** A subset \( T \subset D \) of vertices of a digraph \( D \) is called terminal if for all \( x \) in \( T \) and all \( xy \)-paths in \( D \), \( y \) is also in \( T \). A terminal set \( T \) is proper if it is not \( D \) or empty.

Clearly when \( D \) is a poset, the terminal sets are just the downsets.

**Example 3.2.** The terminal sets of the disjoint union \( \mathcal{P} \) of chains are exactly the downsets

\[ T_x = \{ \alpha e_i \mid i \in [d], \alpha \leq x_i \} \]

for vertices \( x \) of \( \mathcal{P} \). These are also exactly the proper terminal sets of \( \mathcal{P}^\infty \).

We extend the definition of \( \mathcal{D}(\mathcal{P}) \) to the following.

**Definition 3.3** (\( \mathcal{D}(D) \)). Let \( \mathcal{D}(D) \) be the family of proper terminal sets of \( D \), ordered by inclusion.
With this definition, Example 3.2 actually gives an isomorphism.

**Lemma 3.4.** Let \( \mathcal{P} \) be a product of chains. Then

\[
T : \mathcal{P} \rightarrow \mathcal{D}(\mathcal{C}^\infty) : x \mapsto T_x
\]

is an isomorphism. The inverse is \( T \mapsto \bigvee T \), where the join is taken in \( \mathcal{P} \).

**Proof.** We have already observed, in Example 3.2, that \( T \) is a bijection. To see that is an isomorphism simply note that \( T_x \subset T_{x'} \) if and only if \( x \leq x' \). That the join is the inverse is clear: an element \( x \) in \( \mathcal{P} \) is uniquely defined by the set \( T_x \) of coordinates below it.

It is not hard to show that the union and intersection of terminal sets are terminal, so for all digraphs \( D, \mathcal{D}(D) \) is a lattice. We only need this when \( D \) is a spanning supergraph of some \( \mathcal{C}^\infty \), but in this case, we show more. In the following, our first main theorem, we show that \( \mathcal{D}(D) \) is the sublattice \( \mathcal{P} - \mathcal{J}_C(D) \) of \( \mathcal{P} \).

**Theorem 3.5.** Let \( L = \mathcal{P} - \mathcal{J} \) be a sublattice of \( \mathcal{P} \), and \( D = D_C(\mathcal{J}) \). Then the map \( T : L \rightarrow \mathcal{D}(D) \) is a lattice isomorphism.

**Proof.** In the case \( \mathcal{J} \) is empty, we have \( L = \mathcal{P} \) and \( D = \mathcal{C}^\infty \), so the isomorphism is shown in Lemma 3.4.

In the general case, we first observe that \( \mathcal{D}(D) \) is a subposet of \( \mathcal{P} = \mathcal{D}(\mathcal{C}^\infty) \). To see that it is a subset, observe that adding arcs to \( D \) can only remove terminal sets, not create new ones: if a terminal set in \( D \) contains \( x_i e_i \), then it must contain all \( \alpha_i e_i \) for all \( \alpha_i < x_i \), so any terminal set of \( D \) was a terminal set \( T_x \) of \( \mathcal{C}^\infty \) for some \( x \). As \( \mathcal{D}(D) \) is also ordered by inclusion, its order is induced from \( \mathcal{D}(\mathcal{C}^\infty) \), so is a subposet.

Now it is enough to show that \( T : L \rightarrow \mathcal{D}(D) \) is a bijection. We do this by induction on the size of \( \mathcal{J} \). Let \( i[\alpha, \beta] \in \mathcal{J} \), \( \mathcal{J}' = \mathcal{J} - i[\alpha, \beta] \), \( L' = \mathcal{P} - \mathcal{J}' \), and \( A' = A_C(\mathcal{J}') \). By induction we have that \( T \) is bijection from \( L' \) to \( \mathcal{D}(A') \).

We must show for \( T_x \in \mathcal{D}(A') \), that \( T_x \notin \mathcal{D}(A) \) if and only if \( x \in i[\alpha, \beta] \). Now \( T_x \) being a terminal set in \( A' \) but not in \( A = A' \cup \{ (\alpha e_i, (\beta + 1) e_j) \} \) means exactly that \( \alpha e_i \in T_x \) and \( (\beta + 1) e_j \notin T_x \). This is to say \( \alpha < x_i \) but \( (\beta + 1) > x_j \). This means that \( x \in i[\alpha, \beta] \), as needed.

The inverse isomorphism \( T^{-1} \) can be viewed as an embedding of \( \mathcal{D}(D) \) into \( \mathcal{P} \). Viewing the spanning subgraph \( \mathcal{C}^\infty \) of \( D \) as a type of ‘pointed chain decomposition’ of \( D \) the following, which is immediate from the theorem, can be viewed as a generalization of Corollary 1.5. (We make the fact that this is a generalization more explicit with Corollary 5.3).

**Corollary 3.6.** For any spanning supergraph \( D \) of \( \mathcal{C}^\infty \) the map

\[
\mathcal{D}(D) \rightarrow \mathcal{P} : T \mapsto \bigvee T
\]

is an embedding of \( \mathcal{D}(D) \) into \( \mathcal{P} \).

Also, as Theorem 3.5 makes no assumption on the closure of \( \mathcal{J} \), and \( \mathcal{P} - \bigcup \mathcal{J} \) is independent of it, we get the following, which could also be simply verified directly.

**Corollary 3.7.** Let \( \mathcal{J}^c \) be the closure of the family \( \mathcal{J} \). Then \( \mathcal{D}(D_C(\mathcal{J}^c)) \cong \mathcal{D}(D_C(\mathcal{J})) \). Let \( D^T \) be the transitive closure of a reflexive digraph \( D \). Then \( \mathcal{D}(D^T) = \mathcal{D}(D) \).

By Proposition 2.5, this means we can henceforth assume that \( D \) is transitive, and so a preorder.
4. Generalizing antichains to independent sets

A directed path \( x_1 \to x_2 \to \cdots \to x_d \) in a digraph is non-trivial if \( d \geq 1 \). Observe that the directed path \( x \) is trivial, while the directed path, or loop, \( x \to x \), is non-trivial.

**Definition 4.1.** A subset \( I \) of a digraph \( A \) is independent if there is no non-trivial \( xy \)-path in \( A \) for (not necessarily distinct) \( x \) and \( y \) in \( I \).

Observe that by this definition, looped vertices cannot be in any independent sets. Compare the following with Example 3.2.

**Example 4.2.** The independent sets of \( T \) are exactly the antichains

\[
x = (x_1, \ldots, x_d) = \{x_1e_1, \ldots, x_de_d\}
\]

for vertices \( x \) of \( P \).

**Definition 4.3** \((\mathcal{A}_d(A))\). Let \( \mathcal{A}_d(A) \) be the family of independent sets of \( A \) of size \( d \), and order it as follows. For \( I, I' \in \mathcal{A}_d(A) \), let \( I \geq I' \) if for each \( a \in I \) there is an \( aa' \)-path in \( A \) with \( a' \in I' \).

The width \( w(A) \) of a digraph \( A \) is the size of the maximum independent set. Observe that if \( d > w(A) \), then \( \mathcal{A}_d(A) \) is empty. If \( d = w(A) \) and \( A \) is a poset, then \( \mathcal{A}_d(A^{irr}) = A(A) \) where \( A^{irr} \) is the graph we get from \( A \) by removing loops.

**Lemma 4.4.** Let \( \mathcal{T} \) be a disjoint union of transitive tournaments. Then

\[
I : \mathcal{P} \to \mathcal{A}_d(\mathcal{T}) : x \mapsto x := \{x_1e_1, \ldots, x_de_d\}
\]

is an isomorphism.

**Proof.** We have already observed that \( I \) is a bijection. To see that is an isomorphism simply note that there is an \((\alpha e_i, \beta e_j)\)-path in \( \mathcal{T} \) if and only if \( i = j \) and \( \alpha \geq \beta \). So \( x \geq x' \) if and only if \( I(x) \geq I(x') \). \( \square \)

Our goal now is to extend this lemma to Theorem 4.6. We could do this with almost exactly the same proof as in we used in the proof of Theorem 3.5, but it is more satisfying to use Theorem 4.5, our second main theorem.

**Theorem 4.5.** Let \( \mathcal{I} \) be a family of irreducible intervals of \( \mathcal{P} \) and let \( A = A_c(\mathcal{I}) \) and \( D = D_c(\mathcal{I}) \). Then

\[
T : \mathcal{A}_d(A) \to \mathcal{D}(D) : x \mapsto T_x
\]

is a lattice isomorphism.

**Proof.** In case that \( \mathcal{I} \) is empty, \( A = \mathcal{T} \) and \( D = \mathcal{C}^* \), so the result follows immediately from Lemmas 3.4 and 4.4.

We show that the restriction of \( T : x \mapsto T_x \) to \( \mathcal{A}_d(A) \) is a bijection to \( \mathcal{D}(D) \) by showing that \( x \in \mathcal{A}_d(A) \) if and only if \( T_x \in \mathcal{D}(D) \). As it follows from Theorem 3.5 that \( \mathcal{D}(D) \) is a lattice, \( T \) is then a isomorphism of lattices.

Indeed, that \( x \in \mathcal{A}_d(A) \) means there are no non-trivial \((x_ie_i, x_je_j)\)-paths in \( A \). This is equivalent to there being no arcs \( \alpha e_i \to \beta e_i \) in \( A \) for any \( \alpha \leq x_i \) and \( x_j \leq \beta \). But this is equivalent to there being no arcs \( \alpha e_i \to (\beta + 1)e_j \) in \( D \) for any \( \alpha \leq x_i \) and \( x_j \leq \beta \), which is to say any arc originating at some \( x_i \) in \( T_x \) terminates at \( \beta x_j \) for some \( \beta \leq x_j \). This is also in \( T_x \), so this means \( T_x \) is a terminal set. \( \square \)
Our third main theorem is now immediate.

**Theorem 4.6.** Let $L = \mathcal{P} - \cup \emptyset$ be sublattice of $\mathcal{P}$, and $A = A_{C}(\mathcal{J})$. Then the map $I : L \rightarrow k_{d}(A) : x \mapsto \{x_{1}e_{1}, \ldots, x_{d}e_{d}\}$ is a lattice isomorphism.

*Proof.* This is immediate from Theorems 3.5 and 4.5. \hfill \Box

As Corollary 3.7 followed from Theorem 3.5 we get the following from Theorem 4.6.

**Corollary 4.7.** Let $C$ be the closure of the family $\mathcal{J}$. Then $k_{d}(A_{C}(C)) \cong k_{d}(A_{C}(\mathcal{J}))$. Let $A^{T}$ be the transitive closure of a digraph $A$. Then $k_{d}(A^{T}) \cong k_{d}(A)$.

By Proposition 2.6, this allows us to assume that $A$ is transitive, and that $A_{C}(\mathcal{J}) = K_{C}^{-1}(D_{C}(\mathcal{J}))$. We will do so after this section.

By taking the inverse isomorphism $I^{-1}$ in Theorem 4.6 we get the following ‘independent set analogue’ to Corollary 3.6. A special case of it would have come out of [5], but was never made explicit there.

**Corollary 4.8.** For any spanning supergraph of a disjoint union of tournaments $\mathcal{T} = \cup_{i=1}^{d} T_{i}$ the map $k_{d}(A) \rightarrow \mathcal{P} : X \mapsto (X \cap T_{1}, \ldots, X \cap T_{d})$ is a well defined embedding of $k_{d}(A)$ into $\mathcal{P}$.

Though we did not need Theorem 1.6 to prove Theorem 4.6, our generalization of Corollary 4.9. will need it to prove the following generalization of Theorem 1.9.

**Corollary 4.9.** For any digraph $A$, the poset $A_{M}(A)$ of all maximum sized independent sets of $A$ is a distributive lattice.

*Proof.* We can remove arcs from $A$ to get a subgraph $A_{1}$ with the same independent sets, but no directed cycles. Indeed, if removing an arc $x \rightarrow y$ joins two independent sets $I_{x}$ and $I_{y}$, then in particular there was no directed $(y,x)$-path in $D$, contradicting the fact that $x \rightarrow y$ is in a directed cycle.

The poset $P$ we get by adding loops to every vertex of the transitive closure $A^{T}$ of $A_{1}$, has a decomposition $\mathcal{C}$ into $d = w(P)$ chains $C_{1}, \ldots, C_{m}$ by Theorem 1.6. Removing the loops from each of these chains we get a tournament decomposition of $T_{1}, \ldots, T_{m}$ of $A^{T}$. By Corollary 4.8 this gives an isomorphism of $k_{d}(A^{T}) = k_{d}(A^{T})$ to a sublattice of $\mathcal{P}$. This is $A_{M}(A_{1})$ by Corollary 4.7, which is the same as $A_{M}(A)$ by construction. \hfill \Box

This actually proved more. Where $\mathcal{T} = \{T_{1}, \ldots, T_{m}\}$ is the tournament decomposition of $A^{T}$ in the proof and $\mathcal{R} = R_{C}(A)$ for corresponding $\mathcal{C}$, we get that $A = A_{C}(\mathcal{R})$. This gives the following further complement to Theorem 1.10.

**Corollary 4.10.** For any digraph $A$ with $A_{M}(A) = L$, there is a (subdirect) sublattice $\mathcal{P} - \mathcal{R} \cong L$ of a product of chains $\mathcal{P}$ such that $A = A_{C}(\mathcal{R})$.

5. Characterisation of sublattices

A sublattice $L = \mathcal{P} - \cup \emptyset$ of $\mathcal{P}$ is full if it contains $1_{\mathcal{P}}$ and $0_{\mathcal{P}}$. It is subdirect if for each $i \in [d]$, the projection $\pi_{i} : L \rightarrow C_{i}$ is surjective; this is necessarily full. A
Figure 2. Subdirect sublattice $L = \mathbb{Z}_2 \times \mathbb{Z}_2 - [2, 1]_2 - 2[2, 1]_1$

that is not tight.

full subgraph is tight if its covers are covers of $\mathcal{P}$. We call $\mathcal{J}$ full, subdirect or tight if $\mathcal{P} - \mathcal{J}$ is as a sublattice of $\mathcal{P}$.

It was shown in [6] that every tight sublattice of a product of chains is subdirect. The converse was also claimed, but the proof was flawed: indeed, the lattice $L = \mathbb{Z}_2 \times \mathbb{Z}_2 - [2, 1]_2 - 2[2, 1]_1$ shown in Figure 2 is a subdirect sublattice of $\mathbb{Z}_2 \times \mathbb{Z}_2$, but not tight. One notices in this example that $\text{AC}(\mathcal{I})$ is acyclic while $\text{DC}(\mathcal{I})$ is not.

This is indicative of the difference between tight and subdirect sublattices.

5.1. Full sublattices.

Lemma 5.1. Let $\mathcal{P} - \cup \mathcal{J}$ be a full sublattice of $\mathcal{P}$. Let $A = \text{AC}(\mathcal{J})$ and $D = \text{DC}(\mathcal{J})$. Then the following are true.

(i) For any $i$ the vertex $n_i e_i$ in $A$ is a source and the vertex $0 e_i$ is a sink.

(ii) The vertex $\infty$ is a source in $D$ and $0$ is a sink.

(iii) Where $D^*$ is the graph we get from $D$ by removing the vertices $\infty$ and $0$, $T \mapsto T - \{0\}$ is a bijection of the family of proper terminal sets of $D$ to the family of terminal sets of $D^*$.

Proof. By Theorem 4.6 the map $x \mapsto x$ is an isomorphism of $A_d(A)$ to $\mathcal{P} - \cup \mathcal{J}$. By definition this is full if and only if $1_\mathcal{P}$ and $0_\mathcal{P}$ are in $A_d(A)$, which is clearly true if and only if there are no arcs in $A$ terminating at $n_i e_i$ or originating at $0 e_i$, for any $i$. This gives (i). Item (ii) is now immediate using that $D = K_e(A)$.

We now show that (ii) implies (iii). As $\{0\}$ is in any proper terminal set of $D$ and has no out edges, it is clear that for any proper terminal set $T$ of $D$, $T \setminus \{0\}$ is a terminal set of $D^*$. We must show that there are no other terminal sets of $D^*$. Assume $T$ is terminal in $D^*$ but $T \cup \{0\}$ is not terminal in $D$. Then $T$ must have some out arc in $D$ that does not exist in $D^*$. This can only be an arc to $\infty$ but by (ii), there are no such arcs.

We thus make the following definition.

Definition 5.2. For a family $\mathcal{J}$ of irreducible intervals of $\mathcal{P}$, let $D^* = D^*_e(\mathcal{J})$ be the subgraph of $D_e(\mathcal{J})$ induced by the vertices of $\mathcal{E}^* = \mathcal{E}_\infty - \{\infty, 0\}$. Let $\mathcal{D}^*(D^*)$ be the set of terminal subsets of $D^*$.

\[\text{Figure 2. Subdirect sublattice } L = \mathbb{Z}_2 \times \mathbb{Z}_2 - [2, 1]_2 - 2[2, 1]_1\]
In light of item (iii) of the Theorem, we have that , $\mathbb{D}^*(D^*) = \mathbb{D}(D)$ when $\mathcal{J}$ is full. The spanning subgraph $C^\infty$ of $D$ yields a spanning subgraph $C^*$ of $\mathbb{R}^*$, which is exactly a chain decomposition of $\mathbb{R}^*$ thus Corollary 3.6 yields the following, which with Theorem 3.5 yields a one-to-one correspondence between the full sublattices of products of chains, and the chain decomposition of transitive reflexive digraphs, or preorders.

**Corollary 5.3.** For any chain decomposition $\mathcal{C}^*$ of a preorder $D^*$ the map

$$\mathbb{D}^*(D^*) \to \mathcal{P} : T \mapsto \bigvee T$$

is a full embedding of $\mathbb{D}(\mathcal{R}) = \mathbb{D}^*(D^*)$ into $\mathcal{P}$.

A preorder is a poset if it is acyclic. This brings us to the next point.

### 5.2. Tight and subdirect sublattices

A digraph is *acyclic* if its only directed cycles are loops, and is *strongly acyclic* if it it has no directed cycles.

In the proof of the following lemma, we use the following, simple, well-known ideas. The *height* of a lattice is the length of a maximum ascending chain of covers from 0 to 1. For a distributive lattice, every cover is in a maximum ascending chain. It follows that a sublattice $L$ of $\mathcal{P}$ is tight if and only if $L$ and $\mathcal{P}$ have the same height.

**Lemma 5.4.** Let $L = \mathcal{P} - \cup \mathcal{J}$ be a sublattice of $\mathcal{P}$, and $\mathcal{J}$ be closed. Let $D = D_\mathcal{C}(\mathcal{J})$ and let $A = A_\mathcal{C}(\mathcal{J})$. Then the following are true.

(i) $L$ is tight (in $\mathcal{P}$) if and only if $D$ is acyclic.

(ii) $L$ is subdirect if and only if $D$ has no up edges: those of the form $\alpha \varepsilon_i \to (\alpha + 1) \varepsilon_i$.

(iii) $L$ is subdirect if and only if $A$ is irreflexive.

(iv) $L$ is subdirect if and only if $A$ is strongly acyclic.

**Proof.** (i). First assume that $L$ is tight, and assume, towards contradiction that $D$ contains a cycle $\alpha_1 \varepsilon_i \to \alpha_2 \varepsilon_i \to \cdots \to \alpha_\ell \varepsilon_i \to \alpha_1 \varepsilon_i$, for some $\ell \geq 2$. We show that no vertex $x$ in $L$ has $x_i = \alpha_1$, which contradicts the fact that $L$ is tight. Indeed, if $x$ did have $x_i = \alpha_1$, then $x_i \geq \alpha_1$ so by Lemma 2.4 $x_i > \alpha_2$ so $x_i > \alpha_3$ etc., and we get that $x_i > \alpha_1$, a contradiction.

On the other hand, assume that $D$ is acyclic. Then its vertices can be ordered so that all arcs go down. Adding vertices from the bottom of this ordering, one at a time, we get an ascending walk in $L = \mathbb{D}(D)$ from 0 to 1 of size $|D|$, showing that $L$ has height equal to the height of $\mathcal{P}$. Thus $L$ is a tight sublattice.

(ii). There is an edge $\alpha \varepsilon_i \to (\alpha + 1) \varepsilon_i$ in $D$ if and only if there is no terminal set of $D$ containing $\alpha \varepsilon_i$ but not $(\alpha + 1) \varepsilon_i$. This means precisely that $|T_\varepsilon \cap C_\varepsilon| = \alpha$ is not true for any $T_\varepsilon \in \mathbb{D}(D)$. So no element $x \in L$ has $x_i = \alpha$.

The equivalence of (ii) and (iii) is immediate from the fact that $D = K_\mathcal{C}(A)$, and the equivalence of (iii) and (iv) is immediate by the transitivity of $A$.

Using that $\mathcal{J}$ can be assumed closed, so $D_\mathcal{C}(\mathcal{J})$ can be assumed transitive, we get the following by Corollary 1.2 and the observation that on posets $\mathbb{D}^*$ is the same as $\mathcal{D}$.

**Corollary 5.5.** If $L = \mathcal{P} - \cup \mathcal{J}$ is a tight sublattice, then $D_\mathcal{C}(\mathcal{J}) = J_L^\infty$ and $D_\mathcal{C}(\mathcal{J}) = J_L$.
This shows explicitly that Corollary 5.3 is a generalization of Corollary 1.5, and that Corollary 3.6 can in turn be viewed as a generalization.

6. A ONE-TO-ONE CORRESPONDENCE

Our main drive has been to catalog the sublattices of a given product of chains with the digraphs $D$ and $A$, and from this to describe the lattices intrinsically with $D$ and $A_E$. One aspect of the original decomposition theories we loose is the fact that in them, the ‘cataloging’ digraph $J_L$ was unique for a given lattice $L$. We recapture this uniqueness by showing that $J_L^\infty$ is a canonical quotient of $D$ for any $D$ with $D(D) = L$. Via this quotient, chain decompositions of $D^*$ become ‘chain covers’ of $J_L$, extending the correspondence of Corollary 1.8.

6.1. Back To Posets. Let $D$ be a reflexive transitive digraph, and let $\sim$ be the relation on $D$ defined by $x \sim y$ if $x$ and $y$ are in a directed cycle. This is clearly an equivalence relation. The partial ordering, defined on the set $P_D := D/\sim$ of equivalence classes of $\sim$, by $[a] \geq [b]$ if and only if $a \to b$, is well defined because of the transitivity of $D$. Further, it is a poset, which we also denote by $P_D$. (Indeed, $D$ is a preorder, so ‘$\sim$’ is the established non-preference relation on a preorder. Its quotient is known to be a poset.)

Now the quotient homomorphism $q : D \to P_D : a \to [a]$ induces maps between $2^D$ and $2^{P_D}$. For $T \subset D$, we let $[T] := \{[x] \mid x \in T\} \subset P_D$, and for $S \subset P_D$ we let $\cup S = \cup_{[x] \in S} [x]$. Clearly $\cup S = S$ and $\cup T = T$.

**Lemma 6.1.** Let $D$ be a transitive reflexive digraph, then $D(D) \to D(P_D) : T \mapsto [T]$ is an isomorphism with inverse $S \mapsto \cup S$.

**Proof.** Let $T$ be in $D(D)$. Then for $[x] \in [T]$ with $[x] \geq [y]$, we have that $x \to y$ and so $y \in T$. Thus $[y] \in [T]$, giving that $[T] \in D(P_A)$.

Reversing the argument, we get that $\cup S$ is a terminal set if $S$ is; so $T \mapsto [T]$ is a bijection. That it is an isomorphism is clear, as the order on both posets is inclusion, and $[T]$ simply partitions a set $T$ with respect to an underlying fixed partition of $D$.

One can make exactly the same construction for a non-reflexive graph $A$, yielding an acyclic quotient $A_A = A/\sim$, which may or may not have loops. (A class $[x]$ will have a loop if it contains a looped vertex or more than one vertex.) Analogously to Lemma 6.1 one can prove the following.

**Lemma 6.2.** Let $A$ be a transitive digraph, then $\kappa_d(A) \to \kappa_d(A) : A \mapsto \{[a] \mid a \in A\}$ is an isomorphism.

6.2. Embeddings and Chain Covers. In Corollary 3.6 we viewed $\mathcal{C}_\infty$ as a spanning subgraph of $D$, rather than the other way around. Now we take it a step further, viewing it as the image of a bijective homomorphism $\phi : \mathcal{C}_\infty \to D$. As such, the embedding in Corollary 3.6 can be written as $E_\phi : D(D) \to \mathcal{P} : T \mapsto \bigvee \phi^{-1}(T)$. This holds for any bijective homomorphism $\phi : \mathcal{C}_\infty \to D$ to a transitive reflexive graph $D$. Under passage to $P_D$, $\phi$ yields a surjective homomorphism $(q \circ \phi) : \mathcal{C}_\infty \to P_D$. Observing that $(q \circ \phi)^{-1}(S) = \phi^{-1}(\cup S)$, Lemma 6.1 extends the above embedding $E_\phi$ to an embedding $E_\phi : D(P_D) \to \mathcal{P} : S \mapsto \bigvee \phi^{-1}(S)$. 
On the other hand, for an embedding $E : \mathbb{D}(P_D) \to \mathcal{P}$ we define $\phi_E : C^\infty \to P_D$ by

$$\phi_E^{-1}(p) = T_{E(D_p)} - T_{E(D_p - \{p\})}.$$

**Theorem 6.3.** The constructions $E \to \phi_E$ and $\phi \to E_\phi$ give a one-to-one correspondence between the surjective homomorphisms $\phi$ of pointed unions of chains $C^\infty$ to $P_R$, and the embeddings $E$ of $\mathbb{D}(P_R)$ into products of chains.

**Proof.** To see that $E = E_\phi$, it is enough to show it for the join irreducible elements of $\mathbb{D}(P_D)$, i.e., those terminal sets of the form $T_r$. This is clear:

$$E_{\phi_E}(T_r) = \bigvee \phi_\phi^{-1}(T_r) = \bigvee T_{E(T_r)} = E(T_r).$$

To see that $\phi_{x_\phi} = \phi$, let $r$ be in $P_R$. Then

$$\phi_{E_\phi}^{-1}(r) = T_{E_{\phi}(T_r)} - T_{E_{\phi}(T_r - \{r\})}
= T_{\bigvee (\phi^{-1}(T_r) - \{r\})}
= \phi^{-1}(T_r) - \phi^{-1}(T_r - \{r\})
= \phi^{-1}(r)$$

The last equality comes from the fact that $T_r$ is join irreducible in $\mathbb{D}(P_R)$ and the unique terminal set it covers is $T_r - \{r\}$. \qed

Restricting to full embeddings, we can consider the supergraph $D^*$ of $C^*$ instead of $D$. For a full embedding $E$ into $\mathcal{P}$, the surjective homomorphism $\phi_E : C^\infty \to D$, must take only $\infty$ and $0$ of $C^\infty$ to $\infty = \phi_E(\infty)$ and $0 = \phi_E(0)$ respectively. So restriction to $C^*$ yields a surjective homomorphism $\phi_E : C^* \to D^*$. On the other hand, any surjective homomorphism $\phi_E : C^* \to D^*$ can be extended to surjective homomorphism $C^\infty \to D$.

**Definition 6.4.** A *loose chain* of a poset $P$ is a pair $(\phi, C)$ where $C$ is a chain, and $\phi : C \to P$ is a homomorphism. It is a *chain* of $P$ if $\phi$ is injective. A *loose chain cover* of $P$ is a family $\mathcal{L} = \{(\phi_1, C_1), \ldots, (\phi_d, C_d)\}$ of loose chains of $P$ such that the union of the vertex sets of $\phi_i(C_i)$ is the vertex set of $P$. $\mathcal{L}$ is a *chain cover* if all the loose chains in the cover are chains. It is a *chain decomposition* if the chains $\phi_i(C_i)$ are disjoint.

It is clear that a surjective homomorphism $\phi : C^* \to P$ is exactly a loose chain cover $\{(\phi, C_1), \ldots, (\phi, C_d)\}$ of $P$. Thus we have the following corollary to Theorem 6.3.

**Corollary 6.5.** The constructions $E \to \phi_E$ and $\phi \to E_\phi$ give a one-to-one correspondence between full embeddings of $L$ into products of chains and loose chain covers of $J_L$. Subdirect embeddings correspond with chain covers, and tight embeddings correspond with chain decompositions.

**References**


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