An Economic Injury Level (EIL) is a measurement of the fewest number of insect pests that will cause economic damage to a crop or forest. It has been estimated that monitoring pest populations and establishing EILs can reduce pesticide use by 30%–50%.

Accurate population estimates are crucial for determining EILs. A population density of one insect pest can be approximated by

\[ D(t) = \frac{t^2}{90} + \frac{t}{3} \]

pests per plant, where \( t \) is the number of days since initial infestation. What is the rate of change of this population density when the population density is equal to the EIL of 20 pests per plant? Section 2.4 can help answer this question.
Chapter 2 Overview

The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits is to test functions for continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior. They also have special mathematical properties, not otherwise guaranteed.

2.1 Rates of Change and Limits

What you’ll learn about
- Average and Instantaneous Speed
- Definition of Limit
- Properties of Limits
- One-sided and Two-sided Limits
- Sandwich Theorem

...and why
Limits can be used to describe continuity, the derivative, and the integral: the ideas giving the foundation of calculus.

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls free fall.

Average and Instantaneous Speed

A moving body’s average speed during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit time—kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

\[ y = 16t^2 \]

feet in the first \( t \) seconds. The average speed of the rock over any given time interval is the distance traveled, \( \Delta y \), divided by the length of the interval \( \Delta t \). For the first 2 seconds of fall, from \( t = 0 \) to \( t = 2 \), we have

\[
\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32 \text{ ft/sec}. \quad \text{Now try Exercise 1.}
\]

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant \( t = 2 \).

SOLUTION

Solve Numerically We can calculate the average speed of the rock over the interval from time \( t = 2 \) to any slightly later time \( t = 2 + h \) as

\[
\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h}.
\]

We cannot use this formula to calculate the speed at the exact instant \( t = 2 \) because that would require taking \( h = 0 \), and \( 0/0 \) is undefined. However, we can get a good idea of what is happening at \( t = 2 \) by evaluating the formula at values of \( h \) close to 0. When we do, we see a clear pattern (Table 2.1 on the next page). As \( h \) approaches 0, the average speed approaches the limiting value 64 ft/sec.

continued
Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that
\[
\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h} = \frac{64h + 16h^2}{h} = 64 + 16h.
\]
For values of \(h\) different from 0, the expressions on the right and left are equivalent and the average speed is 64 + 16h ft/sec. We can now see why the average speed has the limiting value 64 + 16(0) = 64 ft/sec as \(h\) approaches 0. Now try Exercise 3.

Definition of Limit

As in the preceding example, most limits of interest in the real world can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average speed was not defined at \(h = 0\) but approached the limit 64 as \(h\) approached 0. We were able to see this numerically and to confirm it algebraically by eliminating \(h\) from the denominator. But we cannot always do that. For instance, we can see both graphically and numerically (Figure 2.1) that the values of \(f(x) = \sin x / x\) approach 1 as \(x\) approaches 0.

We cannot eliminate the \(x\) from the denominator of \(\sin x / x\) to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 75.

DEFINITION Limit

Assume \(f\) is defined in a neighborhood of \(c\) and let \(c\) and \(L\) be real numbers. The function \(f\) has limit \(L\) as \(x\) approaches \(c\) if, given any positive number \(\varepsilon\), there is a positive number \(\delta\) such that for all \(x\),

\[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.\]

We write

\[\lim_{x \to c} f(x) = L.\]

The sentence \(\lim_{x \to c} f(x) = L\) is read, “The limit of \(f\) of \(x\) as \(x\) approaches \(c\) equals \(L\).” The notation means that the values \(f(x)\) of the function \(f\) approach or equal \(L\) as the values of \(x\) approach (but do not equal) \(c\). Appendix A3 provides practice applying the definition of limit.

We saw in Example 2 that \(\lim_{h \to 0} (64 + 16h) = 64\).

As suggested in Figure 2.1,

\[\lim_{x \to 0} \frac{\sin x}{x} = 1.\]

Figure 2.2 illustrates the fact that the existence of a limit as \(x \to c\) never depends on how the function may or may not be defined at \(c\). The function \(f\) has limit 2 as \(x \to 1\) even though \(f\) is not defined at 1. The function \(g\) has limit 2 as \(x \to 1\) even though \(g(1) \neq 2\). The function \(h\) is the only one whose limit as \(x \to 1\) equals its value at \(x = 1\).
THEOREM 1

Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

\[
\lim_{x \to c} k = k \quad \text{Limit of the function with constant value } k
\]

and

\[
\lim_{x \to c} x = c, \quad \text{Limit of the identity function at } x = c
\]

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

THEOREM 1 Properties of Limits

If \( L, M, c, \) and \( k \) are real numbers and

\[
\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M,
\]

then

1. **Sum Rule:**

\[
\lim_{x \to c} (f(x) + g(x)) = L + M
\]

The limit of the sum of two functions is the sum of their limits.

2. **Difference Rule:**

\[
\lim_{x \to c} (f(x) - g(x)) = L - M
\]

The limit of the difference of two functions is the difference of their limits.

3. **Product Rule:**

\[
\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M
\]

The limit of a product of two functions is the product of their limits.

4. **Constant Multiple Rule:**

\[
\lim_{x \to c} (k \cdot f(x)) = k \cdot L
\]

The limit of a constant times a function is the constant times the limit of the function.

5. **Quotient Rule:**

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0
\]

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

continued
Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

**EXAMPLE 3 Using Properties of Limits**

Use the observations \( \lim_{x \to c} k = k \) and \( \lim_{x \to c} x = c \), and the properties of limits to find the following limits.

(a) \( \lim_{x \to c} (x^3 + 4x^2 - 3) \)  
(b) \( \lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} \)

**SOLUTION**

(a) \( \lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = c^3 + 4c^2 - 3 \)  
(b) \( \lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)} = \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} = \frac{c^4 + c^2 - 1}{c^2 + 5} \)

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that \( \lim_{x \to c} k = k \) and \( \lim_{x \to c} x = c \), we can immediately work our way to limits of polynomial functions and most rational functions using substitution.

**THEOREM 2 Polynomial and Rational Functions**

1. If \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) is any polynomial function and \( c \) is any real number, then  
   \[ \lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0. \]

2. If \( f(x) \) and \( g(x) \) are polynomials and \( c \) is any real number, then  
   \[ \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0. \]
**EXAMPLE 4 Using Theorem 2**

(a) \( \lim_{x \to 3} \frac{x^2(2 - x)}{x^2} = (3)^2(2 - 3) = -9 \)

(b) \( \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{2 + 2} = \frac{12}{4} = 3 \)

*Now try Exercises 9 and 11.*

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.

**EXAMPLE 5 Using the Product Rule**

Determine \( \lim_{x \to 0} \frac{\tan x}{x} \).

**SOLUTION**

**Solve Graphically** The graph of \( f(x) = \frac{\tan x}{x} \) in Figure 2.3 suggests that the limit exists and is about 1.

**Confirm Analytically** Using the analytic result of Exercise 75, we have

\[
\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1.
\]

*Now try Exercise 27.*

Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

**EXAMPLE 6 Exploring a Nonexistent Limit**

Use a graph to show that

\( \lim_{x \to 2} \frac{x^3 - 1}{x - 2} \)

does not exist.

**SOLUTION**

Notice that the denominator is 0 when \( x \) is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 2.4 of \( f(x) = \frac{(x^3 - 1)}{(x - 2)} \) strongly suggests that as \( x \to 2 \) from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

*Now try Exercise 29.*

**One-sided and Two-sided Limits**

Sometimes the values of a function \( f \) tend to different limits as \( x \) approaches a number \( c \) from opposite sides. When this happens, we call the limit of \( f \) as \( x \) approaches \( c \) from the
right the **right-hand limit** of \( f \) at \( c \) and the limit as \( x \) approaches \( c \) from the left the **left-hand limit** of \( f \) at \( c \). Here is the notation we use:

right-hand: \( \lim_{x \to c^+} f(x) \)  
left-hand: \( \lim_{x \to c^-} f(x) \)

**Theorem 3** One-sided and Two-sided Limits

A function \( f(x) \) has a limit as \( x \) approaches \( c \) if and only if the right-hand and left-hand limits at \( c \) exist and are equal. In symbols,

\[
\lim_{x \to c^-} L = \lim_{x \to c^+} L  
\]

Thus, the greatest integer function \( f(x) = \text{int } x \) of Example 7 does not have a limit as \( x \to 3 \) even though each one-sided limit exists.

**Example 8 Exploring Right- and Left-Hand Limits**

All the following statements about the function \( y = f(x) \) graphed in Figure 2.6 are true.

At \( x = 0 \): \( \lim_{x \to 0^+} f(x) = 1 \).

At \( x = 1 \): \( \lim_{x \to 1^-} f(x) = 0 \) even though \( f(1) = 1 \),

\[
\lim_{x \to 1^-} f(x) = 1,
\]

\( f \) has no limit as \( x \to 1 \). (The right- and left-hand limits at 1 are not equal, so \( \lim_{x \to 1} f(x) \) does not exist.)

At \( x = 2 \): \( \lim_{x \to 2^-} f(x) = 1, \)

\( \lim_{x \to 2^-} f(x) = 1 \) even though \( f(2) = 2 \).

At \( x = 3 \): \( \lim_{x \to 3^-} f(x) = 2 = \lim_{x \to 3} f(x) = \lim_{x \to 3^+} f(x) \).

At \( x = 4 \): \( \lim_{x \to 4^-} f(x) = 1 \).

At noninteger values of \( c \) between 0 and 4, \( f \) has a limit as \( x \to c \).

*Now try Exercise 37.*
Sandwich Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function \( f \) whose values are sandwiched between the values of two other functions, \( g \) and \( h \). If \( g \) and \( h \) have the same limit as \( x \to c \), then \( f \) has that limit too, as suggested by Figure 2.7.

**THEOREM 4 The Sandwich Theorem**

If \( g(x) \leq f(x) \leq h(x) \) for all \( x \neq c \) in some interval about \( c \), and

\[
\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,
\]

then

\[
\lim_{x \to c} f(x) = L.
\]

**EXAMPLE 9 Using the Sandwich Theorem**

Show that \( \lim_{x \to 0} [x^2 \sin(1/x)] = 0 \).

**SOLUTION**

We know that the values of the sine function lie between \(-1\) and \(1\). So, it follows that

\[
x^2 \sin \left( \frac{1}{x} \right) = |x^2| \cdot \sin \left( \frac{1}{x} \right) \leq |x^2| \cdot 1 = x^2
\]

and

\[-x^2 \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2.
\]

Because \( \lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0 \), the Sandwich Theorem gives

\[
\lim_{x \to 0} \left( x^2 \sin \left( \frac{1}{x} \right) \right) = 0.
\]

The graphs in Figure 2.8 support this result.

**Quick Review 2.1** *(For help, go to Section 1.2.)*

In Exercises 1–4, find \( f(2) \).

1. \( f(x) = 2x^3 - 5x^2 + 4 \quad 0 \)
2. \( f(x) = \frac{4x^2 - 5}{x^3 + 4} \quad \frac{11}{12} \)
3. \( f(x) = \sin \left( \frac{\pi x}{2} \right) \quad 0 \)
4. \( f(x) = \begin{cases} 1, & x < 2 \\ \frac{1}{x^2 - 1}, & x \geq 2 \end{cases} \quad \frac{1}{3} \)

In Exercises 5–8, write the inequality in the form \( a < x < b \).

5. \( x < 4 \quad -4 < x < 4 \)
6. \( x < c^2 \quad -c^2 < x < c^2 \)
7. \( |x - 2| < 3 \quad -1 < x < 5 \)
8. \( |x - c| < d^2 \quad c - d^2 < x < c + d^2 \)

In Exercises 9 and 10, write the fraction in reduced form.

9. \( \frac{x^2 - 3x - 18}{x + 3} \quad \frac{x - 6}{x + 1} \)
10. \( \frac{2x^2 - x}{2x^2 + x - 1} \quad \frac{x}{x + 1} \)
Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first $t$ seconds.

1. Find the average speed during the first 3 seconds of fall. $48 \text{ ft/sec}$
2. Find the average speed during the first 4 seconds of fall. $64 \text{ ft/sec}$
3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically. $96 \text{ ft/sec}$
4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically. $128 \text{ ft/sec}$

In Exercises 5 and 6, use $\lim_{x \to c} k = k$, $\lim_{x \to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x \to 2} (2x^3 - 3x^2 + x - 1) = 2c^3 - 3c^2 + c - 1$
6. $\lim_{x \to 2} \frac{x^4 - x^3 + 1}{x^3 + 9} = \frac{c^4 - c^3 + 1}{c^3 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x \to -1/2} 3x^2(2x - 1) = -\frac{3}{2}$
8. $\lim_{x \to 4} (x + 3)^{1998} = 1$
9. $\lim_{x \to 1} (3x^3 + 3x^2 - 2x - 17) = -15$
10. $\lim_{x \to 2} \frac{y^2 + 5y + 6}{y + 2} = 5$
11. $\lim_{x \to 3} \frac{y^2 + 4y + 3}{y^2 - 3} = 0$
12. $\lim_{x \to 1/2} \int x = 0$
13. $\lim_{x \to 2} (x - 6)^{2/3} = 4$
14. $\lim_{x \to 2} \sqrt{x + 3} = \sqrt{5}$

In Exercises 15–18, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

15. $\lim_{x \to 2} \sqrt{x - 2}$ Expression not defined at $x = 2$. There is no limit.
16. $\lim_{x \to 0} \frac{1}{x^2}$ Expression not defined at $x = 0$. There is no limit.
17. $\lim_{x \to 0} \frac{|x|}{x}$ Expression not defined at $x = 0$. There is no limit.
18. $\lim_{x \to 0} \frac{(4 + x)^2 - 16}{x}$ Limit = 8.

In Exercises 19–28, determine the limit graphically. Confirm algebraically.

19. $\lim_{x \to 0} \frac{x - 1}{x^2 - 1} = \frac{1}{2}$
20. $\lim_{t \to 2} \frac{t^2 - 3t + 2}{t^2 - 4} = \frac{1}{4}$
21. $\lim_{x \to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \frac{1}{2}$
22. $\lim_{x \to 0} \frac{2 + x}{x} = 2$
23. $\lim_{x \to 0} \frac{(2 + x)^3 - 8}{x} = 12$
24. $\lim_{x \to 0} \frac{\sin x}{2x^2 - x} = -1$
25. $\lim_{x \to 0} \frac{x}{\sin x} = 1$
26. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
27. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
28. $\lim_{x \to 0} \frac{3 \sin x}{3x} = 4$

In Exercises 29 and 30, use a graph to show that the limit does not exist.

29. $\lim_{x \to -1} \frac{x^2 - 4}{x - 1}$
30. $\lim_{x \to 2} \frac{x + 1}{x^2 - 4}$

In Exercises 31–36, determine the limit.

31. $\lim_{x \to 0} \frac{\sin x}{x} = 0$
32. $\lim_{x \to 0} \frac{x}{x - 1} = 1$
33. $\lim_{x \to 0} \frac{\sin x}{x} = 0$
34. $\lim_{x \to 1} \frac{x}{x - 1} = 1$
35. $\lim_{x \to 0} \frac{x}{|x|} = 1$
36. $\lim_{x \to 0} \frac{x}{|x|} = -1$

In Exercises 37 and 38, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

37. (a) $\lim_{x \to -1} f(x) = 1$ True (b) $\lim_{x \to 0} f(x) = 0$ True
(c) $\lim_{x \to 0} f(x) = 1$ False (d) $\lim_{x \to 0} f(x)$ does not exist. False
(e) $\lim_{x \to 0} f(x)$ exists True (f) $\lim_{x \to 0} f(x) = 0$ True
(g) $\lim_{x \to 0} f(x) = 1$ False (h) $\lim_{x \to 0} f(x) = 1$ False
(i) $\lim_{x \to 0} f(x) = 0$ False (j) $\lim_{x \to 0} f(x) = 2$ False

38. (a) $\lim_{x \to 0^+} f(x) = 1$ True (b) $\lim_{x \to 0^-} f(x)$ does not exist. False
(c) $\lim_{x \to 0^+} f(x) = 2$ False (d) $\lim_{x \to 0^-} f(x) = 2$ True
(e) $\lim_{x \to 0} f(x) = 1$ True (f) $\lim_{x \to 0} f(x)$ does not exist. True
(g) $\lim_{x \to 0} f(x)$ exists True
(h) $\lim_{x \to 0} f(x)$ exists at every $c$ in $(-1, 1)$. True
(i) $\lim_{x \to 0} f(x)$ exists at every $c$ in $(1, 3)$. True

29. Answers will vary. One possible graph is given by the window $[-4.7, 4.7]$ by $[-15, 15]$ with Xscl = 1 and Yscl = 5.
30. Answers will vary. One possible graph is given by the window $[-4.7, 4.7]$ by $[-15, 15]$ with Xscl = 1 and Yscl = 5.
In Exercises 39–44, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

39. (a) \( \lim_{x \to 3} f(x) \) 3
   (b) \( \lim_{x \to 3} f(x) \) -2
   (c) \( \lim_{x \to 3} f(x) \) No limit
   (d) \( f(3) \) 1

40. (a) \( \lim_{t \to -4} g(t) \) 5
   (b) \( \lim_{t \to -4} g(t) \) 2
   (c) \( \lim_{t \to -4} g(t) \) No limit
   (d) \( g(-4) \) 2

41. (a) \( \lim_{h \to 0} f(h) \) -4
   (b) \( \lim_{h \to 0} f(h) \) -4
   (c) \( \lim_{h \to 0} f(h) \) -4
   (d) \( f(0) \) -4

42. (a) \( \lim_{x \to -2} p(x) \) 3
   (b) \( \lim_{x \to -2} p(x) \) 3
   (c) \( \lim_{x \to -2} p(x) \) 3
   (d) \( p(-2) \) 3

43. (a) \( \lim_{x \to 0} F(x) \) 4
   (b) \( \lim_{x \to 0} F(x) \) -3
   (c) \( \lim_{x \to 0} F(x) \) No limit
   (d) \( F(0) \) 4

44. (a) \( \lim_{x \to 2} G(x) \) 1
   (b) \( \lim_{x \to 2} G(x) \) 1
   (c) \( \lim_{x \to 2} G(x) \) 1
   (d) \( G(2) \) 3

In Exercises 45–48, match the function with the table.

45. \( y_1 = \frac{x^2 + x - 2}{x - 1} \) (c)
46. \( y_1 = \frac{x^2 - x - 2}{x - 1} \) (b)
47. \( y_1 = \frac{x^2 - 2x + 1}{x - 1} \) (d)
48. \( y_1 = \frac{x^2 + x - 2}{x + 1} \) (a)

\[
\begin{array}{c|c|c}
X & Y_1 & Y_2 \\
\hline
.7 & 2.7 & 7.3667 \\
.6 & 2.8 & 7.3667 \\
.9 & 2.9 & 1.826 \\
1 & 3.1 & 1.826 \\
1.1 & 3.2 & 1.826 \\
1.2 & 3.3 & 1.826 \\
1.3 & 3.4 & 1.826 \\
\end{array}
\]

49. Assume that \( \lim_{x \to 4} f(x) = 0 \) and \( \lim_{x \to 4} g(x) = 3 \).
   (a) \( \lim_{x \to 4} (g(x) + 3) \) 6
   (b) \( \lim_{x \to 4} x f(x) \) 0
   (c) \( \lim_{x \to 4} \frac{g^2(x)}{g(x)} \) 9
   (d) \( \lim_{x \to 4} \frac{g(x)}{f(x) - 1} \) -3

50. Assume that \( \lim_{x \to 0} f(x) = 7 \) and \( \lim_{x \to 0} g(x) = -3 \).
   (a) \( \lim_{x \to 0} (f(x) + g(x)) \) 4
   (b) \( \lim_{x \to 0} (f(x) \cdot g(x)) \) -21
   (c) \( \lim_{x \to 0} g(x) \) -12
   (d) \( \lim_{x \to 0} \frac{f(x)}{g(x)} \) 7

In Exercises 51–54, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of \( f \).
(b) Determine \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \).
(c) **Writing to Learn** Does \( \lim_{x \to c} f(x) \) exist? If so, what is it? If not, explain.

51. \( c = 2, f(x) = \begin{cases} 
3 - x, & x < 2 \\
\frac{x}{2} + 1, & x \geq 2 
\end{cases} \)
   (b) Right-hand: 2  Left-hand: 1
   (c) No, because the two one-sided limits are different.

52. \( c = 2, f(x) = \begin{cases} 
3 - x, & x < 2 \\
x/2, & x \geq 2 
\end{cases} \)
   (b) Right-hand: 1  Left-hand: 1
   (c) Yes. The limit is 1.

53. \( c = 1, f(x) = \begin{cases} 
1 - x, & x < 1 \\
x^3 - 2x + 5, & x \geq 1 
\end{cases} \)
   (b) Right-hand: 4  Left-hand: no limit
   (c) No, because the left-hand limit doesn’t exist.

54. \( c = -1, f(x) = \begin{cases} 
1 - x^2, & x \neq -1 \\
2, & x = -1 
\end{cases} \)
   (b) Right-hand: 0  Left-hand: 0
   (c) Yes, the limit is 0.
In Exercises 55–58, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of .
(b) At what points c in the domain of f does \( \lim_{x \to c} f(x) \) exist?
(c) At what points c does only the left-hand limit exist?
(d) At what points c does only the right-hand limit exist?

55. \( f(x) = \begin{cases} \sin x, & -2\pi \leq x < 0 \\ \cos x, & 0 \leq x \leq 2\pi \\ \end{cases} \)

56. \( f(x) = \begin{cases} \cos x, & -\pi \leq x < 0 \\ \sec x, & 0 \leq x \leq \pi \\ \end{cases} \)

57. \( f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \\ \end{cases} \)

58. \( f(x) = \begin{cases} x, & -1 \leq x < 0, \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ (b) (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \\ 0, & x < -1, \text{ or } x > 1 \\ \end{cases} \)

In Exercises 59–62, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

59. \( \lim_{x \to 0} x \sin x = 0 \)

60. \( \lim_{x \to 0} x^2 \sin x = 0 \)

61. \( \lim_{x \to 0} x^2 \sin \frac{1}{x^2} = 0 \)

62. \( \lim_{x \to 0} x^2 \cos \frac{1}{x^2} = 0 \)

63. **Free Fall** A water balloon dropped from a window high above the ground falls \( y = 4.9t^2 \) m in \( t \) sec. Find the balloon’s

(a) average speed during the first 3 sec of fall. 14.7 m/sec
(b) speed at the instant \( t = 3 \). 29.4 m/sec

64. **Free Fall on a Small Airless Planet** A rock released from rest to fall on a small airless planet falls \( y = gt^2 \) m in \( t \) sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g. \( \frac{3}{4} \)
(b) Find the average speed for the fall. 5 m/sec
(c) With what speed did the rock hit the bottom? 10 m/sec

66. True. \( \lim_{x \to 0} \left( \frac{x + \sin x}{x} \right) = \lim_{x \to 0} \left( 1 + \frac{\sin x}{x} \right) = 1 + \lim_{x \to 0} \frac{\sin x}{x} = 2 \)

68. **Multiple Choice** What is the value of \( \lim_{x \to 1} f(x) \)?
   - (A) 5/2
   - (B) 3/2
   - (C) 1
   - (D) 0
   - (E) does not exist

69. **Multiple Choice** What is the value of \( \lim_{x \to 1} f(x) \)?
   - (A) 5/2
   - (B) 3/2
   - (C) 1
   - (D) 0
   - (E) does not exist

70. **Multiple Choice** What is the value of \( f(1) \)?
   - (A) 5/2
   - (B) 3/2
   - (C) 1
   - (D) 0
   - (E) does not exist

**Explorations**
In Exercises 71–74, complete the following tables and state what you believe \( \lim_{x \to 0} f(x) \) to be.

(a) \[
\begin{array}{cccc}
   x & -0.1 & -0.01 & -0.001 & -0.0001 & \ldots \\
\end{array}
\]

(b) \[
\begin{array}{cccc}
   x & 0.1 & 0.01 & 0.001 & 0.0001 & \ldots \\
\end{array}
\]

71. \( f(x) = x \sin \frac{1}{x} \)

72. \( f(x) = \sin \frac{1}{x} \)

73. \( f(x) = \frac{10^x - 1}{x} \)

74. \( f(x) = x \sin (\ln |x|) \)

75. **Group Activity** To prove that \( \lim_{\theta \to 0} (\sin \theta)/\theta = 1 \) when \( \theta \) is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

(a) To show that the right-hand limit is 1, explain why we can restrict our attention to \( 0 < \theta < \pi/2 \). Because the right-hand limit at zero depends only on the values of the function for positive \( x \)-values near zero.

(b) Use the figure to show that

\[
\text{area of } \triangle OAP = \frac{1}{2} \sin \theta, \\
\text{area of sector } OAP = \frac{\theta}{2}, \\
\text{area of } \triangle OAT = \frac{1}{2} \tan \theta. \\
\]

Use: area of triangle = \( \frac{1}{2} \)(base)(height)

area of circular sector = \( \frac{1}{2} \)(angle)(radius)^2

(c) Use part (b) and the figure to show that for \( 0 < \theta < \pi/2 \),

\[
\frac{1}{2} \sin \theta < \frac{\theta}{2} < \frac{1}{2} \tan \theta.
\]

This is how the areas of the three regions compare.
(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form
\[
1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.
\]
Multiply by 2 and divide by $\sin \theta$.

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form
\[
\cos \theta < \frac{\sin \theta}{\theta} < 1.
\]
Take reciprocals, remembering that all of the values involved are positive.

(f) Use the Sandwich Theorem to show that
\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.
\]

(g) Show that $(\sin \theta)/\theta$ is an even function.

(h) Use part (g) to show that
\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.
\]

(i) Finally, show that
\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.
\] The two one-sided limits both exist and are equal to 1.

75. (f) The limits for $\cos \theta$ and 1 are both equal to 1. Since $\frac{\sin \theta}{\theta}$ is between them, it must also have a limit of 1.

(g) $\frac{\sin (-\theta)}{-\theta} = -\frac{\sin \theta}{\theta} = \frac{\sin \theta}{\theta}$

(h) If the function is symmetric about the $y$-axis, and the right-hand limit at zero is 1, then the left-hand limit at zero must also be 1.

76. **Controlling Outputs** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x \to 2} f(x) = 2 = f(2)$. The limit can be found by substitution.

(b) Use a graph to estimate values for $a$ and $b$ so that $1.8 < f(x) < 2.2$ provided $a < x < b$. One possible answer: $a = 1.75, b = 2.28$

(c) Use a graph to estimate values for $a$ and $b$ so that $1.99 < f(x) < 2.01$ provided $a < x < b$. One possible answer: $a = 1.99, b = 2.01$

77. **Controlling Outputs** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$. $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

(b) Use a graph to estimate an interval $(a, b)$ about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$. One possible answer: $a = 0.305, b = 0.775$

(c) Use a graph to estimate an interval $(a, b)$ about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$. One possible answer: $a = 0.513, b = 0.535$

78. **Limits and Geometry** Let $P(a, a^2)$ be a point on the parabola $y = x^2, a > 0$. Let $O$ be the origin and $(0, b)$ the $y$-intercept of the perpendicular bisector of line segment $OP$. Find $\lim_{P \to O} b$. 

\[
\frac{1}{2}
\]
2.2 Limits Involving Infinity

What you’ll learn about

• Finite Limits as \( x \to \pm \infty \)
• Sandwich Theorem Revisited
• Infinite Limits as \( x \to a \)
• End Behavior Models
• “Seeing” Limits as \( x \to \pm \infty \)

... and why

Limits can be used to describe the behavior of functions for numbers large in absolute value.

![Graph of \( f(x) = \frac{x}{\sqrt{x^2 + 1}} \)]

\( [-10, 10] \) by \([-1.5, 1.5] \)

(a)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.7071</td>
</tr>
<tr>
<td>2</td>
<td>0.8944</td>
</tr>
<tr>
<td>3</td>
<td>0.9487</td>
</tr>
<tr>
<td>4</td>
<td>0.9701</td>
</tr>
<tr>
<td>5</td>
<td>0.9806</td>
</tr>
<tr>
<td>6</td>
<td>0.9864</td>
</tr>
</tbody>
</table>

\( Y = \frac{x}{\sqrt{x^2 + 1}} \)

(b)

![Graph of \( f(x) = 1/x \)]

\([-6, 6]\) by \([-4, 4] \)

\( f(x) = 1/x \) for \( x \to 0 \)

**DEFINITION** Horizontal Asymptote

The line \( y = b \) is a **horizontal asymptote** of the graph of a function \( y = f(x) \) if either

\[
\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.
\]

The graph of \( f(x) = 2 + \frac{1}{x} \) has the single horizontal asymptote \( y = 2 \) because

\[
\lim_{x \to \infty} 2 + \frac{1}{x} = 2 \quad \text{and} \quad \lim_{x \to -\infty} 2 + \frac{1}{x} = 2.
\]

A function can have more than one horizontal asymptote, as Example 1 demonstrates.

**EXAMPLE 1 Looking for Horizontal Asymptotes**

Use graphs and tables to find \( \lim_{x \to \infty} f(x) \), \( \lim_{x \to -\infty} f(x) \), and identify all horizontal asymptotes of \( f(x) = \frac{x}{\sqrt{x^2 + 1}} \).

**SOLUTION**

**Solve Graphically** Figure 2.10a shows the graph for \(-10 \leq x \leq 10\). The graph climbs rapidly toward the line \( y = 1 \) as \( x \) moves away from the origin to the right. On our calculator screen, the graph soon becomes indistinguishable from the line. Thus \( \lim_{x \to \infty} f(x) = 1 \). Similarly, as \( x \) moves away from the origin to the left, the graph drops rapidly toward the line \( y = -1 \) and soon appears to overlap the line. Thus \( \lim_{x \to -\infty} f(x) = -1 \). The horizontal asymptotes are \( y = 1 \) and \( y = -1 \).
Confirm Numerically The table in Figure 2.10b confirms the rapid approach of $f(x)$ toward 1 as $x \to \infty$. Since $f$ is an odd function of $x$, we can expect its values to approach $-1$ in a similar way as $x \to -\infty$.

Now try Exercise 5.

Sandwich Theorem Revisited

The Sandwich Theorem also holds for limits as $x \to \pm \infty$.

EXAMPLE 2 Finding a Limit as $x$ Approaches Infinity

Find $\lim_{x \to \infty} f(x)$ for $f(x) = \frac{\sin x}{x}$.

SOLUTION

Solve Graphically and Numerically The graph and table of values in Figure 2.11 suggest that $y = 0$ is the horizontal asymptote of $f$.

Confirm Analytically We know that $-1 \leq \sin x \leq 1$. So, for $x > 0$ we have

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}.$$  

Therefore, by the Sandwich Theorem,

$$0 = \lim_{x \to \infty} \left( -\frac{1}{x} \right) = \lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0.$$  

Since $(\sin x)/x$ is an even function of $x$, we can also conclude that

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$  

Now try Exercise 9.

Limits at infinity have properties similar to those of finite limits.

THEOREM 5 Properties of Limits as $x \to \pm \infty$

If $L$, $M$, and $k$ are real numbers and

$$\lim_{x \to \pm \infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm \infty} g(x) = M,$$

then

1. Sum Rule: $\lim_{x \to \pm \infty} (f(x) + g(x)) = L + M$

2. Difference Rule: $\lim_{x \to \pm \infty} (f(x) - g(x)) = L - M$

3. Product Rule: $\lim_{x \to \pm \infty} (f(x) \cdot g(x)) = L \cdot M$

4. Constant Multiple Rule: $\lim_{x \to \pm \infty} (k \cdot f(x)) = k \cdot L$

5. Quotient Rule: $\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

6. Power Rule: If $r$ and $s$ are integers, $s \neq 0$, then

$$\lim_{x \to \pm \infty} (f(x))^{rs} = L^{rs}$$

provided that $L^{rs}$ is a real number.
We can use Theorem 5 to find limits at infinity of functions with complicated expressions, as illustrated in Example 3.

**Example 3 Using Theorem 5**

Find \( \lim_{x \to \infty} \frac{5x + \sin x}{x} \).

**Solution**

Notice that \( \frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x} \).

So,

\[
\lim_{x \to \infty} \frac{5x + \sin x}{x} = \lim_{x \to \infty} 5 + \lim_{x \to \infty} \frac{\sin x}{x} \quad \text{(Sum Rule)}
\]

\[
= 5 + 0 = 5. \quad \text{(Known Values)}
\]

Now try Exercise 25.

**Exploration 1 Exploring Theorem 5**

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let \( f(x) = 5x + \sin x \) and \( g(x) = x \). Do the limits as \( x \to \infty \) of \( f \) and \( g \) exist? Can we apply the Quotient Rule to \( \lim_{x \to \infty} f(x)/g(x) \)? Explain. Does the limit of the quotient exist?

2. Let \( f(x) = \sin^2 x \) and \( g(x) = \cos^2 x \). Describe the behavior of \( f \) and \( g \) as \( x \to \infty \). Can we apply the Sum Rule to \( \lim_{x \to \infty} (f(x) + g(x)) \)? Explain. Does the limit of the sum exist?

3. Let \( f(x) = \ln (2x) \) and \( g(x) = \ln (x + 1) \). Find the limits as \( x \to \infty \) of \( f \) and \( g \). Can we apply the Difference Rule to \( \lim_{x \to \infty} (f(x) - g(x)) \)? Explain. Does the limit of the difference exist?

4. Based on parts 1–3, what advice might you give about applying Theorem 5?

**Infinite Limits as \( x \to a \)**

If the values of a function \( f(x) \) outgrow all positive bounds as \( x \) approaches a finite number \( a \), we say that \( \lim_{x \to a} f(x) = \infty \). If the values of \( f \) become large and negative, exceeding all negative bounds as \( x \to a \), we say that \( \lim_{x \to a} f(x) = -\infty \).

Looking at \( f(x) = 1/x \) (Figure 2.9, page 70), we observe that

\[
\lim_{x \to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x \to 0^-} 1/x = -\infty.
\]

We say that the line \( x = 0 \) is a *vertical asymptote* of the graph of \( f \).

**Definition Vertical Asymptote**

The line \( x = a \) is a *vertical asymptote* of the graph of a function \( y = f(x) \) if either

\[
\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.
\]
EXAMPLE 4  Finding Vertical Asymptotes

Find the vertical asymptotes of \( f(x) = \frac{1}{x^2} \). Describe the behavior to the left and right of each vertical asymptote.

**SOLUTION**

The values of the function approach \( \infty \) on either side of \( x = 0 \).

\[
\lim_{x \to 0^-} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x^2} = \infty.
\]

The line \( x = 0 \) is the only vertical asymptote.  

Now try Exercise 27.

We can also say that \( \lim_{x \to 0}(1/x^2) = \infty \). We can make no such statement about \( 1/x \).

EXAMPLE 5  Finding Vertical Asymptotes

The graph of \( f(x) = \tan x = (\sin x)/(\cos x) \) has infinitely many vertical asymptotes, one at each point where the cosine is zero. If \( a \) is an odd multiple of \( \pi/2 \), then

\[
\lim_{x \to a^-} \tan x = -\infty \quad \text{and} \quad \lim_{x \to a^+} \tan x = \infty,
\]

as suggested by Figure 2.12.

Now try Exercise 31.

You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that \( \lim_{x \to 0} (\sin x)/x = 1 \).

End Behavior Models

For numerically large values of \( x \), we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

EXAMPLE 6  Modeling Functions For \( |x| \) Large

Let \( f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6 \) and \( g(x) = 3x^4 \). Show that while \( f \) and \( g \) are quite different for numerically small values of \( x \), they are virtually identical for \( |x| \) large.

**SOLUTION**

Solve Graphically   The graphs of \( f \) and \( g \) (Figure 2.13a), quite different near the origin, are virtually identical on a larger scale (Figure 2.13b).

Confirm Analytically   We can test the claim that \( g \) models \( f \) for numerically large values of \( x \) by examining the ratio of the two functions as \( x \to \pm \infty \). We find that

\[
\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = \lim_{x \to \pm \infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4}
\]

\[
= \lim_{x \to \pm \infty} \left( 1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4} \right)
\]

\[
= 1,
\]

convincing evidence that \( f \) and \( g \) behave alike for \( |x| \) large.  

Now try Exercise 39.
If one function provides both a left and right end behavior model, it is simply called an end behavior model. Thus, \( g(x) = \frac{3x^4}{3x^2 - 5x + 6} \) is an end behavior model for \( f(x) = \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^2 - 5x + 6} \) (Example 6).

In general, \( g(x) = a_n x^n \) is an end behavior model for the polynomial function \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, a_n \neq 0 \). Overall, the end behavior of all polynomials behave like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

**EXAMPLE 7 Finding End Behavior Models**

Find an end behavior model for

(a) \( f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7} \)

(b) \( g(x) = \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5} \)

**SOLUTION**

(a) Notice that \( 2x^5 \) is an end behavior model for the numerator of \( f \), and \( 3x^2 \) is one for the denominator. This makes

\[
\frac{2x^5}{3x^2} = \frac{2}{3} x^3
\]

an end behavior model for \( f \).

(b) Similarly, \( 2x^3 \) is an end behavior model for the numerator of \( g \), and \( 5x^3 \) is one for the denominator of \( g \). This makes

\[
\frac{2x^3}{5x^3} = \frac{2}{5}
\]

an end behavior model for \( g \).

Notice in Example 7b that the end behavior model for \( g \), \( y = 2/5 \), is also a horizontal asymptote of the graph of \( g \), while in 7a, the graph of \( f \) does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function’s right and left end behavior models need not be the same function.

**EXAMPLE 8 Finding End Behavior Models**

Let \( f(x) = x + e^{-x} \). Show that \( g(x) = x \) is a right end behavior model for \( f \) while \( h(x) = e^{-x} \) is a left end behavior model for \( f \).

**SOLUTION**

On the right,

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x + e^{-x}}{x} = \lim_{x \to \infty} \left( 1 + \frac{e^{-x}}{x} \right) = 1 \text{ because } \lim_{x \to \infty} \frac{e^{-x}}{x} = 0.
\]

continued
Section 2.2 Limits Involving Infinity

On the left,
\[
\lim_{x \to \infty} \frac{f(x)}{h(x)} = \lim_{x \to \infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x \to \infty} \left( \frac{x}{e^{-x}} + 1 \right) = 1 \text{ because } \lim_{x \to \infty} \frac{x}{e^{-x}} = 0.
\]

The graph of \( f(x) \) in Figure 2.14 supports these end behavior conclusions.

Now try Exercise 45.

“Seeing” Limits as \( x \to \pm \infty \)

We can investigate the graph of \( y = f(x) \) as \( x \to \pm \infty \) by investigating the graph of \( y = f(1/x) \) as \( x \to 0 \).

EXAMPLE 9 Using Substitution

Find \( \lim_{x \to \infty} \sin (1/x) \).

SOLUTION

Figure 2.15a suggests that the limit is 0. Indeed, replacing \( \lim_{x \to \infty} \sin (1/x) \) by the equivalent \( \lim_{x \to 0^+} \sin x = 0 \) (Figure 2.15b), we find

\[
\lim_{x \to \infty} \sin 1/x = \lim_{x \to 0^+} \sin x = 0.
\]

Now try Exercise 49.

Figure 2.15 The graphs of (a) \( f(x) = \sin (1/x) \) and (b) \( g(x) = f(1/x) = \sin x \). (Example 9)

Quick Review 2.2 (For help, go to Section 1.2 and 1.5.)

In Exercises 1–4, find \( f^{-1} \) and graph \( f, f^{-1}, \) and \( y = x \) in the same square viewing window.

1. \( f(x) = 2x - 3 \) \hspace{1cm} 2. \( f(x) = e^x \) \hspace{1cm} 3. \( f(x) = \tan^{-1} x \) \hspace{1cm} 4. \( f(x) = \cot^{-1} x \)

In Exercises 5 and 6, find the quotient \( q(x) \) and remainder \( r(x) \) when \( f(x) \) is divided by \( g(x) \).

5. \( f(x) = 2x^3 - 3x^2 + x - 1, \hspace{1cm} g(x) = 3x^3 + 4x - 5 \)

6. \( f(x) = 2x^2 - x^3 + x - 1, \hspace{1cm} g(x) = x^3 - x^2 + 1 \)

In Exercises 7–10, write a formula for (a) \( f(-x) \) and (b) \( f(1/x) \). Simplify where possible.

7. \( f(x) = \cos x \) \hspace{1cm} (a) \( f(-x) = \cos x \) \hspace{1cm} (b) \( f(1/x) = \cos (1/x) \)

8. \( f(x) = e^{-x} \) \hspace{1cm} (a) \( f(-x) = e^x \) \hspace{1cm} (b) \( f(1/x) = e^{-1/x} \)

9. \( f(x) = \ln x \) \hspace{1cm} (a) \( f(-x) = -\ln (-x) \) \hspace{1cm} (b) \( f(1/x) = -x \ln x \)

10. \( f(x) = \left(x + \frac{1}{x}\right) \sin x \) \hspace{1cm} (a) \( f(-x) = \left(x + \frac{1}{x}\right) \sin x \) \hspace{1cm} (b) \( f(1/x) = \left(\frac{1}{x} + x\right) \sin \left(\frac{1}{x}\right) \)
Section 2.2 Exercises

In Exercises 1–8, use graphs and tables to find (a) \( \lim_{x \to a} f(x) \) and (b) \( \lim_{x \to a^-} f(x) \) (c) Identify all horizontal asymptotes.

1. \( f(x) = \cos \frac{1}{x} \) (a) 1 (b) 1 (c) \( y = 1 \)
2. \( f(x) = \frac{\sin 2x}{x} \) (a) 0 (b) 0 (c) \( y = 0 \)
3. \( f(x) = \frac{e^{-x}}{x} \) (a) 0 (b) \(-\infty \) (c) \( y = 0 \)
4. \( f(x) = \frac{3x^3 - x + 1}{x + 3} \) (a) \( \infty \) (b) \( y = 0 \)
5. \( f(x) = \frac{3x + 1}{|x| + 2} \) (a) 3 (b) \(-3 \) (c) \( y = 3, y = -3 \)
6. \( f(x) = \frac{2x - 1}{|x| - 3} \) (a) 2 (b) \(-2 \) (c) \( y = 2, y = -2 \)
7. \( f(x) = \frac{x}{|x|} \) (a) 1 (b) \(-1 \) (c) \( y = 1, y = -1 \)
8. \( f(x) = \frac{1}{|x| + 1} \) (a) 1 (b) 1 (c) \( y = 1 \)

In Exercises 9–12, find the limit and confirm your answer using the Sandwich Theorem.

9. \( \lim_{x \to \infty} \frac{1 - \cos x}{x^2} = 0 \)
10. \( \lim_{x \to \infty} \frac{1 - \cos x}{x^2} = 0 \)
11. \( \lim_{x \to \infty} \frac{\sin x}{x} = 0 \)
12. \( \lim_{x \to \infty} \frac{\sin (x^2)}{x} = 0 \)

In Exercises 13–20, use graphs and tables to find the limits.

13. \( \lim_{x \to 2^-} \frac{1}{x - 2} = \infty \)
14. \( \lim_{x \to 2^+} \frac{x}{x - 2} = -\infty \)
15. \( \lim_{x \to 3^-} \frac{x}{x + 3} = -\infty \)
16. \( \lim_{x \to 3^+} \frac{x}{x + 3} = -\infty \)
17. \( \lim_{x \to 0^+} \frac{\tan x}{x} = 0 \)
18. \( \lim_{x \to 0^+} \frac{\csc x}{x} = 0 \)
19. \( \lim_{x \to 0^-} \frac{\csc x}{x} = \infty \)
20. \( \lim_{x \to x \rightarrow (\pi/2)} \sec x = -\infty \)

In Exercises 21–26, find \( \lim_{x \to a} y \) and \( \lim_{x \to a^-} y \). Both are 5

21. \( y = \left( \frac{2 - x}{x + 1} \right) \left( \frac{x^2}{5 + x^2} \right) \)
22. \( y = \left( \frac{2 - x}{x + 1} \right) \left( \frac{5x^2 - 1}{x^2} \right) \)
23. \( y = \frac{\cos (1/x)}{1 + (1/x)} \) Both are 1
24. \( y = \frac{2x + \sin x}{x} \) Both are 2
25. \( y = \frac{\sin x}{2x^2 + x} \) Both are 0
26. \( y = \frac{\sin x + 2 \sin x}{2x^2} \) Both are 0

In Exercises 27–34, (a) find the vertical asymptotes of the graph of \( f(x) \) (b) Describe the behavior of \( f(x) \) to the left and right of each vertical asymptote.

27. \( f(x) = \frac{1}{x^2 - 4} \) (a) \( x = \pm 2 \) (b) \( x \rightarrow -\infty \)
28. \( f(x) = \frac{x^2 - 1}{2x + 4} \) (a) \( x = -2 \) (b) \( x \rightarrow -\infty \)
29. \( f(x) = \frac{x^2 - 2x}{x + 1} \) (a) \( x = -1 \) (b) \( x \rightarrow -\infty \)
30. \( f(x) = \frac{1 - x(a)}{x^2 - 5x + 3} \) (a) \( x = \frac{1}{a} \) (b) \( x \rightarrow -\infty \)
31. \( f(x) = \cot x \) (a) \( x = km, k \) any integer
32. \( f(x) = \sec x \) (a) \( x = \frac{n\pi}{2} \) (b) \( x \rightarrow -\infty \)
33. \( f(x) = \tan x \) (a) \( x = \frac{n\pi}{2} \) (b) \( x \rightarrow -\infty \)
34. \( f(x) = \cot x \) (a) \( x = \frac{n\pi}{2} \) (b) \( x \rightarrow -\infty \)

In Exercises 35–38, match the function with the graph of its end behavior model.

35. \( y = \frac{2x^3 - 3x^2 + 1}{x + 3} \) (a)
36. \( y = \frac{x^5 - x^4 + x + 1}{2x^2 + x - 3} \) (b)
37. \( y = \frac{2x^3 - x^2 + x - 1}{2 - x} \) (c)
38. \( y = \frac{x^4 - 3x^3 + x^2 - 1}{1 - x^2} \) (d)

In Exercises 39–44, (a) find a power function end behavior model for \( f(x) \) (b) Identify any horizontal asymptotes.

39. \( f(x) = 3x^2 - 2x + 1 \) (a) \( 3x^2 \) (b) None
40. \( f(x) = -4x^3 + x^2 - 2x - 1 \) (a) \(-4x^3 \) (b) None
41. \( f(x) = \frac{2x^2 + 3x - 5}{x^2 + 2x + 1} \) (a) \(-2x \) (b) \( y = 0 \)
42. \( f(x) = \frac{1}{x^2 - 4} \) (a) \(-2 \) (b) \( y = 3 \)
43. \( f(x) = \frac{2x^2 - x + 3}{x - 2} \) (a) \( 2x^2 \) (b) None
44. \( f(x) = \frac{-x^4 + 2x^2 + x - 3}{x - 2} \) (a) \(-x^4 \) (b) None

In Exercises 45–48, find (a) a simple basic function as a right end behavior model and (b) a simple basic function as a left end behavior model for the function.

45. \( y = e^{-x} - 2x \) (a) \( e^{-x} \) (b) \(-2x \)
46. \( y = x + \ln |x| \) (a) \( x \) (b) \( x \)
47. \( y = x \) and \( y = x^2 + \sin x \) (a) \( x \) (b) \( x^2 \)

In Exercises 49–52, use the graph of \( y = f(1/x) \) to find \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a^-} f(x) \).

49. \( f(x) = xe^x \) (a) \( e \) (b) \( e \)
50. \( f(x) = x^2e^{-x} \) (a) \( 0 \) (b) \( 0 \)
51. \( f(x) = \ln |x| \) (a) \( \infty \) (b) \( -\infty \)
52. \( f(x) = x \) (a) \( 1 \) (b) \( -1 \)

In Exercises 53 and 54, find the limit of \( f(x) \) as (a) \( x \to -\infty \), (b) \( x \to \infty \), (c) \( x \to 0^- \), and (d) \( x \to 0^+ \).

53. \( f(x) = \frac{1}{x} \) (a) \( x \to 0^- \) (b) \( -1 \) (c) \( 0 \) (d) \( 1 \)
54. \( f(x) = \frac{x - 1}{x^2} \) (a) \( x \to 0^+ \) (b) \( 1 \) (c) \( x \to 0^- \) (d) \( 0 \)

**Group Activity** In Exercises 55 and 56, sketch a graph of a function \( y = f(x) \) that satisfies the stated conditions. Include any asymptotes.

55. \( \lim_{x \to 0} f(x) = 2 \) (a) \( x \to 0^- \) (b) \( x \to 0^+ \)
56. \( \lim_{x \to 0} f(x) = 2 \) (a) \( x \to 0^- \) (b) \( x \to 0^+ \)
65. **Exploring Properties of Limits** Find the limits of \( f, g, \) and \( fg \) as \( x \to c. \)

(a) \( f(x) = \frac{1}{x}, \quad g(x) = x, \quad c = 0 \)
\[ f \to -\infty \text{ as } x \to 0^-, \quad f \to \infty \text{ as } x \to 0^+; \quad g \to 0, \quad fg \to 0 \]
\[ f \to -\infty \text{ as } x \to 0^-, \quad f \to -\infty \text{ as } x \to 0^+; \quad g \to 0, \quad fg \to -8 \]
(b) \( f(x) = -\frac{2}{x^2}, \quad g(x) = 4x^3, \quad c = 0 \)

58. Yes. The limit of \( (f + g) \) will be the same as the limit of \( f \). This is because adding numbers that are very close to zero has little effect on the value of \( (f + g) \) since the values of \( g \) are becoming arbitrarily large.
Chapter 2  Limits and Continuity

2.3 Continuity

What you’ll learn about
• Continuity at a Point
• Continuous Functions
• Algebraic Combinations
• Composites
• Intermediate Value Theorem for Continuous Functions

... and why
Continuous functions are used to describe how a body moves through space and how the speed of a chemical reaction changes with time.

Continuity at a Point

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function’s values are likely to have been at the times we did not measure (Figure 2.16). In doing so, we are assuming that we are working with a continuous function, a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Any function \( y = f(x) \) whose graph can be sketched in one continuous motion without lifting the pencil is an example of a continuous function.

Continuous functions are the functions we use to find a planet’s closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies (Figure 2.17). As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

To understand continuity, we need to consider a function like the one in Figure 2.18, whose limits we investigated in Example 8, Section 2.1.

Figure 2.17 The laser was developed as a result of an understanding of the nature of the atom.

Figure 2.18 The function is continuous on \([0, 4]\) except at \(x = 1\) and \(x = 2\). (Example 1)

EXAMPLE 1 Investigating Continuity

Find the points at which the function \( f \) in Figure 2.18 is continuous, and the points at which \( f \) is discontinuous.

SOLUTION

The function \( f \) is continuous at every point in its domain \([0, 4]\) except at \(x = 1\) and \(x = 2\).

At these points there are breaks in the graph. Note the relationship between the limit of \( f \) and the value of \( f \) at each point of the function’s domain.

Points at which \( f \) is continuous:

At \( x = 0 \), \( \lim_{x \to 0^+} f(x) = f(0) \).

At \( x = 4 \), \( \lim_{x \to 4^-} f(x) = f(4) \).

At \( 0 < c < 4, c \neq 1, 2 \), \( \lim_{x \to c} f(x) = f(c) \).

continued
Continuity at a Point

Interior Point: A function \( y = f(x) \) is continuous at an interior point \( c \) of its domain if

\[
\lim_{x \to c} f(x) = f(c).
\]

Endpoint: A function \( y = f(x) \) is continuous at a left endpoint \( a \) or is continuous at a right endpoint \( b \) of its domain if

\[
\lim_{x \to a^-} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^+} f(x) = f(b),
\]

respectively.

Points at which \( f \) is discontinuous:

At \( x = 1 \), \( \lim_{x \to 1} f(x) \) does not exist.

At \( x = 2 \), \( \lim_{x \to 2} f(x) = 1 \), but \( 1 \neq f(2) \).

At \( c < 0, c > 4 \), these points are not in the domain of \( f \).

To define continuity at a point in a function’s domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit). (Figure 2.19)

DEFINITION  Continuity at a Point

Interior Point: A function \( y = f(x) \) is **continuous at an interior point** \( c \) of its domain if

\[
\lim_{x \to c} f(x) = f(c).
\]

Endpoint: A function \( y = f(x) \) is **continuous at a left endpoint** \( a \) or is **continuous at a right endpoint** \( b \) of its domain if

\[
\lim_{x \to a^-} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^+} f(x) = f(b),
\]

respectively.

If a function \( f \) is not continuous at a point \( c \), we say that \( f \) is **discontinuous** at \( c \) and \( c \) is a **point of discontinuity** of \( f \). Note that \( c \) need not be in the domain of \( f \).

**EXAMPLE 2** Finding Points of Continuity and Discontinuity

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

**SOLUTION**

For the function to be continuous at \( x = c \), the limit as \( x \to c \) must exist and must equal the value of the function at \( x = c \). The greatest integer function is discontinuous at every integer. For example,

\[
\lim_{x \to 3^-} \text{int } x = 2 \quad \text{and} \quad \lim_{x \to 3^+} \text{int } x = 3
\]

so the limit as \( x \to 3 \) does not exist. Notice that \( \text{int } 3 = 3 \). In general, if \( n \) is any integer,

\[
\lim_{x \to n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \text{int } x = n,
\]

so the limit as \( x \to n \) does not exist.

The greatest integer function is continuous at every other real number. For example,

\[
\lim_{x \to 1.5} \text{int } x = 1 = \text{int } 1.5.
\]

In general, if \( n - 1 < c < n, n \) an integer, then

\[
\lim_{x \to c} \text{int } x = n - 1 = \text{int } c.
\]

Now try Exercise 7.
Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at \( x = 0 \). The function in (b) would be continuous if it had \( f(0) = 1 \). The function in (c) would be continuous if \( f(0) \) were 1 instead of 2. The discontinuities in (b) and (c) are removable. Each function has a limit as \( x \to 0 \), and we can remove the discontinuity by setting \( f(0) \) equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: \( \lim_{x \to 0} f(x) \) does not exist and there is no way to improve the situation by changing \( f \) at 0. The step function in (d) has a jump discontinuity; the one-sided limits exist but have different values. The function \( f(x) = 1/x^2 \) in (e) has an infinite discontinuity. The function in (f) has an oscillating discontinuity: it oscillates and has no limit as \( x \to 0 \).

**Figure 2.21** The function in part (a) is continuous at \( x = 0 \). The functions in parts (b)–(f) are not.
Continuous Functions

A function is continuous on an interval if and only if it is continuous at every point of the interval. A continuous function is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, \( y = \frac{1}{x} \) is not continuous on \([-1, 1]\).

**EXAMPLE 3 Identifying Continuous Functions**

The reciprocal function \( y = \frac{1}{x} \) (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at \( x = 0 \) because it is not defined there.

Polynomial functions \( f \) are continuous at every real number \( c \) because \( \lim_{x \to c} f(x) = f(c) \). Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function \( y = |x| \) is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like \( y = \sqrt[n]{x} \) (\( n \) a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions.

**Algebraic Combinations**

As you may have guessed, algebraic combinations of continuous functions are continuous wherever they are defined.
Composites

All composites of continuous functions are continuous. This means composites like
\[ y = \sin(x^2) \quad \text{and} \quad y = |\cos x| \]
are continuous at every point at which they are defined. The idea is that if \( f(x) \) is continuous at \( x = c \) and \( g(x) \) is continuous at \( x = f(c) \), then \( g \circ f \) is continuous at \( x = c \) (Figure 2.23). In this case, the limit as \( x \to c \) is \( g(f(c)) \).

**THEOREM 6  Properties of Continuous Functions**

If the functions \( f \) and \( g \) are continuous at \( x = c \), then the following combinations are continuous at \( x = c \).

1. Sums: \( f + g \)
2. Differences: \( f - g \)
3. Products: \( f \cdot g \)
4. Constant multiples: \( k \cdot f \), for any number \( k \)
5. Quotients: \( f/g \), provided \( g(c) \neq 0 \)

**THEOREM 7  Composite of Continuous Functions**

If \( f \) is continuous at \( c \) and \( g \) is continuous at \( f(c) \), then the composite \( g \circ f \) is continuous at \( c \).

**EXAMPLE 4  Using Theorem 7**

Show that \( y = \frac{x \sin x}{x^2 + 2} \) is continuous.

**SOLUTION**

The graph (Figure 2.24) of \( y = |(x \sin x)/(x^2 + 2)| \) suggests that the function is continuous at every value of \( x \). By letting
\[ g(x) = |x| \quad \text{and} \quad f(x) = \frac{x \sin x}{x^2 + 2}, \]
we see that \( y \) is the composite \( g \circ f \).

We know that the absolute value function \( g \) is continuous. The function \( f \) is continuous by Theorem 6. Their composite is continuous by Theorem 7. **Now try Exercise 33.**
Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the intermediate value property. A function is said to have the intermediate value property if it never takes on two values without taking on all the values in between.

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function \( y = f(x) \) that is continuous on a closed interval \([a, b]\) takes on every value between \(f(a)\) and \(f(b)\). In other words, if \(y_0\) is between \(f(a)\) and \(f(b)\), then \(y_0 = f(c)\) for some \(c\) in \([a, b]\).

The continuity of \(f\) on the interval is essential to Theorem 8. If \(f\) is discontinuous at even one point of the interval, the theorem’s conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be connected, a single, unbroken curve, like the graph of \(\sin x\). It will not have jumps like those in the graph of the greatest integer function \(\text{int } x\), or separate branches like we see in the graph of \(1/x\).

Most graphers can plot points (dot mode). Some can turn on pixels between plotted points to suggest an unbroken curve (connected mode). For functions, the connected format basically assumes that outputs vary continuously with inputs and do not jump from one value to another without taking on all values in between.

EXAMPLE 5 Using Theorem 8

Is any real number exactly 1 less than its cube?

SOLUTION

We answer this question by applying the Intermediate Value Theorem in the following way. Any such number must satisfy the equation \(x = x^3 - 1\) or, equivalently, \(x^3 - x - 1 = 0\). Hence, we are looking for a zero value of the continuous function \(f(x) = x^3 - x - 1\) (Figure 2.26). The function changes sign between 1 and 2, so there must be a point \(c\) between 1 and 2 where \(f(c) = 0\).

Now try Exercise 46.
Quick Review 2.3  

(For help, go to Sections 1.2 and 2.1.)

1. Find \( \lim_{x \to 1} \frac{3x^2 - 2x + 1}{x^3 + 4} \).

2. Let \( f(x) = \int x \). Find each limit. (a) \(-2\)  (b) \(-1\)  (c) No limit  (d) \(-1\)
   (a) \( \lim_{x \to -1} f(x) \)  (b) \( \lim_{x \to -1} f(x) \)  (c) \( \lim_{x \to 0} f(x) \)  (d) \( f(-1) \)

3. Let \( f(x) = \frac{x^2 - 4x + 5}{4 - x} \), \( x < 2 \) and \( x \geq 2 \).
   Find each limit. (a) 1  (b) 2  (c) No limit  (d) 2
   (a) \( \lim_{x \to 2} f(x) \)  (b) \( \lim_{x \to 2} f(x) \)  (c) \( \lim_{x \to 2} f(x) \)  (d) \( f(2) \)

In Exercises 4–6, find the remaining functions in the list of functions:

4. \( f(x) = \frac{2x - 1}{x + 5} \), \( g(x) = \frac{1}{x} + 1 \)
   \( f \circ g \) \( x \neq 0 \), \( x \neq -5 \)

Section 2.3 Exercises

5. All points not in the domain, i.e., all \( x < -3/2 \)

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. \( y = \frac{1}{x + 2} \), \( x = -2 \), infinite discontinuity
2. \( y = \frac{x + 1}{x^2 - 4x + 3} \), \( x = 1 \) and \( x = 3 \), both infinite discontinuities
3. \( y = \frac{1}{x^2 + 1} \) None
4. \( y = |x - 1| \) None
5. \( y = \sqrt{2x + 3} \) None
6. \( y = \sqrt{2x - 1} \) None
7. \( y = |x|/\sqrt{2} \), \( x = 0 \), jump discontinuity
8. \( y = \cot x \) \( x = k\pi \) for all integers \( k \), infinite discontinuity
9. \( y = e^{\sqrt{x}} \), \( x = 0 \), infinite discontinuity
10. \( y = \ln (x + 1) \) All points not in the domain, i.e., all \( x < -1 \)

In Exercises 11–18, use the function \( f \) defined and graphed below to answer the questions.

\[
f(x) = \begin{cases} 
2x, & 0 < x < 1 \\
1, & x = 1 \\
-2x + 4, & 1 < x < 2 \\
0, & 2 < x < 3 \\
\end{cases}
\]

11. (a) Does \( f(-1) \) exist? Yes
    (b) Does \( \lim_{x \to -1} f(x) \) exist? Yes
    (c) Does \( \lim_{x \to 1} f(x) = f(-1) \) ? Yes
    (d) Is \( f \) continuous at \( x = -1 \)? Yes

12. (a) Does \( f(1) \) exist? Yes
    (b) Does \( \lim_{x \to 1} f(x) \) exist? Yes
    (c) Does \( \lim_{x \to 1} f(x) = f(1) \) ? No
    (d) Is \( f \) continuous at \( x = 1 \)? No

13. (a) Is \( f \) defined at \( x = 2 \)? (Look at the definition of \( f \)) No
    (b) Is \( f \) continuous at \( x = 2 \)? No

14. At what values of \( x \) is \( f \) continuous? Everywhere in \( (1, 3) \) except for \( x = 0, 1, 2 \)

15. What value should be assigned to \( f(2) \) to make the extended function continuous at \( x = 2 \)? 0

16. What new value should be assigned to \( f(1) \) to make the new function continuous at \( x = 1 \)? 2

17. Writing to Learn Is it possible to extend \( f \) to be continuous at \( x = 0 \)? If so, what value should the extended function have there? If not, why not? No, because the right-hand and left-hand limits are not the same at zero.

18. Writing to Learn Is it possible to extend \( f \) to be continuous at \( x = 3 \)? If so, what value should the extended function have there? If not, why not? Yes. Assign the value 0 to \( f(3) \).

In Exercises 19–24, (a) find each point of discontinuity. (b) Which of the discontinuities are removable? not removable? Give reasons for your answers.

19. \( f(x) = \begin{cases} 
3 - x, & x < 2 \\
x + 1, & x > 2
\end{cases} \)
    (a) \( x = 2 \) (b) Not removable, the one-sided limits are different.

20. \( f(x) = \begin{cases} 
3 - x, & x < 2 \\
x + 2, & x > 2
\end{cases} \)
    (a) \( x = 2 \) (b) Removable, assign the value 1 to \( f(2) \).

21. \( f(x) = \begin{cases} 
\frac{1}{x - 1}, & x < 1 \\
\frac{1}{x - 1}, & x > 1
\end{cases} \)
    (a) \( x = 1 \) (b) Not removable, it’s an infinite discontinuity.

22. \( f(x) = \begin{cases} 
1 - x^2, & x ≠ -1 \\
x, & x = -1
\end{cases} \)
    (a) \( x = -1 \) (b) Removable, assign the value 0 to \( f(-1) \).
23. All points not in the domain along with \( x = 0, 1 \)
(b) \( x = 0 \) is a removable discontinuity, assign \( f(0) = 0 \). \( x = 1 \) is not removable, the two-sided limits are different.

24. All points not in the domain along with \( x = 1, 2 \)
(b) \( x = 1 \) is not removable, the one-sided limits are different. \( x = 2 \) is a removable discontinuity, assign \( f(2) = 1 \).

In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. \( f(x) = \frac{x^2 - 9}{x + 3}, \ x = -3 \)
26. \( f(x) = \frac{x^3 - 1}{x^2 - 1}, \ x = 1 \)
27. \( f(x) = \frac{\sin x}{x}, \ x = 0 \)
28. \( f(x) = \frac{\sin 4x}{x}, \ x = 0 \)
29. \( f(x) = \frac{x - 4}{\sqrt{x} - 2}, \ x = 4 \)
30. \( f(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}, \ x = 2 \)

In Exercises 31 and 32, explain why the given function is continuous.

31. \( f(x) = \frac{1}{x - 3} \)
32. \( g(x) = \frac{1}{\sqrt{x - 1}} \)

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. \( f(x) = \sqrt{\frac{x}{x + 1}} \)
34. \( f(x) = \sin (x^2 + 1) \)
35. \( f(x) = \cos (\sqrt{1 - x}) \)
36. \( f(x) = \tan \left( \frac{x^2}{x^2 + 4} \right) \)

Group Activity

37. \( y = \frac{1}{\sqrt{x} + 2} \)
38. \( y = x^2 + \sqrt{4 - x} \)
39. \( y = |x^2 - 4x| \)
40. \( y = \begin{cases} \frac{x^2 - 1}{x - 1} & x \neq 1 \\ 2, & x = 1 \end{cases} \)

In Exercises 41–44, sketch a possible graph for a function \( f \) that has the stated properties.

41. \( f(3) \) exists but \( \lim_{x \to 3} f(x) \) does not.
42. \( f(-2) \) exists, \( \lim_{x \to -2} f(x) = f(-2) \), but \( \lim_{x \to -2} f(x) \) does not exist.
43. \( f(4) \) exists, \( \lim_{x \to 4} f(x) \) exists, but \( f \) is not continuous at \( x = 4 \).
44. \( f(x) \) is continuous for all \( x \) except \( x = 1 \), where \( f \) has a nonremovable discontinuity.

45. Solving Equations

Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.
\( x = -0.724 \) and \( x = 1.221 \)

46. Solving Equations

Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.
\( x = -1.521 \)

47. Continuous Function

Find a value for \( a \) so that the function
\( f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases} \)
is continuous.
\( a = \frac{4}{3} \)

48. Continuous Function

Find a value for \( a \) so that the function
\( f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ ax^2 + 1, & x > 2 \end{cases} \)
is continuous.
\( a = 3 \)

49. Continuous Function

Find a value for \( a \) so that the function
\( f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \geq -1 \end{cases} \)
is continuous.
\( a = 4 \)

50. Continuous Function

Find a value for \( a \) so that the function
\( f(x) = \begin{cases} x^3, & x < 1 \\ x^3 - a, & x \geq 1 \end{cases} \)
is continuous.
\( a = -1 \)

51. Writing to Learn

Explain why the equation \( e^{-x} = x \) has at least one solution.

52. Salary Negotiation

A welder’s contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of $36,500.
(a) Show that Luisa’s salary is given by
\( y = 36,500(1.035)^{t}, \)
where \( t \) is the time, measured in years, since Luisa signed the contract.
(b) Graph Luisa’s salary function. At what values of \( t \) is it continuous?

53. Airport Parking

Valuepark charge $1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is $7.25.
\( f(x) = \begin{cases} -1.10 \int_{-x}^{0}, & 0 \leq x \leq 6 \\ 7.25, & 6 < x \leq 24 \end{cases} \)

(a) Write a formula that gives the charge for \( x \) hours with \( 0 \leq x \leq 24 \). (Hint: See Exercise 52.)
(b) Graph the function in part (a). At what values of \( x \) is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems. False. Consider \( f(x) = 1/x \) which is continuous and has a point of discontinuity at \( x = 0 \).

54. True or False

A continuous function cannot have a point of discontinuity. Justify your answer.

55. True or False

It is possible to extend the definition of a function \( f \) at a jump discontinuity \( x = a \) so that \( f \) is continuous at \( x = a \). Justify your answer. True. If \( f \) has a jump discontinuity at \( x = a \), then \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \) so \( f \) is not continuous at \( x = a \).

56. True or False

It is possible to extend the definition of a function \( f \) at a removable discontinuity \( x = a \) so that \( f \) is continuous at \( x = a \). Justify your answer. True. If \( f \) has a removable discontinuity at \( x = a \), then \( \lim_{x \to a} f(x) \) exists and \( f \) is continuous at \( x = a \).
56. **Multiple Choice** On which of the following intervals is $f(x) = \frac{1}{\sqrt{x}}$ not continuous? B
   (A) $(0, \infty)$       (B) $[0, \infty)$       (C) $(0, 2)$
   (D) $(1, 2)$       (E) $[1, \infty)$

57. **Multiple Choice** Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x - 1}$? E
   (A) $x = -1$       (B) $x = -1/2$       (C) $x = 0$
   (D) $x = 1/2$       (E) $x = 1$

58. **Multiple Choice** Which of the following statements about the function
   $$f(x) = \begin{cases} 
   2x, & 0 < x < 1 \\
   1, & x = 1 \\
   -x + 3, & 1 < x < 2 
   \end{cases}$$
   is not true? A
   (A) $f(1)$ does not exist.
   (B) $\lim_{x \to 0^+} f(x)$ exists.
   (C) $\lim_{x \to 2} f(x)$ exists.
   (D) $\lim_{x \to 1} f(x)$ exists.
   (E) $\lim_{x \to 1} f(x) = f(1)$

59. **Multiple Choice** Which of the following points of discontinuity of
   $$f(x) = \frac{x(x - 1)(x - 2)^2(x + 1)^2(x - 3)^2}{x(x - 1)(x - 2)(x + 1)^2(x - 3)^2}$$
   is not removable? E
   (A) $x = -1$       (B) $x = 0$       (C) $x = 1$
   (D) $x = 2$       (E) $x = 3$

**Exploration**

60. Let $f(x) = \left(1 + \frac{1}{x}\right)^x$.
   
   Domain of $f$: $(-\infty, -1) \cup (0, \infty)$
   (a) Find the domain of $f$.
   (b) Draw the graph of $f$.
   (c) Writing to Learn Explain why $x = -1$ and $x = 0$ are points of discontinuity of $f$. Because $f$ is undefined there due to division by 0.
   (d) Writing to Learn Are either of the discontinuities in part (c) removable? Explain. $x = 0$: removable, right-hand limit is 1; not removable, infinite discontinuity $x = -1$: not removable, infinite discontinuity
   (e) Use graphs and tables to estimate $\lim_{x \to \infty} f(x)$. 2.718 or $e$

**Extending the Ideas**

61. **Continuity at a Point** Show that $f(x)$ is continuous at $x = a$ if and only if
   This is because $\lim_{h \to 0} f(a + h) = \lim_{h \to 0} f(x)$. $\lim_{h \to 0} f(a + h) = f(a)$.

62. **Continuity on Closed Intervals** Let $f$ be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all $x$ in $[a, b]$ or $f(x) < 0$ for all $x$ in $[a, b]$.

63. **Properties of Continuity** Prove that if $f$ is continuous on an interval, then so is $|f|$.

64. **Everywhere Discontinuous** Give a convincing argument that the following function is not continuous at any real number.
   $$f(x) = \begin{cases} 
   1, & \text{if } x \text{ is rational} \\
   0, & \text{if } x \text{ is irrational} 
   \end{cases}$$
Section 2.4 Rates of Change and Tangent Lines

Rates of Change and Tangent Lines

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The average rate of change of a quantity over a period of time is the amount of change divided by the time it takes. In general, the average rate of change of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of \( f(x) = x^3 - x \) over the interval \([1, 3]\).

SOLUTION

Since \( f(1) = 0 \) and \( f(3) = 24 \), the average rate of change over the interval \([1, 3]\) is

\[
\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.
\]

Now try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies (Drosophila) grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

Secant to a Curve

A line through two points on a curve is a secant to the curve.

EXAMPLE 2 Growing Drosophila in a Laboratory

Use the points \( P(23, 150) \) and \( Q(45, 340) \) in Figure 2.27 to compute the average rate of change and the slope of the secant line \( PQ \).

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of \( 340 - 150 = 190 \) flies in \( 45 - 23 = 22 \) days.

The average rate of change in the population \( p \) from day 23 to day 45 was

\[
\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day,}
\]

or about 9 flies per day.
Chapter 2  Limits and Continuity

This average rate of change is also the slope of the secant line through the two points \( P \) and \( Q \) on the population curve. We can calculate the slope of the secant \( PQ \) from the coordinates of \( P \) and \( Q \).

\[
\text{Secant slope: } \frac{\Delta y}{\Delta x} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}
\]

\text{Now try Exercise 7.}

As suggested by Example 2, \textit{we can always think of an average rate of change as the slope of a secant line.} In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant \( PQ \) change as we back \( Q \) along the curve toward \( P \). The results for four positions of \( Q \) are shown in Figure 2.28.

\[8.6 \text{ flies/day}\]

In terms of population, what we see as \( Q \) approaches \( P \) is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at \( P \) \((17.5 \text{ flies per day})\). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day \( t = 23 \).

\textbf{Tangent to a Curve}

The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function \( y = f(x) \) is changing with respect to \( x \) at any particular value \( x = a \) to be the slope of the tangent to the curve \( y = f(x) \) at \( x = a \). But how are we to define the tangent line at an arbitrary point \( P \) on the curve and find its slope from the formula \( y = f(x) \)? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century’s major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

1. We start with what we can calculate, namely, the slope of a secant through \( P \) and a point \( Q \) nearby on the curve.
2. We find the limiting value of the secant slope (if it exists) as \( Q \) approaches \( P \) along the curve.

3. We define the slope of the curve at \( P \) to be this number and define the tangent to the curve at \( P \) to be the line through \( P \) with this slope.

**EXAMPLE 3 Finding Slope and Tangent Line**

Find the slope of the parabola \( y = x^2 \) at the point \( P(2, 4) \). Write an equation for the tangent to the parabola at this point.

**SOLUTION**

We begin with a secant line through \( P(2, 4) \) and a nearby point \( Q(2 + h, (2 + h)^2) \) on the curve (Figure 2.29).

![Figure 2.29](image)

The slope of the tangent to the parabola \( y = x^2 \) at \( P(2, 4) \) is 4.

We then write an expression for the slope of the secant line and find the limiting value of this slope as \( Q \) approaches \( P \) along the curve.

\[
\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 4}{h} = \frac{h^2 + 4h + 4 - 4}{h} = h + 4.
\]

The limit of the secant slope as \( Q \) approaches \( P \) along the curve is

\[
\lim_{Q \to P} \text{(secant slope)} = \lim_{h \to 0} (h + 4) = 4.
\]

Thus, the slope of the parabola at \( P \) is 4.

The tangent to the parabola at \( P \) is the line through \( P(2, 4) \) with slope \( m = 4 \).

\[
y - 4 = 4(x - 2) \\
y = 4x - 8 + 4 \\
y = 4x - 4
\]

**Slope of a Curve**

To find the tangent to a curve \( y = f(x) \) at a point \( P(a, f(a)) \) we use the same dynamic procedure. We calculate the slope of the secant line through \( P \) and a point \( Q(a + h, f(a + h)) \). We then investigate the limit of the slope as \( h \to 0 \) (Figure 2.30). If the limit exists, it is the slope of the curve at \( P \) and we define the tangent at \( P \) to be the line through \( P \) having this slope.
Chapter 2  Limits and Continuity

DEFINITION  Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at $P$ is the line through $P$ with this slope.

**EXAMPLE 4  Exploring Slope and Tangent**

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of $a$?

**SOLUTION**

(a) The slope at $x = a$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{1}{a + h} - \frac{1}{a}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a + h)}{a(a + h)}$$

$$= \lim_{h \to 0} \frac{-h}{h} \cdot \frac{1}{a(a + h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a + h)} = -\frac{1}{a^2}.$$

(b) The slope will be $-1/4$ if

$$\frac{-1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4$$

Multiply by $-4a^2$.

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.31).

(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As $a$ moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal.

*Now try Exercise 19.*

The expression

$$\frac{f(a + h) - f(a)}{h}$$

is the **difference quotient of $f$ at $a$**. Suppose the difference quotient has a limit as $h$ approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function’s rate of change with respect to $x$ at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 3.
Normal to a Curve

The normal line to a curve at a point is the line perpendicular to the tangent at that point.

**EXAMPLE 5 Finding a Normal Line**

Write an equation for the normal to the curve \( f(x) = 4 - x^2 \) at \( x = 1 \).

**SOLUTION**

The slope of the tangent to the curve at \( x = 1 \) is

\[
\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{4 - (1 + h)^2 - 3}{h} = \lim_{h \to 0} \frac{4 - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0} \frac{-h(2 + h)}{h} = -2.
\]

Thus, the slope of the normal is \( \frac{1}{2} \), the negative reciprocal of \(-2\). The normal to the curve at \((1, f(1)) = (1, 3)\) is the line through \((1, 3)\) with slope \( m = \frac{1}{2} \).

\[
y - 3 = \frac{1}{2}(x - 1) \\
y = \frac{1}{2}x - \frac{1}{2} + 3 \\
y = \frac{1}{2}x + \frac{5}{2}
\]

You can support this result by drawing the graphs in a square viewing window.

*Now try Exercise 11 (c, d).*

**Speed Revisited**

The function \( y = 16t^2 \) that gave the distance fallen by the rock in Example 1, Section 2.1, was the rock’s position function. A body’s average speed along a coordinate axis (here, the \( y \)-axis) for a given period of time is the average rate of change of its position \( y = f(t) \). Its instantaneous speed at any time \( t \) is the instantaneous rate of change of position with respect to time at time \( t \), or

\[
\lim_{h \to 0} \frac{f(t + h) - f(t)}{h}.
\]

We saw in Example 1, Section 2.1, that the rock’s instantaneous speed at \( t = 2 \) sec was 64 ft/sec.

**EXAMPLE 6 Investigating Free Fall**

Find the speed of the falling rock in Example 1, Section 2.1, at \( t = 1 \) sec.

**SOLUTION**

The position function of the rock is \( f(t) = 16t^2 \). The average speed of the rock over the interval between \( t = 1 \) and \( t = 1 + h \) sec was

\[
\frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).
\]

The rock’s speed at the instant \( t = 1 \) was

\[
\lim_{h \to 0} 16(h + 2) = 32 \text{ ft/sec}.
\]

*Now try Exercise 27.*
Quick Review 2.4  (For help, go to Section 1.1.)

In Exercises 1 and 2, find the increments $\Delta x$ and $\Delta y$ from point $A$ to point $B$.

1. $A(-5, 2), \quad B(3, 5)$
   \[ \Delta x = 8, \quad \Delta y = 3 \]

2. $A(1, 3), \quad B(a, b)$
   \[ \Delta x = a - 1, \quad \Delta y = b - 3 \]

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(–2, 3), \quad (5, –1) \quad \text{Slope} = -\frac{4}{7}$
4. $(–3, –1), \quad (3, 3) \quad \text{Slope} = \frac{2}{3}$

In Exercises 5–9, find the increments for these slopes.

5. $f(x) = e^x$  
   (a) $[–2, 0]$,  
   (b) $[1, 3]$
6. $f(x) = \cot x$  
   (a) $[\pi/4, 3\pi/4]$,  
   (b) $[\pi/6, \pi/2]$
7. $f(x) = 2 + \cos x$  
   (a) $[0, \pi]$ ,  
   (b) $[–\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

(a) Estimate the slopes of the secants $PQ_1, PQ_2, PQ_3,$ and $PQ_4,$ arranging them in order of a table. What is the appropriate unit for these slopes?
(b) Estimate the speed at point $P$.

7. Accelerating from a Standstill  
   The figure shows the distance-time graph for a 1994 Ford Mustang Cobra accelerating from a standstill.

8. Lunar Data  
   The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.

9. Through $(1, 6)$ and $(4, -1)$  
   \[ y = \frac{7}{3}x + \frac{25}{3} \]

10. Through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$  
    \[ y = -\frac{3}{4}x + \frac{19}{4} \]

11. Through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$  
    \[ y = \frac{4}{3}x + \frac{8}{3} \]

12. Through $(–1, 3)$ and parallel to $2x + 3y = 5$  
    \[ y = \frac{2}{3}x + \frac{7}{3} \]

13. For what value of $b$ will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?  
    \[ b = \frac{19}{3} \]
18. No. The function is discontinuous at \( x = \frac{3\pi}{4} \). The left-hand limit of the difference quotient doesn’t exist.

17. \( f(x) = \begin{cases} 
\frac{1}{x}, & x \leq 2 \\
\frac{4 - x}{4}, & x > 2 
\end{cases} \) at \( x = 2 \). Yes. The slope is \(-\frac{1}{4}\).

18. \( f(x) = \begin{cases} 
\sin x, & 0 \leq x < 3\pi/4 \\
\cos x, & 3\pi/4 \leq x \leq 2\pi 
\end{cases} \) at \( x = 3\pi/4 \).

In Exercises 19–22, (a) find the slope of the curve at \( x = a \).

(b) Writing to Learn Describe what happens to the tangent at \( x = a \) as \( a \) changes.

19. \( y = x^2 + 2 \)

(a) \( 2a \) (b) The slope of the tangent steadily increases as \( a \) increases.

20. \( y = \frac{2}{x} \)

(a) \(-\frac{1}{a^2}\) (b) The slope of the tangent is always negative.

21. \( y = \frac{1}{x - 1} \)

(a) \(-\frac{1}{(a - 1)^2}\) (b) The slope of the tangent is always negative. The tangents are very steep near \( x = 0 \) and nearly horizontal as \( a \) moves away from the origin.

22. \( y = 9 - x^2 \)

(a) \(-2a\) (b) The slope of the tangent steadily decreases as \( a \) increases.

23. Free Fall An object is dropped from the top of a 100-m tower. Its height above ground after \( t \) sec is \( 100 - 4.9t^2 \) m. How fast is it falling 2 sec after it is dropped? \( 19.6 \) m/sec

24. Rocket Launch At \( t \) sec after lift-off, the height of a rocket is \( 3t^2 \) ft. How fast is the rocket climbing after 10 sec? \( 60 \) ft/sec

25. Area of Circle What is the rate of change of the area of a circle with respect to the radius when the radius is \( r = 3 \) in.? \( 6\pi \) in²/sec

26. Volume of Sphere What is the rate of change of the volume of a sphere with respect to the radius when the radius is \( r = 2 \) in.? \( 16\pi \) in³/sec

27. Free Fall on Mars The equation for free fall at the surface of Mars is \( x = 1.86t^2 \) m with \( t \) in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at \( t = 1 \) sec.

\[ 3.72 \text{ m/sec} \]

28. Free Fall on Jupiter The equation for free fall at the surface of Jupiter is \( x = 11.44t^2 \) m with \( t \) in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at \( t = 2 \) sec.

\[ 45.76 \text{ m/sec} \]

29. Horizontal Tangent At what point is the tangent to \( f(x) = x^2 + 4x - 1 \) horizontal? \((-2, -5)\)

30. Horizontal Tangent At what point is the tangent to \( f(x) = 3 - 4x - x^2 \) horizontal? \((-2, 7)\)

31. Finding Tangents and Normals

(a) Find an equation for each tangent to the curve \( y = 1/(x - 1) \) that has slope \(-1\). (See Exercise 21.) At \( x = 0; y = -x - 1 \)
At \( x = 2; y = -x + 3 \)
(b) Find an equation for each normal to the curve \( y = 1/(x - 1) \) that has slope \( 1 \). At \( x = 0; y = x - 1 \)
At \( x = 2; y = x + 1 \)

32. Finding Tangents Find the equations of all lines tangent to \( y = 9 - x^2 \) that pass through the point \((1, 12)\).
At \( x = -1; y = 2x + 10 \)
At \( x = 3; y = -6x + 18 \)

33. Table 2.2 gives the amount of federal spending in billions of dollars for national defense for several years.

<table>
<thead>
<tr>
<th>Year</th>
<th>National Defense Spending ($ billions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>299.3</td>
</tr>
<tr>
<td>1995</td>
<td>272.1</td>
</tr>
<tr>
<td>1999</td>
<td>274.9</td>
</tr>
<tr>
<td>2000</td>
<td>294.5</td>
</tr>
<tr>
<td>2001</td>
<td>305.5</td>
</tr>
<tr>
<td>2002</td>
<td>348.6</td>
</tr>
<tr>
<td>2003</td>
<td>404.9</td>
</tr>
</tbody>
</table>


(a) Find the average rate of change in spending from 1990 to 1995. \(-5.4\) billion dollars per year

(b) Find the average rate of change in spending from 2000 to 2001. \(11.0\) billion dollars per year

(c) Find the average rate of change in spending from 2002 to 2003. \(56.3\) billion dollars per year

(d) Let \( x = 0 \) represent 1990, \( x = 1 \) represent 1991, and so forth. Find the quadratic regression equation for the data and superimpose its graph on a scatter plot of the data.

\[ y = 2.177x - 22.315x + 306.443 \]

(e) Compute the average rates of change in parts (a), (b), and (c) using the regression equation.

(f) Use the regression equation to find how fast the spending was growing in 2003. \(34.3\) billion dollars per year

(g) Writing to Learn Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on this equation.

One possible reason is that the war in Iraq and increased spending to prevent terrorist attacks in the U.S. caused an unusual increase in defense spending.

34. Table 2.3 gives the amount of federal spending in billions of dollars for agriculture for several years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Agriculture Spending ($ billions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>12.0</td>
</tr>
<tr>
<td>1995</td>
<td>9.8</td>
</tr>
<tr>
<td>1999</td>
<td>23.0</td>
</tr>
<tr>
<td>2000</td>
<td>36.6</td>
</tr>
<tr>
<td>2001</td>
<td>26.4</td>
</tr>
<tr>
<td>2002</td>
<td>22.0</td>
</tr>
<tr>
<td>2003</td>
<td>22.6</td>
</tr>
</tbody>
</table>


(a) Let \( x = 0 \) represent 1990, \( x = 1 \) represent 1991, and so forth. Make a scatter plot of the data.

(b) Let \( P \) represent the point corresponding to 2003, \( Q_1 \) the point corresponding to 2000, \( Q_2 \) the point corresponding to 2001, and \( Q_3 \) the point corresponding to 2002. Find the slope of the secant line \( PQ_i \) for \( i = 1, 2, 3 \).

Slope of \( PQ_1 = -4.7 \), Slope of \( PQ_2 = -1.9 \), Slope of \( PQ_3 = 0.6 \).
Standardized Test Questions

35. **True or False** If the graph of a function has a tangent line at \( x = a \), then the graph also has a normal line at \( x = a \). Justify your answer. **True.**

36. **True or False** The graph of \( f(x) = |x| \) has a tangent line at \( x = 0 \). Justify your answer. **False.**

37. **Multiple Choice** If the line \( L \) tangent to the graph of a function \( f \) at the point \((2, 5)\) passes through the point \((-1, -3)\), what is the slope of \( L \)?
   - (A) \(-3/8\)
   - (B) \(3/8\)
   - (C) \(-8/3\)
   - (D) \(8/3\)
   - (E) undefined
   **(D)**

38. **Multiple Choice** Find the average rate of change of \( f(x) = x^2 + x \) over the interval \([1, 3]\).
   - (A) \(-5\)
   - (B) \(1/5\)
   - (C) \(1/4\)
   - (D) \(4\)
   - (E) 5
   **(A)**

39. **Multiple Choice** Which of the following is an equation of the tangent to the graph of \( f(x) = 2x/ax = 1? \)
   - (A) \(y = -2x\)
   - (B) \(y = 2x\)
   - (C) \(y = -2x + 4\)
   - (D) \(y = -x + 3\)
   - (E) \(y = x + 3\)
   **(C)**

40. **Multiple Choice** Which of the following is an equation of the normal to the graph of \( f(x) = 2x/ax = 1? \)
   - (A) \(y = 1/2x + 3/2\)
   - (B) \(y = 1/2x\)
   - (C) \(y = 1/2x + 2\)
   - (D) \(y = -1/2x + 2\)
   - (E) \(y = 2x + 5\)
   **(B)**

**Explorations**

In Exercises 41 and 42, complete the following for the function.
(a) Compute the difference quotient \( f(1 + h) - f(1)/h \).

41. (a) \(\frac{e^{1+h} - e}{h}\)
   (b) Limit = 2.718 (c) They’re about the same. (d) Yes, it has a tangent whose slope is about \( e \).

42. (a) \(\frac{2x + 3 - 2}{h}\)
   (b) Limit = 1.386 (c) They’re about the same. (d) Yes, it has a tangent whose slope is about 4 ln.

**Quick Quiz for AP* Preparation: Sections 2.3 and 2.4**

1. **Multiple Choice** Which of the following values is the average rate of \( f(x) = \sqrt{x} + 1 \) over the interval \((0, 3)\)?
   - (A) \(-3\)
   - (B) \(-1\)
   - (C) \(-1/3\)
   - (D) \(1/3\)
   - (E) 3
   **(D)**

2. **Multiple Choice** Which of the following statements is false for the function
   \[ f(x) = \begin{cases} 
   3/4x, & 0 \leq x < 4 \\
   2, & x = 4 \\
   -x + 7, & 4 < x \leq 6 \\
   1, & 6 < x < 8
   \end{cases} \]
   - (A) \(\lim_{x \to 4} f(x)\) exists
   - (B) \(f(4)\) exists
   - (C) \(\lim_{x \to 6} f(x)\) exists
   - (D) \(\lim_{x \to 8}, f(x)\) exists
   - (E) \(f\) is continuous at \( x = 4 \)
   **(D)**

3. **Multiple Choice** Which of the following is an equation for the tangent line to \( f(x) = 9 - x^2 \) at \( x = 2? \)
   - (A) \(y = 1/4x + 9/2\)
   - (B) \(y = -4x + 13\)
   - (C) \(y = -4x - 3\)
   - (D) \(y = 4x - 3\)
   - (E) \(y = 4x + 13\)
   **(B)**

4. **Free Response** Let \( f(x) = 2x - x^2 \).
   (a) Find \( f(3) \).
   (b) Find \( f(3 + h) \).
   (c) Find \( \frac{f(3 + h) - f(3)}{h} \).
   (d) Find the instantaneous rate of change of \( f \) at \( x = 3 \).
   **(a) \(-3\) (b) \(-3 - 4h - h^2\) (c) \(-4 - h\) (d) \(-4\)**
Chapter 2 Key Terms

average rate of change (p. 87)
average speed (p. 59)
connected graph (p. 83)
Constant Multiple Rule for Limits (p. 61)
continuity at a point (p. 78)
continuous at an endpoint (p. 79)
continuous at an interior point (p. 79)
continuous extension (p. 81)
continuous on an interval (p. 81)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
der free fall (p. 91)
horizontal asymptote (p. 70)
infinite discontinuity (p. 80)
instantaneous rate of change (p. 91)
instantaneous speed (p. 91)
intermediate value property (p. 83)
Intermediate Value Theorem for Continuous Functions (p. 83)
jump discontinuity (p. 80)
left end behavior model (p. 74)
left-hand limit (p. 64)
limit of a function (p. 60)
normal to a curve (p. 91)
oscillating discontinuity (p. 80)
point of discontinuity (p. 79)
Power Rule for Limits (p. 71)
Product Rule for Limits (p. 61)
Properties of Continuous Functions (p. 82)
Quotient Rule for Limits (p. 61)
removable discontinuity (p. 80)
right end behavior model (p. 74)
right-hand limit (p. 64)
Sandwich Theorem (p. 65)
secant to a curve (p. 87)
slope of a curve (p. 89)
Sum Rule for Limits (p. 61)
tangent line to a curve (p. 88)
two-sided limit (p. 64)
vertical asymptote (p. 72)
vertical tangent (p. 94)

Chapter 2 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, find the limits.

1. \( \lim_{x \to 2} (x^3 - 2x^2 + 1) \)
2. \( \lim_{x \to -2} \frac{x^3 + 1}{3x^2 - 2x + 5} \)
3. \( \lim_{x \to 4} \sqrt{1 - 2x} \)
4. \( \lim_{x \to \infty} \sqrt{9 - x^2} \)
5. \( \lim_{x \to 0} \frac{2 + x^2}{x} \)
6. \( \lim_{x \to -\infty} \frac{2x^2 + 3}{5x^2 + 7} \)
7. \( \lim_{x \to \infty} \frac{x^4 + x^3}{12x^3 + 128} \)
8. \( \lim_{x \to 0} \frac{\sin x}{4x} \)
9. \( \lim_{x \to 0} \frac{x \csc x + 1}{x \csc x} \)
10. \( \lim_{x \to 0} e^x \sin x \)
11. \( \lim_{x \to \pi/2} \ln (2x - 1) \)
12. \( \lim_{x \to \pi/2} \ln (2x - 1) \)
13. \( \lim_{x \to \infty} e^{-x} \cos x \)
14. \( \lim_{x \to \infty} \frac{x + \sin x}{x + \cos x} \)

In Exercises 15–20, determine whether the limit exists on the basis of the graph of \( f(x) \). The domain of \( f \) is the set of real numbers.

15. \( \lim_{x \to a} f(x) \) Limit exists
16. \( \lim_{x \to a} f(x) \) Limit exists
17. \( \lim_{x \to a} f(x) \) Limit exists
18. \( \lim_{x \to a} f(x) \) Doesn’t exist
19. \( \lim_{x \to a} f(x) \) Limit exists
20. \( \lim_{x \to a} f(x) \) Limit exists

In Exercises 21–24, determine whether the function \( f \) used in Exercises 15–20 is continuous at the indicated point.

21. \( x = a \) Yes
22. \( x = b \) No
23. \( x = c \) No
24. \( x = d \) Yes
25. Determine
(a) \( \lim_{x \to -3} g(x) \). 1 
(b) \( g(3) \). 1.5
(c) whether \( g(x) \) is continuous at \( x = 3 \). No
(d) the points of discontinuity of \( g(x) \). \( g \) is discontinuous at \( x = 3 \) (and points not in domain).
(e) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not. Yes, can remove discontinuity at \( x = 3 \) by assigning the value 1 to \( g(3) \).

26. Determine
(a) \( \lim_{x \to -1} k(x) \). 1.5 
(b) \( \lim_{x \to -1} k(x) \). 0 
(c) \( k(1) \). 0
(d) whether \( k(x) \) is continuous at \( x = 1 \). No
(e) the points of discontinuity of \( k(x) \). \( k \) is discontinuous at \( x = 1 \) (and points not in domain).
(f) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not. Discontinuity at \( x = 1 \) is not removable because the two one-sided limits are different.

27. \( f(x) = \frac{x + 3}{x + 2} \)

28. \( f(x) = \frac{x - 1}{x^2 (x + 2)} \)

In Exercises 27 and 28, (a) find the vertical asymptotes of the graph of \( y = f(x) \), and (b) describe the behavior of \( f(x) \) to the left and right of any vertical asymptote.

29. \( f(x) = \begin{cases} 
1, & x \leq -1 \\
-x, & -1 < x < 0 \\
1, & x = 0 \\
-x, & 0 < x < 1 \\
1, & x \geq 1 
\end{cases} \)

(a) Find the right-hand and left-hand limits of \( f \) at \( x = -1 \), 0, and 1.
(b) Does \( f \) have a limit as \( x \) approaches \(-1\)? \( 0 \)? \( 1 \)? If so, what is it? If not, why not?
(c) Is \( f \) continuous at \( x = -1 \)? \( 0 \)? \( 1 \)? Explain.

30. \( f(x) = \begin{cases} 
|x^3 - 4x|, & x < 1 \\
|x^2 - 2x - 2|, & x \geq 1 
\end{cases} \)

(a) Find the right-hand and left-hand limits of \( f \) at \( x = 1 \). 1 
Left-hand limit = 3 Right-hand limit = 3
(b) Does \( f \) have a limit as \( x \to 1 \)? If so, what is it? If not, why not? No, because the two one-sided limits are different.
(c) At what points is \( f \) continuous? Every place except for \( x = 1 \)
(d) At what points is \( f \) discontinuous? At \( x = 1 \)

In Exercises 29 and 30, find the right-hand and left-hand limits of \( f \) at \( x = 1 \).

31. \( f(x) = \frac{x + 1}{4 - x^2} \)

32. \( g(x) = \sqrt{3x + 2} \)
There are no points of discontinuity.

In Exercises 33–36, find (a) a power function end behavior model and (b) any horizontal asymptotes.

33. \( f(x) = -\frac{2x + 1}{x^2 - 2x + 1} \)
(a) \( 2 \) 
(b) \( y = 0 \) (x-axis)

34. \( f(x) = \frac{2x^2 + 5x - 1}{x^2 + 2x} \)
(a) \( 2 \) 
(b) \( y = 2 \)

35. \( f(x) = \frac{x^3 - 4x^2 + 3x + 3}{3} \)
(a) \( \sqrt[3]{3} \) 
(b) None

36. \( f(x) = \frac{x^4 - 3x^2 + x - 1}{x^3 - x + 1} \)
(a) \( 1 \) 
(b) None

In Exercises 37 and 38, find (a) a right end behavior model and (b) a left end behavior model for the function.

37. \( f(x) = x + e^x \) 
(a) \( e^x \) 
(b) \( x \)

38. \( f(x) = \ln |x| + \sin x \) 
(a) \( \ln |x| \) 
(b) \( \ln |x| \)

Group Activity In Exercises 39 and 40, what value should be assigned to \( k \) to make \( f \) a continuous function?

39. \( f(x) = \begin{cases} 
\frac{x^2 + 2x - 15}{x - 3}, & x \neq 3 \\
k, & x = 3 
\end{cases} \)

40. \( f(x) = \begin{cases} 
\frac{\sin x}{2x}, & x \neq 0 \\
k, & x = 0 
\end{cases} \)

Group Activity In Exercises 41 and 42, sketch a graph of a function \( f \) that satisfies the given conditions.

41. \( \lim_{x \to \infty} f(x) = 3 \), \( \lim_{x \to -\infty} f(x) = \infty \)

42. \( \lim_{x \to 3} f(x) = \infty \), \( \lim_{x \to -3} f(x) = -\infty \)

43. Average Rate of Change Find the average rate of change of \( f(x) = 1 + \sin x \) over the interval \([0, \pi/2]\). \( \frac{2}{\pi} \)

44. Rate of Change Find the instantaneous rate of change of the volume \( V = (1/3)\pi r^2H \) of a cone with respect to the radius \( r \) at \( r = a \) if the height \( H \) does not change. \( \frac{2}{3}\pi aH \)

45. Rate of Change Find the instantaneous rate of change of the surface area \( S = 6x^2 \) of a cube with respect to the edge length \( x \) at \( x = a \). \( 12a \)

46. Slope of a Curve Find the slope of the curve \( y = x^2 - x - 2 \) at \( x = a \). \( 2a - 1 \)

47. Tangent and Normal Let \( f(x) = x^2 - 3x \) and \( P = (1, f(1)) \). Find (a) the slope of the curve \( y = f(x) \) at \( P \), (b) an equation of the tangent at \( P \), and (c) an equation of the normal at \( P \).
(a) \(-1\) 
(b) \( y = -x + 1 \) 
(c) \( y = x - 3 \)
48. **Horizontal Tangents** At what points, if any, are the tangents to the graph of \( f(x) = x^2 - 3x \) horizontal? (See Exercise 47.) \( \frac{3}{2}, -\frac{9}{4} \)

49. **Bear Population** The number of bears in a federal wildlife reserve is given by the population equation

\[
p(t) = \frac{200}{1 + 7e^{-0.1t}},
\]

where \( t \) is in years.

(a) **Writing to Learn** Find \( p(0) \). Give a possible interpretation of this number. 25. Perhaps this is the number of bears placed in the reserve when it was established.

(b) Find \( \lim_{t \to \infty} p(t) \).

(c) **Writing to Learn** Give a possible interpretation of the result in part (b).

50. **Taxi Fares** Bluetop Cab charges $3.20 for the first mile and $1.35 for each additional mile or part of a mile.

(a) Write a formula that gives the charge for \( x \) miles with \( 0 \leq x < 20 \), \( f(x) = \frac{3.20 + 1.35x}{2} \), \( 0 < x \leq 20 \), \( f(x) = \frac{3.20 + 1.35x}{2} \), \( x = 0 \), \( f(x) = \frac{3.20 + 1.35x}{2} \).

(b) Graph the function in (a). At what values of \( x \) is \( f \) discontinuous? \( f \) is discontinuous at integer values of \( x: 0, 1, 2, \ldots, 19 \).

51. **Table 2.4** Population of Florida

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>15,487</td>
</tr>
<tr>
<td>1999</td>
<td>15,759</td>
</tr>
<tr>
<td>2000</td>
<td>15,983</td>
</tr>
<tr>
<td>2001</td>
<td>16,355</td>
</tr>
<tr>
<td>2002</td>
<td>16,692</td>
</tr>
<tr>
<td>2003</td>
<td>17,019</td>
</tr>
</tbody>
</table>


(a) Let \( x = 0 \) represent 1990, \( x = 1 \) represent 1991, and so forth. Make a scatter plot for the data.

(b) Let \( P \) represent the point corresponding to 2003, \( Q_1 \) the point corresponding to 1998, \( Q_2 \) the point corresponding to 1999, \ldots, and \( Q_8 \) the point corresponding to 2002. Find the slope of the secant the \( PQ_i \) for \( i = 1, 2, 3, 4, 5 \).

(c) Predict the rate of change of population in 2003.

(d) Find a linear regression equation for the data, and use it to calculate the rate of the population in 2003.

49. (e) Perhaps this is the maximum number of bears which the reserve can support due to limitations of food, space, or other resources. Or, perhaps the number is capped at 200 and excess bears are moved to other locations.

51. (b) Slope of \( PQ_1 = 306.4 \); slope of \( PQ_2 = 315 \); slope of \( PQ_3 = 345.3 \); slope of \( PQ_4 = 332 \); slope of \( PQ_5 = 327 \).

(c) We use the average rate of change in the population from 2002 to 2003 which is 327,000.

(d) \( y = 309.457x + 12966.533 \), rate of change is 309 thousand because rate of change of a linear function is its slope.

52. **Limit Properties** Assume that
\[
\lim_{x \to a} f(x) = 3, \quad \lim_{x \to a} g(x) = 2, \quad \lim_{x \to a} h(x) = 1.
\]

And that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Find \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \).

\( \lim_{x \to a} f(x) = 3/2, \lim_{x \to a} g(x) = 1/2 \)

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**AP* Examination Preparation**

You should solve the following problems without using a graphing calculation.

53. **Free Response** Let \( f(x) = \frac{x}{|x^2 - 9|} \).

(a) Find the domain of \( f \).

(b) Write an equation for each vertical asymptote of the graph of \( f \).

(c) Write an equation for each horizontal asymptote of the graph of \( f \).

(d) Is \( f \) odd, even, or neither? Justify your answer.

(e) Find all values of \( x \) for which \( f \) is discontinuous and classify each discontinuity as removable or nonremovable.

54. **Free Response** Let \( f(x) = \frac{x^3 - 2x^2 + 1}{x^2 + 3} \).

(a) Find \( \lim_{x \to a} f(x) \), \( \lim_{x \to a} f(x) = \lim_{x \to a} -x^2 + 2x - 3 = 4 - 2a^2 \).

(b) Find \( \lim_{x \to a} f(x) \), \( \lim_{x \to a} f(x) = \lim_{x \to a} 4 - 2x^2 = -4 \).

(c) Find all values of \( a \) that make \( f \) continuous at \( 2 \). Justify your answer.

55. **Free Response** Let \( f(x) = \frac{x^3 - 2x^2 + 1}{x^2 + 3} \).

(a) Find all zeros of \( f \).

(b) Find a right end behavior model \( g(x) \) for \( f \) \( g(x) = x \).

(c) Determine \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} g(x) \).

\[
\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x) = \lim_{x \to \infty} x^3 - 2x^2 + 1 = \infty.
\]

53. (d) Odd, because \( f(-x) = \frac{-x}{|x^2 - 9|} = -\frac{x}{|x^2 - 9|} = -f(x) \) for all \( x \) in the domain.

54. (e) For \( \lim_{x \to 2} f(x) \) to exist, we must have \( 4 - 2a^2 = -4 \), so \( a = \pm 2 \). If \( a = \pm 2 \), then \( \lim_{x \to 2} f(x) = \lim_{x \to 2} f(x) = 2 \) = -4, making \( f \) continuous at 2 by definition.

55. (a) The zeros of \( f(x) = \frac{x^3 - 2x^2 + 1}{x^2 + 3} \) are the same as the zeros of the polynomial \( x^3 - 2x^2 + 1 \). By inspection, one such zero is \( x = 1 \). Divide \( x^3 - 2x^2 + 1 \) by \( x - 1 \) to get \( x^2 - x^2 - 1 \), which has zeros \( \frac{1 \pm \sqrt{5}}{2} \) by the quadratic formula. Thus, the zeros of \( f \) are \( \frac{1 + \sqrt{5}}{2} \) and \( \frac{1 - \sqrt{5}}{2} \).