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Boundary problem for the refined analytic torsion on compact manifolds with boundary

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Abstract. The refined analytic torsion was introduced by Braverman and Kappeler on an odd dimensional closed Riemannian manifold in 2000's as an analytic analogue of the Turaev torsion. It is defined by using the graded zeta-determinant of the odd signature operator and is described as an element of the determinant line for cohomologies. In this note we briefly discuss the boundary problem for the refined analytic torsion on a compact Riemannian manifold with boundary by introducing well posed boundary conditions for the odd signature operator. We also discuss the gluing formula for the refined analytic torsion on a closed Riemannian manifold with respect to our chosen boundary conditions.

1 Introduction

The Reidemeister torsion was introduced by Reidemeister and Franz in 1930's on a finite CW-complex with an orthogonal representation of its fundamental group ([18], [25]). It is a homeomorphic invariant, not a homotopy invariant. They used this invariant to classify the lens spaces up to homeomorphism. Turaev extended the Reidemeister torsion to a torsion for a general representation of a fundamental group, called Turaev torsion, by using new concepts such as Euler structure and homology orientation ([23], [24]). Turaev torsion was developed further by works of Farber and Turaev ([11], [12]).

Key words and phrases: Refined analytic torsion, Zeta-determinant, Eta-invariant, Odd signature operator, Gluing formula.

The analytic torsion was introduced in 1970's by Ray and Singer to recover the Reidemeister torsion by analytic and differential geometric way. They defined the analytic torsion by the zeta-determinants of Hodge Laplacians acting on flat bundle valued differential forms on closed Riemannian manifolds ([21], [22]). They conjectured the equality of the Reidemeister and analytic torsions. Cheeger and Müller proved the equality of two torsions on a closed manifold independently ([10], [19]).

On the same line of analytic torsion, the analytic analogues of the Turaev torsion have been studied in two ways. One is the method of Burghelea-Haller ([2], [3]) and the other one is the method of Braverman-Kappeler ([4], [5]). Burghelea and Haller constructed the complex valued analytic torsion by using a Euler structure, Mathai-Quillen form and a non-degenerated bilinear form acting on a flat vector bundle on a closed manifold. They defined non-selfadjoint Laplacians from the non-degenerate bilinear form and constructed a complex valued analytic torsion. On the other hand, Braverman and Kappeler constructed a (sign) refined analytic torsion on a closed Riemannian manifold by using the odd signature operator acting on complex flat bundle valued differential forms. It is described as an element of the determinant line of cohomologies. If all the cohomologies vanish, the refined analytic torsion is a complex number. If the odd signature operator comes from an acyclic Hermitian connection, the refined analytic torsion is a complex number, whose modulus part is the classical Ray-Singer analytic torsion and the phase part is the rho invariant, the difference of two eta invariants ([1]).

In this note we discuss briefly the boundary value problem of the refined analytic torsion on a compact Riemannian manifold with boundary. For this purpose we are going to introduce well posed boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ and discuss the refined analytic torsion subject to the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ ([13]). We next discuss the gluing problem of the refined analytic torsion with respect to the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ ([14]). Vertman also studied the boundary value problem and gluing problem for the refined analytic torsion in [26] and [27]] but his method is completely different from ours. In this note we restrict ourselves to the case of Hermitian connections for simplicity.

2 Odd signature operator on a compact manifold with boundary

Let (M, g^M) be a compact oriented m-dimensional Riemannian manifold with boundary Y ($m = 2r - 1$, odd integer), where g^M is assumed to be a product metric near the boundary Y. We denote by (E, ∇) a flat complex vector bundle $E \to M$ with a flat connection ∇. We extend ∇ to a covariant derivative

$$\nabla : \Omega^{\bullet}(M, E) \to \Omega^{\bullet+1}(M, E).$$

We choose a Hermitian inner product h^E on E. For ϕ, $\psi \in C^{\infty}(E)$, if ∇ satisfies

$$dh^E(\phi, \psi) = h^E(\nabla\phi, \psi) + h^E(\phi, \nabla\psi),$$

∇ is called a Hermitian connection. All through this note we assume that ∇ is a Hermitian connection for some Hermitian inner product h^E. Using the Hodge star operator $*_M$, we define the involution $\Gamma = \Gamma(g^M) : \Omega^\bullet(M,E) \to \Omega^{m-\bullet}(M,E)$ by

$$\Gamma\omega := i^r(-1)^{\frac{q(q+1)}{2}} *_M \omega, \qquad \omega \in \Omega^q(M,E),$$

where $r = \frac{m+1}{2}$. Then $\Gamma^2 = \mathrm{Id}$. The odd signature operator $\mathcal{B}$ is defined by

$$\mathcal{B} \;=\; \mathcal{B}(\nabla, g^M) \;:=\; \Gamma\,\nabla \;+\; \nabla\,\Gamma : \Omega^\bullet(M,E) \;\longrightarrow\; \Omega^\bullet(M,E). \tag{2.1}$$

Then $\mathcal{B}$ is an elliptic differential operator of order 1. Let N be a collar neighborhood of Y which is isometric to $[0,1) \times Y$. Then there is a natural isomorphism

$$\Psi : \Omega^p(N, E|_N) \to C^\infty([0,1), \Omega^p(Y, E|_Y) \oplus \Omega^{p-1}(Y, E|_Y)), \tag{2.2}$$

defined by $\Psi(\omega_1 + du \wedge \omega_2) = (\omega_1, \omega_2)$, where u is a coordinate normal to the boundary Y. Using the product structure we can induce a flat connection $\nabla^Y : \Omega^\bullet(Y, E|_Y) \to \Omega^\bullet(Y, E|_Y)$ from ∇ and a Hodge star operator $*_Y : \Omega^\bullet(Y, E|_Y) \to \Omega^{m-1-\bullet}(Y, E|_Y)$ from $*_M$. We define two involutions β, Γ^Y by

$$\beta : \Omega^p(Y, E|_Y) \to \Omega^p(Y, E|_Y), \quad \beta(\omega) = (-1)^p\omega$$
$$\Gamma^Y : \Omega^p(Y, E|_Y) \to \Omega^{m-1-p}(Y, E|_Y), \quad \Gamma^Y(\omega) = i^{r-1}(-1)^{\frac{p(p+1)}{2}} *_Y \omega.$$

It is straightforward that

$$\beta^2 = \mathrm{Id}, \qquad \Gamma^Y\Gamma^Y = \mathrm{Id}\,.$$

The odd signature operator $\mathcal{B}$ is expressed on N, under the isomorphism (2.2), by

$$\mathcal{B} = -i\beta\Gamma^Y \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla_{\partial u} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \left(\nabla^Y + \Gamma^Y\nabla^Y\Gamma^Y\right) \right\}, \tag{2.3}$$

where ∂u is the inward normal derivative to the boundary Y on N. We denote

$$\mathcal{B}_Y := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \left(\nabla^Y + \Gamma^Y\nabla^Y\Gamma^Y\right), \qquad \mathcal{H}^\bullet(Y, E|_Y) := \ker \mathcal{B}_Y^2.$$

Then $\mathcal{B}_Y$ is a self-adjoint elliptic operator on Y and hence $\mathcal{H}^\bullet(Y, E|_Y)$ is a finite dimensional vector space. We have

$$\Omega^\bullet(Y, E|_Y) = \mathrm{Im}\,\nabla^Y \oplus \mathrm{Im}\,\Gamma^Y\nabla^Y\Gamma^Y \oplus \mathcal{H}^\bullet(Y, E|_Y).$$

If $\nabla\phi = \Gamma\nabla\Gamma\phi = 0$ for $\phi \in \Omega^\bullet(M,E)$, simple computation shows that ϕ is expressed, near the boundary Y, by

$$\phi = \nabla^Y\phi_{\tan} + \phi_{\tan,h} + du \wedge (\Gamma^Y\nabla^Y\Gamma^Y\phi_{\mathrm{nor}} + \phi_{\mathrm{nor},h}), \tag{2.4}$$

where $\phi_{\tan,h},\ \phi_{\mathrm{nor},h} \in \mathcal{H}^\bullet(Y, E|_Y)$.
We define $\mathcal{K}$ by

$$\mathcal{K} := \{\phi_{\tan,h} \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla\phi = \Gamma\nabla\Gamma\phi = 0\},$$

where ϕ has the form (2.4). Then

$$\Gamma^Y \mathcal{K} = \{\phi_{\mathrm{nor},h} \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla\phi = \Gamma\nabla\Gamma\phi = 0\},$$

and we have

$$\mathcal{K} \oplus \Gamma^Y \mathcal{K} = \mathcal{H}^\bullet(Y, E|_Y),$$

which shows that $(\mathcal{H}^\bullet(Y, E|_Y),\ \langle\ ,\ \rangle_Y,\ -i\beta\Gamma^Y)$ is a symplectic vector space with Lagrangian subspaces $\mathcal{K}$ and $\Gamma^Y\mathcal{K}$. We denote by

$$\mathcal{L}_0 = \begin{pmatrix} \mathcal{K} \\ \mathcal{K} \end{pmatrix}, \qquad \mathcal{L}_1 = \begin{pmatrix} \Gamma^Y\mathcal{K} \\ \Gamma^Y\mathcal{K} \end{pmatrix},$$

and define the orthogonal projections $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1} : \Omega^\bullet(Y, E|_Y) \oplus \Omega^\bullet(Y, E|_Y) \to \Omega^\bullet(Y, E|_Y) \oplus \Omega^\bullet(Y, E|_Y)$ by

$$\operatorname{Im} \mathcal{P}_{-,\mathcal{L}_0} = \begin{pmatrix} \operatorname{Im} \nabla^Y \oplus \mathcal{K} \\ \operatorname{Im} \nabla^Y \oplus \mathcal{K} \end{pmatrix}, \qquad \operatorname{Im} \mathcal{P}_{+,\mathcal{L}_1} = \begin{pmatrix} \operatorname{Im} \Gamma^Y\nabla^Y\Gamma^Y \oplus \Gamma^Y\mathcal{K} \\ \operatorname{Im} \Gamma^Y\nabla^Y\Gamma^Y \oplus \Gamma^Y\mathcal{K} \end{pmatrix}.$$

Then $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$ are pseudodifferential operators and give well-posed boundary conditions for $\mathcal{B}$ and the refined analytic torsion. We denote by $\mathcal{B}_{\mathcal{P}_{-,\mathcal{L}_0}}$ and $\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}}$ the realizations of $\mathcal{B}$ and $\mathcal{B}^2_q$ with respect to $\mathcal{P}_{-,\mathcal{L}_0}$, *i.e.*

$$\begin{aligned} \operatorname{Dom}\left(\mathcal{B}_{\mathcal{P}_{-,\mathcal{L}_0}}\right) &= \{\psi \in \Omega^\bullet(M, E) \mid \mathcal{P}_{-,\mathcal{L}_0}(\psi|_Y) = 0\} =: \Omega^\bullet(M, E)_{\mathcal{P}_{-,\mathcal{L}_0}}, \\ \operatorname{Dom}\left(\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}}\right) &= \{\psi \in \Omega^q(M, E) \mid \mathcal{P}_{-,\mathcal{L}_0}(\psi|_Y) = 0,\ \mathcal{P}_{-,\mathcal{L}_0}((\mathcal{B}\psi)|_Y) = 0\} \\ &=: \Omega^q(M, E)_{\mathcal{P}_{-,\mathcal{L}_0}} \end{aligned}$$

We define $\mathcal{B}_{\mathcal{P}_{+,\mathcal{L}_1}}$, $\mathcal{B}^2_{q,\mathcal{P}_{+,\mathcal{L}_1}}$, and $\mathcal{B}^2_{q,\mathrm{abs}}$, $\mathcal{B}^2_{q,\mathrm{rel}}$ in the similar way. Then we have the following result (Lemma 2.11 in [13]).

Lemma 2.1.

$$\ker \mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}} = \ker \mathcal{B}^2_{q,\mathrm{rel}} = H^q(M, Y; E), \qquad \ker \mathcal{B}^2_{q,\mathcal{P}_{+,\mathcal{L}_1}} = \ker \mathcal{B}^2_{q,\mathrm{abs}} = H^q(M; E).$$

Since the odd signature operator $\mathcal{B}$ subject to the boundary condition $\mathcal{P}_{-,\mathcal{L}_0}$ or $\mathcal{P}_{+,\mathcal{L}_1}$ is a self-adjoint operator, we can choose $\frac{\pi}{2}$ and π for the Agmon angles of $\mathcal{B}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}$ and $\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}$, respectively. We fix $\frac{\pi}{2}$ and π for the Agmon angles. Then, for $\mathfrak{D} = \mathcal{P}_{-,\mathcal{L}_0}$ or $\mathcal{P}_{+,\mathcal{L}_1}$ we define the zeta function $\zeta_{\mathcal{B}^2_{q,\mathfrak{D}}}(s)$ and eta function $\eta_{\mathcal{B}_{\mathrm{even},\mathfrak{D}}}(s)$ by

$$\begin{aligned} \zeta_{\mathcal{B}^2_{q,\mathfrak{D}}}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\operatorname{Tr} e^{-t\mathcal{B}^2_{q,\mathfrak{D}}} - \dim\ker \mathcal{B}^2_{q,\mathfrak{D}} \right) dt \\ \eta_{\mathcal{B}_{\mathrm{even},\mathfrak{D}}}(s) &= \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \operatorname{Tr}\left(\mathcal{B} e^{-t\mathcal{B}^2_{\mathrm{even},\mathfrak{D}}} \right) dt. \end{aligned} \tag{2.5}$$

It was shown in [13] that $\zeta_{\mathcal{B}^2_{q,\mathfrak{D}}}(s)$ and $\eta_{\mathcal{B}_{\mathrm{even},\mathfrak{D}}}(s)$ have regular values at $s=0$. We define the zeta-determinant and eta-invariant by

$$\begin{aligned} \log \operatorname{Det} \mathcal{B}^2_{q,\mathfrak{D}} &:= -\zeta'_{\mathcal{B}^2_{q,\mathfrak{D}}}(0), \\ \eta(\mathcal{B}_{\mathrm{even},\mathfrak{D}}) &:= \frac{1}{2}\left(\eta_{\mathcal{B}_{\mathrm{even},\mathfrak{D}}}(0) + \dim\ker \mathcal{B}_{\mathrm{even},\mathfrak{D}}\right). \end{aligned} \tag{2.6}$$

We put

$$\Omega^q_-(M,E) = \operatorname{Im}\nabla \cap \Omega^q(M,E), \qquad \Omega^q_+(M,E) = \operatorname{Im}\Gamma\nabla\Gamma \cap \Omega^q(M,E),$$
$$\Omega^{\mathrm{even}}_\pm(M,E) = \sum_{q=\mathrm{even}} \Omega^q_\pm(M,E),$$

and denote by $\mathcal{B}^\pm_{\mathrm{even}}$ the restriction of $\mathcal{B}_{\mathrm{even}}$ to $\Omega^{\mathrm{even}}_\pm(M,E)$. The graded zeta-determinant $\operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}_{\mathrm{even},\mathfrak{D}})$ of $\mathcal{B}_{\mathrm{even}}$ with respect to the boundary condition $\mathfrak{D}$ is defined by

$$\operatorname{Det}_{\mathrm{gr}}(\mathcal{B}_{\mathrm{even},\mathfrak{D}}) = \frac{\operatorname{Det} \mathcal{B}^+_{\mathrm{even},\mathfrak{D}}}{\operatorname{Det}\left(-\mathcal{B}^-_{\mathrm{even},\mathfrak{D}}\right)}.$$

We next define the projections $\widetilde{\mathcal{P}}_0, \widetilde{\mathcal{P}}_1 : \Omega^\bullet(Y, E|_Y) \oplus \Omega^\bullet(Y, E|_Y) \to \Omega^\bullet(Y, E|_Y) \oplus \Omega^\bullet(Y, E|_Y)$ as follows. For $\phi \in \Omega^q(M,E)$

$$\widetilde{\mathcal{P}}_0(\phi|_Y) = \begin{cases} \mathcal{P}_{-,\mathcal{L}_0}(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{+,\mathcal{L}_1}(\phi|_Y) & \text{if } q \text{ is odd,} \end{cases} \qquad \widetilde{\mathcal{P}}_1(\phi|_Y) = \operatorname{Id} - \widetilde{\mathcal{P}}_0(\phi|_Y).$$

Then, $\log \operatorname{Det}_{\mathrm{gr}}(\mathcal{B}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}})$ is described as follows ([13]).

(1) $$\log \operatorname{Det}_{\mathrm{gr}}(\mathcal{B}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}}) = \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q \cdot \log \operatorname{Det} \mathcal{B}^2_{q,\widetilde{\mathcal{P}}_0}$$
$$-i\pi\, \eta(\mathcal{B}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}}) + \frac{\pi i}{2}\left(\frac{1}{4}\sum_{q=0}^{m-1}\zeta_{\mathcal{B}^2_{Y,q}}(0) + \sum_{q=0}^{r-2}(r-1-q)(l_q^+ - l_q^-)\right),$$

(2) $$\log \operatorname{Det}_{\mathrm{gr}}(\mathcal{B}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}) = \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q \cdot \log \operatorname{Det} \mathcal{B}^2_{q,\widetilde{\mathcal{P}}_1}$$
$$-i\pi\, \eta(\mathcal{B}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}) - \frac{\pi i}{2}\left(\frac{1}{4}\sum_{q=0}^{m-1}\zeta_{\mathcal{B}^2_{Y,q}}(0) + \sum_{q=0}^{r-2}(r-1-q)(l_q^+ - l_q^-)\right),$$

where $l_q := \dim\ker \mathcal{B}^2_{Y,q}$, $l_q^+ := \dim \mathcal{K} \cap \ker \mathcal{B}^2_{Y,q}$, $l_q^- := \dim \Gamma^Y \mathcal{K} \cap \ker \mathcal{B}^2_{Y,q}$, and $l_q = l_q^+ + l_q^-$, $l_q^- = l^+_{m-1-q}$.

As a final ingredient we consider the trivial connection $\nabla^{\mathrm{trivial}}$ acting on the trivial bundle $M \times C$ and define the trivial odd signature operator $\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even}} : \Omega^{\mathrm{even}}(M, C) \to \Omega^{\mathrm{even}}(M, C)$ in the same way as (2.1). The eta invariant $\eta(\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}})$ associated to $\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even}}$ and the boundary condition $\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}$ is defined in the same way as in (2.5) and (2.6) by simply replacing $\mathcal{B}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}$ with $\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}$. When ∇ is acyclic in the de Rham complex, the refined analytic torsions $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}}(M, g^M, \nabla)$ and $\rho_{an,\mathcal{P}_{+,\mathcal{L}_1}}(M, g^M, \nabla)$ subject to the boundary condition $\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}$ are defined by

$$\begin{aligned}
\log \rho_{an,\mathcal{P}_{-,\mathcal{L}_0}}(M, g^M, \nabla) &:= \log \mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}}) + \frac{\pi i}{2}(\mathrm{rank}\, E)\eta_{\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}}}(0),\\
\log \rho_{an,\mathcal{P}_{+,\mathcal{L}_1}}(M, g^M, \nabla) &:= \log \mathrm{Det}_{\mathrm{gr},\theta}(\mathcal{B}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}) + \frac{\pi i}{2}(\mathrm{rank}\, E)\eta_{\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}}(0).
\end{aligned}$$

3 Refined analytic torsion on a compact manifold with boundary

In this section, we briefly explain the determinant line for cohomologies and the canonical element, following the presentation of [5]. We refer to [5] for more details. Let $(C^\bullet, \partial, \Gamma)$ be a finite complex consisting of finite dimensional vector spaces as follows.

$$(C^\bullet,\ \partial,\ \Gamma)\ :\ 0 \longrightarrow C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^{m-1} \xrightarrow{\partial} C^m \longrightarrow 0.$$

Here $\Gamma : C^\bullet \to C^\bullet$ is an involution, called a chirality operator, satisfying $\Gamma(C^q) = C^{m-q}$ for $0 \le q \le m$. We define the determinant line $\mathrm{Det}(C^\bullet)$ of $(C^\bullet, \partial, \Gamma)$ by

$$\begin{aligned}
\mathrm{Det}(C^q) &= \wedge^{\dim C^q} C^q, \qquad \mathrm{Det}(C^q)^{(-1)} := \mathrm{Hom}\big(\,\mathrm{Det}(C^q), \mathbb{C}\,\big)\\
\mathrm{Det}(C^\bullet) &= \bigotimes_{q=0}^{m} \mathrm{Det}(C^q)^{(-1)^q}, \quad m = 2r-1,
\end{aligned}$$

We extend the chirality operator Γ to an operator of determinant lines $\Gamma : \mathrm{Det}(C^\bullet) \to \mathrm{Det}(C^\bullet)$ and choose arbitrary non-zero elements $c_q \in \mathrm{Det}(C^q)$. Then we define the canonical element c_Γ by

$$\begin{aligned}
c_\Gamma \quad := \quad & (-1)^{\mathcal{R}(C^\bullet)} \cdot c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_{r-1}^{(-1)^{r-1}} \otimes\\
& (\Gamma c_{r-1})^{(-1)^r} \otimes (\Gamma c_{r-2})^{(-1)^{r-1}} \otimes \cdots\cdots \otimes (\Gamma c_1) \otimes (\Gamma c_0)^{(-1)} \in \mathrm{Det}(C^\bullet),
\end{aligned}$$

where $(-1)^{\mathcal{R}(C^\bullet)}$ is a normalization factor defined by (cf. (4-2) in [5])

$$\mathcal{R}(C^\bullet) = \frac{1}{2}\sum_{q=0}^{r-1} \dim C^q \cdot \left(\dim C^q + (-1)^{r+q}\right),$$

and $c^{-1} \in \mathrm{Det}(C^q)^{-1}$ is the unique dual element of $c \in \mathrm{Det}(C^q)$ such that $c^{-1}(c) = 1$. In fact, c_Γ does not depend on the choice of c_q's. We denote by $H^\bullet(\partial)$ the cohomology of the complex $(C^\bullet, \partial)$. Then there is a canonical isomorphism

$$\phi_{C^\bullet} : \mathrm{Det}(C^\bullet) \to \mathrm{Det}(H^\bullet(\partial)).$$

We refer to Subsection 2.4 in [5] for the definition of the isomorphism $\phi_{C^\bullet}$. The refined torsion of the pair $(C^\bullet, \Gamma)$ is the element ρ_Γ in $\mathrm{Det}(H^\bullet(\partial))$ defined by

$$\rho_\Gamma \;=\; \rho_{C^\bullet,\Gamma} \;:=\; \phi_{C^\bullet}(c_\Gamma) \in \mathrm{Det}(H^\bullet(\partial)). \tag{3.1}$$

We now go back to the de Rham complex. For $0 \le \lambda \in R$ we denote by $\Omega^q_{\mathcal{P}_{-,\mathcal{L}_0},[0,\lambda]}(M,E)$, $\Omega^q_{\mathcal{P}_{-,\mathcal{L}_0},(\lambda,\infty)}(M,E)$ the subspaces of $\Omega^q_{\mathcal{P}_{-,\mathcal{L}_0}}(M,E)$ spanned by eigenforms of $\mathcal{B}^2$ whose eigenvalues belong to $[0,\lambda]$, (λ,∞), respectively. We denote by $\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0},[0,\lambda]}$, $\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0},(\lambda,\infty)}$ the restrictions of $\mathcal{B}^2$ to $\Omega^q_{\mathcal{P}_{-,\mathcal{L}_0},[0,\lambda]}(M,E)$, $\Omega^q_{\mathcal{P}_{-,\mathcal{L}_0},(\lambda,\infty)}(M,E)$, respectively. We define $\Omega^q_{\mathcal{P}_{+,\mathcal{L}_1},[0,\lambda]}(M,E)$, $\Omega^q_{\mathcal{P}_{+,\mathcal{L}_1},(\lambda,\infty)}(M,E)$ $\mathcal{B}^2_{q,\mathcal{P}_{+,\mathcal{L}_1},[0,\lambda]}$, $\mathcal{B}^2_{q,\mathcal{P}_{+,\mathcal{L}_1},(\lambda,\infty)}$, $\Omega^{\mathrm{even}}_{\mathcal{P}_{-,\mathcal{L}_0},(\lambda,\infty)}(M,E)$, $\Omega^{\mathrm{even}}_{\mathcal{P}_{+,\mathcal{L}_1},(\lambda,\infty)}(M,E)$, $\mathcal{B}^{\mathrm{even}}_{\mathcal{P}_{-,\mathcal{L}_0},(\lambda,\infty)}$, $\mathcal{B}^{\mathrm{even}}_{\mathcal{P}_{+,\mathcal{L}_1},(\lambda,\infty)}$, in the same way. Then ∇ maps $\Omega^q_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1,[0,\lambda]}(M,E)$ to $\Omega^q_{\widetilde{\mathcal{P}}_1/\widetilde{\mathcal{P}}_0,[0,\lambda]}(M,E)$ and $\left(\Omega^\bullet_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1,[0,\lambda]}(M,E),\nabla\right)$ is a subcomplex of $\left(\Omega^\bullet_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M,E),\nabla\right)$ which computes the same cohomologies. Moreover, $\left(\Omega^\bullet_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1,[0,\lambda]}(M,E),\nabla\right)$ is a finite complex consisting of finite dimensional vector spaces. Hence we can define the refined torsion element $\rho_{\widetilde{\mathcal{P}}_0,[0,\lambda]}$ and $\rho_{\widetilde{\mathcal{P}}_1,[0,\lambda]}$ as (3.1). We finally define the refined analytic torsions $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}}(M,g^M,\nabla)$ and $\rho_{an,\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ as follows.

$$\begin{aligned}
\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}}(M,g^M,\nabla) \;&:=\; \rho_{\widetilde{\mathcal{P}}_0,[0,\lambda]} \cdot \mathrm{Det}_{\mathrm{gr}}(\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}}) \cdot e^{\frac{i\pi}{2}(\mathrm{rank}\,E)\,\eta_{\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}}}(0)}, \\
\rho_{an,\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla) \;&:=\; \rho_{\widetilde{\mathcal{P}}_1,[0,\lambda]} \cdot \mathrm{Det}_{\mathrm{gr}}(\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}) \cdot e^{\frac{i\pi}{2}(\mathrm{rank}\,E)\,\eta_{\mathcal{B}^{\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}}(0)}.
\end{aligned}$$

The refined analytic torsions $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}}(M,g^M,\nabla)$ and $\rho_{an,\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ satisfy the following properties. We refer to [13] for more details.

(1) The right hand side of the definition of $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ does not depend on the choice of λ and hence $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ is well defined.

(2) $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}}(M,g^M,\nabla)$ is an element of $\mathrm{Det}(H^\bullet_{\widetilde{\mathcal{P}}_0}(M,E))$ and $\rho_{an,\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ is an element of $\mathrm{Det}(H^\bullet_{\widetilde{\mathcal{P}}_1}(M,E))$.

(3) If all cohomologies vanish, then the above definitions coincide with the definition given in Section 2.

(4) $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ are independent of the choice of the Agmon angles and invariant of the change of metrics in the interior of M.

(5) $\rho_{an,\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}}(M,g^M,\nabla)$ may depend on the choice of metrics on the boundary Y.

4 Gluing formula for the refined analytic torsion

In this section we discuss briefly the gluing formula for the refined analytic torsion with respect to the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}$, which is shown in [14]. Let $(\widehat{M}, g^{\widehat{M}})$ be a closed Riemannian manifold of dimension $m = 2r-1$ and $\widehat{E} \to \widehat{M}$ be a flat vector bundle with a flat connection ∇. We denote by Y a hypersurface of $\widehat{M}$ such that $\widehat{M} - Y$ has two components, whose closures are denoted by M_1 and M_2, *i.e.* $\widehat{M} = M_1 \cup_Y M_2$. We assume that $g^{\widehat{M}}$ is a product metric near Y and that ∇ is a Hermitian connection. Let ∂u be the unit normal vector field on a collar neighborhood of Y such that ∂u is outward on M_1 and inward on M_2. We denote by $\mathcal{B}^{\widehat{M}}$ the odd signature operator on $\widehat{M}$ and denote by $\mathcal{B}^{M_1}$, $\mathcal{B}^{M_2}$ (E_1, E_2, g^{M_1}, g^{M_2}) the restriction of $\mathcal{B}^{\widehat{M}}$ ($\widehat{E}$, $g^{\widehat{M}}$) to M_1, M_2. We impose the boundary condition $\mathcal{P}_{+,\mathcal{L}_1}$ on M_1 and $\mathcal{P}_{-,\mathcal{L}_0}$ on M_2. In this section we assume that all cohomologies involved vanish. In particular, the tangential operator $\mathcal{B}_Y$ is invertible. Then computations using the BFK-gluing formula ([7], [16], [17]) and adiabatic limit method ([28], [29]) lead to the following result. We refer to [14] for the proof.

Theorem 4.1. *We assume that for each* $0 \leq q \leq m$, $i = 1, 2$, $H^q(\widehat{M}, \widehat{E}) = H^q(M_i, Y; E_i) = H^q(M_i; E_i) = 0$. *Then,*

$$\begin{aligned} &\frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1} \cdot q \cdot \Big(\log \operatorname{Det}(\mathcal{B}^{M_1}_{q,\widetilde{\mathcal{P}}_1})^2 + \log \operatorname{Det}(\mathcal{B}^{M_2}_{q,\widetilde{\mathcal{P}}_0})^2\Big) \\ = \; &\frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1} \cdot q \cdot \Big(\log \operatorname{Det}(\mathcal{B}^{M_1}_{q,\mathrm{abs}})^2 + \log \operatorname{Det}(\mathcal{B}^{M_2}_{q,\mathrm{rel}})^2\Big). \end{aligned}$$

We next define a one parameter family of orthogonal projections $\widetilde{P}(\theta)$ by

$$\widetilde{P}(\theta) = \Pi_{>} cos\theta + \mathcal{P}_{-} sin\theta + \frac{1}{2}(1 - cos\theta - sin\theta)\,\mathrm{Id}, \quad (0 \leq \theta \leq \frac{\pi}{2}).$$

Then $\widetilde{P}(\theta)$ is a smooth curve connecting the APS boundary condition $\Pi_{>}$ and $\mathcal{P}_{-}$. Here $\Pi_{>}$ is the orthogonal projection to the subspace spanned by positive eigenforms of $\mathcal{B}_Y$. We denote the Calderón projector for $\mathcal{B}^{M_1}$ and $\mathcal{B}^{M_2}$ by $\mathcal{C}_{M_1}$, $\mathcal{C}_{M_2}$. For $i = 1, 2$, we also denote the spectral flow for $(\mathcal{B}^{M_i}_{\widetilde{P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]}$ and Maslov index for $(\widetilde{P}(\theta),\ \mathcal{C}_{M_i})_{\theta\in[0,\frac{\pi}{2}]}$ by $\mathrm{SF}(\mathcal{B}^{M_i}_{\widetilde{P}(\theta)})_{\theta\in[0,\frac{\pi}{2}]}$ and $\mathrm{Mas}(\widetilde{P}(\theta),\ \mathcal{C}_{M_i})_{\theta\in[0,\frac{\pi}{2}]}$. We refer to [9], [15] and [20] for the definitions of the Calderón projector, the spectral flow and Maslov index. Then computations following methods in [6] yield the following result. We refer to [14] for the proof.

Theorem 4.2. *We assume that for each* $0 \leq q \leq m$ *and* $i = 1, 2$, $H^q(M_i; E|_{M_i}) = H^q(M_i, Y; E|_{M_i}) = \{0\}$. *Then :*

(1) $\eta(\mathcal{B}^{M_2}_{\mathcal{P}_-}) - \eta(\mathcal{B}^{M_2}_{\Pi_>}) \;=\; \mathrm{SF}(\mathcal{B}^{M_2}_{\widetilde{P}(\theta)})_{\theta\in[0,\frac{\pi}{2}]} \;=\; \mathrm{Mas}(\widetilde{P}(\theta),\ \mathcal{C}_{M_2})_{\theta\in[0,\frac{\pi}{2}]}$.

(2) $\eta(\mathcal{B}^{M_1}_{\mathcal{P}_+}) - \eta(\mathcal{B}^{M_1}_{\Pi_<}) \;=\; \mathrm{SF}(\mathcal{B}^{M_1}_{\mathrm{Id}-\widetilde{P}(\theta)})_{\theta\in[0,\frac{\pi}{2}]} \;=\; \mathrm{Mas}(\mathrm{Id}-\widetilde{P}(\theta),\ \mathcal{C}_{M_1})_{\theta\in[0,\frac{\pi}{2}]}$.

(3) $\mathrm{Mas}(\widetilde{P}(\theta),\ \mathcal{C}_{M_2})_{\theta\in[0,\frac{\pi}{2}]} = \mathrm{Mas}(\mathrm{Id}-\widetilde{P}(\theta),\ \mathcal{C}_{M_1})_{\theta\in[0,\frac{\pi}{2}]}$.

We assume that all cohomologies involved vanish. Then the following results are well known ([8], [6], [15]).

$$(1)\quad \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q\cdot \log\mathrm{Det}_{2\theta}(\mathcal{B}^{\widehat{M}}_q)^2 = \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q\cdot\Big(\log\mathrm{Det}_{2\theta}(\mathcal{B}^{M_1}_{q,\mathrm{abs}})^2 + \log\mathrm{Det}_{2\theta}(\mathcal{B}^{M_2}_{q,\mathrm{rel}})^2\Big).$$

$$(2)\quad \eta(\mathcal{B}^{\hat{M}}) \;=\; \eta(\mathcal{B}^{M_1}_{\Pi_<}) + \eta(\mathcal{B}^{M_2}_{\Pi_>}).$$

Theorem 4.1 and 4.2 together with the above results lead to the gluing formula of the refined analytic torsion as follows.

$$\begin{aligned}
&\log\rho_{an,\mathcal{P}_+}(M_1,g^{M_1},\nabla) + \log\rho_{an,\mathcal{P}_-}(M_2,g^{M_1},\nabla)\\
&= \log\mathrm{Det}_{\mathrm{gr}}(\mathcal{B}^{M_1}_{\mathrm{even},\mathcal{P}_+}) + \ \log\mathrm{Det}_{\mathrm{gr}}(\mathcal{B}^{M_2}_{\mathrm{even},\mathcal{P}_-})\\
&\quad + \ \frac{\pi i}{2}(\mathrm{rank}\,E)\left(\eta_{\mathcal{B}^{M_1,\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_+}}(0) + \eta_{\mathcal{B}^{M_2,\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_-}}(0)\right)\\
&= \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q\cdot\Big(\log\mathrm{Det}(\mathcal{B}^{M_1}_{q,\widetilde{\mathcal{P}}_1})^2 + \log\mathrm{Det}(\mathcal{B}^{M_2}_{q,\widetilde{\mathcal{P}}_0})^2\Big) + \ \frac{\pi i}{2}(\mathrm{rank}\,E)\cdot\\
&\quad \left(\eta_{\mathcal{B}^{M_1,\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_+}}(0) + \eta_{\mathcal{B}^{M_2,\mathrm{trivial}}_{\mathrm{even},\mathcal{P}_-}}(0)\right) - i\pi\ \left(\eta(\mathcal{B}^{M_1}_{\mathrm{even},\mathcal{P}_{+,\mathcal{L}_1}}) + \ \eta(\mathcal{B}^{M_2}_{\mathrm{even},\mathcal{P}_{-,\mathcal{L}_0}})\right)\\
&= \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q\cdot\Big(\log\mathrm{Det}(\mathcal{B}^{M_1}_{q,\mathrm{abs}})^2 + \log\mathrm{Det}(\mathcal{B}^{M_2}_{q,\mathrm{rel}})^2\Big) + \ \frac{\pi i}{2}(\mathrm{rank}\,E)\cdot\\
&\quad \left(\eta_{\mathcal{B}^{M_1,\mathrm{trivial}}_{\mathrm{even},\Pi_<}}(0) + \eta_{\mathcal{B}^{M_2,\mathrm{trivial}}_{\mathrm{even},\Pi_>}}(0)\right) - i\pi\ \left(\eta(\mathcal{B}^{M_1}_{\mathrm{even},\Pi_<}) + \ \eta(\mathcal{B}^{M_2}_{\mathrm{even},\Pi_>})\right)\\
&= \frac{1}{2}\sum_{q=0}^{m}(-1)^{q+1}\cdot q\cdot\log\mathrm{Det}(\mathcal{B}^{\widehat{M}}_q)^2 + \ \frac{\pi i}{2}(\mathrm{rank}\,E)\cdot\eta_{\mathcal{B}^{\widehat{M},\mathrm{trivial}}_{\mathrm{even}}}(0) - i\pi\ \eta(\mathcal{B}^{\widehat{M}}_{\mathrm{even}})\\
&= \log\rho_{an}(\widehat{M},g^{\widehat{M}},\nabla) \qquad (\mathrm{mod}\,2\pi i Z).
\end{aligned}$$

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