SEMICIRCULANT PRECONDITIONING OF ELLIPTIC OPERATORS*

SANG DONG KIM[†] AND SEYMOUR V. PARTER[‡]

Abstract. In this work we consider the semicirculant preconditioning of elliptic differential operators of the form

$$Lu := -\epsilon \Delta u + au_x + bu_y + cu$$

in two cases: $0 < \epsilon \ll 1$ and $\epsilon \equiv 1$. The paper [*Numer. Math.*, 81 (1998), pp. 211–249] provided extremely interesting and useful results in the first case. On the other hand, those appear to contradict basic results on preconditioning given in [*SIAM J. Numer. Anal.*, 27 (1990), pp. 656–694]. We reobtain the results of [*Numer. Math.*, 81 (1998), pp. 211–249] by a new approach which we believe to be more transparent. We also clarify the situation regarding the apparent contradiction with [*SIAM J. Numer. Anal.*, 27 (1990), pp. 656–694]. Finally, we describe the distribution of the preconditioned eigenvalues in the uniformly elliptic case, $\epsilon \equiv 1$.

Key words. preconditioning, difference equations, limiting operator, convection-diffusion equation

AMS subject classifications. 65N, 65F

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1. Introduction. This work was motivated by the paper [LH], which discussed the semicirculant preconditioning of two-dimensional convection diffusion equations. The results on the distribution of the eigenvalues of the preconditioned system are extremely interesting. On the other hand, these results seemed to contradict a fundamental principle enunciated in [MP]. The basic results of [MP] imply that if one preconditions a discrete elliptic operator L_h with Dirichlet boundary conditions by another discrete elliptic operator B_h and one has results such as those of [LH], then B_h must also impose Dirichlet boundary conditions. However, it would appear that the semicirculant preconditioner has some Dirichlet boundary conditions and some periodic boundary conditions. Finally, since part of the attraction of the semicirculant preconditioner is the fact that this preconditioner is easily inverted, one is led to a study of semicirculant preconditioners for the uniformly elliptic case.

In this paper we clarify the apparent contradiction between the results of [LH] and [MP]. Simply put, the limiting operator of the difference schemes is *not* elliptic. It is the limiting hyperbolic operator when ϵ , the singular perturbation parameter, goes to zero. Since the solution of the elliptic convection-diffusion equation converges to the solution of this hyperbolic equation, the method and the preconditioning approach are reasonable. Of course, there is a boundary layer in both the solution of the differential equation and the solution of the difference equation. This method does not capture the correct boundary layer. However, that is a small point.

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[†]Department of Mathematics, Kyungpook National University, Taegu, 702-701, Korea (skim@knu.ac.kr). The research of this author was supported by Korea Research Foundation grant KRF-2002-070-C00014.

[‡]Department of Computer Sciences, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706 (parter@cs.wisc.edu).

While this clarification is both scientifically important and satisfying, it is not the main thrust of our work. Our major effort is directed at the development and exposition of a method for studying these problems which we believe is both more transparent and elementary. Using this approach we both reobtain the results of [LH] and describe the distribution of the eigenvalues of the semicirculant preconditioned system in the uniformly elliptic case. As we might expect from the results of [CC] for the full circulant preconditioner, roughly $(\frac{1}{h})$ of these eigenvalues grow like $O(\frac{1}{h})$. Still, as we shall see, for many of those eigenvalues the coefficients of the growth are quite small. Thus, given the ease of inverting the preconditioner semicirculant, preconditioning may be a useful approach for some problems. While this work focuses on the distributions of the eigenvalues, we are well aware that in these nonsymmetric problems eigenvalues do not tell the whole story as compared with the effectiveness of the preconditioning strategy; see [G].

Let $f \in C(\bar{\Omega})$, and consider the equations

$$(1.1) \quad Lu = -\epsilon \Delta u + au_x + bu_y + du = f \quad \text{in } \Omega,$$

$$(1.2) \quad u = 0 \quad \text{on } \partial\Omega.$$

Here Ω is the unit square $[0, 1] \times [0, 1]$ and the coefficients a, b, d , and ϵ are constant. Moreover,

$$(1.3) \quad d \geq 0, \quad \epsilon > 0.$$

Let m_1, m_2 be positive integers, and set

$$(1.4) \quad h_1 = \frac{1}{m_1 + 1}, \quad h_2 = \frac{1}{m_2 + 1}, \quad \varphi = \frac{h_2}{h_1}.$$

The usual centered second order finite-difference scheme which approximates (1.1) is given by

$$(1.5) \quad -\frac{\epsilon}{h_2^2} \{ Cu_{k-1,j} + Au_{k,j} + Bu_{k+1,j} + \gamma u_{k,j-1} + \alpha u_{k,j} + \beta u_{k,j+1} \} = f_{k,j}, \quad 1 \leq k \leq m_1, \quad 1 \leq j \leq m_2,$$

where

$$(1.6) \quad A = -2\varphi^2,$$

$$(1.7) \quad B = \varphi^2 \left(1 - \frac{ah_1}{2\epsilon} \right), \quad C = \varphi^2 \left(1 + \frac{ah_1}{2\epsilon} \right),$$

$$(1.8) \quad \alpha = -2 - \frac{dh_2^2}{\epsilon},$$

$$(1.9) \quad \beta = 1 - \frac{bh_2}{2\epsilon}, \quad \gamma = 1 + \frac{bh_2}{2\epsilon}.$$

Here $u_{k,j}$ is the value of the approximant at the point (x_k, y_j) and

$$(1.10) \quad f_{k,j} = f(x_k, y_j).$$

Throughout this paper we assume the mesh Peclet number condition

$$(1.11) \quad \left| \frac{bh_2}{2\epsilon} \right| < 1, \quad \left| \frac{ah_1}{2\epsilon} \right| < 1.$$

The difference operator L_h on the left-hand side of (1.5) is easily described as

$$(1.12) \quad L_h := -\frac{\epsilon}{h_2^2} \{T_2 \otimes I_{m_1} + I_{m_2} \otimes T_1\},$$

where I_{m_k} are the appropriate identity matrices, and T_1 and T_2 are tridiagonal matrices of order m_1 and m_2 , respectively. In particular,

$$(1.13) \quad T_1 = \begin{bmatrix} A & B & & & \\ C & A & B & & \\ & \ddots & \ddots & \ddots & \\ & & & C & A & B \\ & & & & C & A \end{bmatrix}$$

and

$$(1.14) \quad T_2 = \begin{bmatrix} \alpha & \beta & & & \\ \gamma & \alpha & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & & \gamma & \alpha & \beta \\ & & & & \gamma & \alpha \end{bmatrix}.$$

The semicircular preconditioner is given by

$$(1.15) \quad S := -\frac{\epsilon}{h_2^2} \{\overline{C} \otimes I_{m_1} + I_{m_2} \otimes T_1\},$$

where $\overline{C}$ is the circulant

$$(1.16) \quad \overline{C} = \begin{bmatrix} \alpha & \beta & & & \gamma \\ \gamma & \alpha & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & & \beta & \gamma & \alpha \\ \beta & & & \gamma & \alpha \end{bmatrix} = T_2 + \begin{bmatrix} 0 & \cdots & 0 & \gamma \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ \beta & 0 & \cdots & 0 \end{bmatrix}.$$

Our basic problem is the study of the eigenvalues λ which satisfy

$$(1.17) \quad \lambda S U = (S - Q \otimes I_{m_1}) U,$$

where

$$(1.18) \quad Q := -\frac{\epsilon}{h_2^2} \begin{bmatrix} 0 & \cdots & 0 & \gamma \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ \beta & 0 & \cdots & 0 \end{bmatrix}.$$

The reader familiar with [LH] will realize that we interchanged the roles of x and y . That small change leads to this form of the error term Q , which appears to be simpler than the corresponding term in [LH].

Of course, we can ignore the $(-\epsilon/h_2^2)$ term and deal with the eigenvalue problem

$$(1.19) \quad \lambda S_0 U = (S_0 - Q_0 \otimes I_{m_1})U,$$

where

$$(1.20) \quad S_0 = \{\overline{C} \otimes I_{m_1} + I_{m_2} \otimes T_1\}$$

and

$$(1.21) \quad Q_0 = \begin{bmatrix} 0 & \cdots & 0 & \gamma \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ \beta & 0 & \cdots & 0 \end{bmatrix}.$$

Since Q_0 is of rank 2, it is clear that there will be $m_1 m_2 - 2m_1$ eigenvalues λ which are exactly equal to one. We then focus on the remaining $2m_1$ eigenvalues.

In our approach we reduce the problem to m_1 problems of the form

$$(1.22) \quad \lambda C_0 u = T_0 u,$$

where T_0 is a particular $m_2 \times m_2$ tridiagonal matrix and C_0 is a related $m_2 \times m_2$ circulant. This is done as follows. We rewrite (1.17) as

$$(1.23) \quad \lambda \{\overline{C} \otimes I_{m_1} + I_{m_2} \otimes T_1\}U = \{T_2 \otimes I_{m_1} + I_{m_2} \otimes T_1\}U.$$

Assume that (τ, F) is an eigenpair for T_1 . That is,

$$(1.24) \quad T_1 F = \tau F.$$

We seek an eigenvector of (1.23) in the form $u \otimes F$. We obtain

$$(1.25) \quad \lambda[(\overline{C} + \tau I_{m_2})u \otimes F] = (T_2 + \tau I_{m_2})u \otimes F.$$

Thus, we set

$$\begin{aligned} C_0 &= \overline{C} + \tau I_{m_2}, \\ T_0 &= T_2 + \tau I_{m_2}. \end{aligned}$$

Since T_1 has m_1 distinct eigenvalues, and hence m_1 linearly independent eigenvectors, we have indeed found m_1 such simpler (one-dimensional) problems.

In dealing with these problems we consider two distinct cases.

Case 1. $0 < E \ll 1$. This is the case considered in [LH]. In this work, as in [LH], we assume

$$(1.26) \quad \frac{h_s}{\epsilon} = G_s, \quad \text{a constant, } s = 1, 2.$$

In this case the eigenvalues of (1.23) fall into three groups. There are exactly $m_1 m_2 - 2m_1$ eigenvalues equal to one. And, as $h_1, h_2 \rightarrow 0$, m_1 of the remaining eigenvalues cluster about an interval (a_1, b_1) with

$$\frac{1}{2} \leq a_1 < b_1 < 1$$

while the other m_1 eigenvalues cluster about a finite interval (c_1, d_1) with

$$1 < c_1 < d_1 < \infty.$$

These are exactly the results of [LH].

In this case the difference approximation is a poor approximation to the elliptic convection-diffusion equation [F], [WH]. On the other hand, if we imagine a sequence (or family) of computations in which (1.26) holds and $h_1 \rightarrow 0$, $h_2 \rightarrow 0$, the solutions of the difference equations converge—in every subdomain away from the edges which have the boundary layers—to an appropriate solution of the reduced hyperbolic equation

$$(1.27) \quad au_x + bu_z + du = f.$$

We will prove this in the appendix. However, it is important to point out that while that proof can be extended from the constant coefficient case to some problems with variable coefficients there are many important cases for which it cannot be extended. An example of such a case is the case of an interior “stagnation point,” i.e., a point $(\bar{x}, \bar{y})$ at which

$$a(\bar{x}, \bar{y}) = b(\bar{x}, \bar{y}) = 0.$$

Case 2. $\epsilon = 1$. In this case we are dealing with a uniformly elliptic problem and simply let $h_1 \rightarrow 0$, $h_2 \rightarrow 0$.

In this case there are exactly $m_2 m_1 - 2m_1$ eigenvalues which are equal to one. There are m_1 eigenvalues in the interval $(.38, 1)$. We believe the correct interval is $(.5, 1)$, but we cannot prove this sharper result. There are m_1 eigenvalues greater than one. The larger of these grows like cm_2 . However, many of the coefficients of growth are small. Indeed, $[m_1/2]$ of these eigenvalues are in the interval $[1, 1 + \frac{1+\sqrt{2}}{2\varphi^2}]$. These results are important for most regular problems. In addition, they are relevant for convection-diffusion equations where (1.26) does not hold. For example, the paper [LW] draws its inspiration and motivation from [LH]. However, we believe our results are equally relevant to those computations.

Analytically, the distinction between the two cases is that in Case 1 the limiting operator is *not* elliptic while in Case 2 the limiting operator is elliptic. Algebraically, the distinction concerns the eigenvalues τ . As we shall see in section 5, the eigenvalues in Case 1 satisfy

$$0 < 2\varphi^2(1 - \delta) \leq |\tau_j| \leq 2\varphi^2(1 + \delta),$$

where

$$\delta = \sqrt{1 - \left(\frac{aG_1}{2}\right)^2}.$$

On the other hand, in Case 2 the eigenvalues τ_j range from $O(h^2)$ to $O(1)$. It is the small eigenvalues τ_j which lead to the large eigenvalues λ .

In section 2 we develop the basic theory for finding the eigenvalues of $C_0^{-1}T_0$. In section 3 we use the elementary theory of one-dimensional difference equations to further extend the theory and obtain the required asymptotic estimates needed to deal with the reaction-diffusion equations. In section 4 we turn to an analysis of the problem in the case $\epsilon = 1$, i.e., the uniformly elliptic case. In section 5 we apply the results of sections 2, 3, and 4 to resolve the two-dimensional problems in both cases. In section 6 we discuss some computational results.

2. The basic theory. In this section we turn to the study of (1.22) and develop the theory for finding the eigenvalues λ of the matrix $C_0^{-1}T_0$. With this in mind we replace α by

$$(2.1) \quad \hat{\alpha} = -2(1 + D_0).$$

We are interested in a wide range of values of D_0 , not necessarily small. The matrix T_0 is the $m_2 \times m_2$ tridiagonal matrix given by

$$(2.2) \quad T_0 = \begin{bmatrix} \hat{\alpha} & \beta & & & \\ \gamma & \hat{\alpha} & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & & \beta & \\ & & & \gamma & \hat{\alpha} \end{bmatrix}$$

and C_0 is the circulant

$$(2.3) \quad C_0 = T_0 + Q_0.$$

Thus, we are concerned with the eigenvalues λ of

$$(2.4) \quad \lambda C_0 U = T U = (C_0 - Q_0)U$$

or

$$(2.5) \quad \lambda U = (I - C_0^{-1}Q_0)U.$$

Therefore $(m_2 - 2)$ eigenvalues are exactly one and there are two nontrivial eigenvalues which are of the form

$$(2.6) \quad \lambda = 1 - \rho,$$

where ρ is a nonzero eigenvalue of the problem

$$(2.7) \quad \rho C_0 U = Q_0 U.$$

We begin our discussion with the following problem: find two m_2 vectors v and w such that

$$(2.8) \quad C_0 v = \begin{bmatrix} \sigma \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad C_0 w = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \sigma \end{bmatrix},$$

where σ will be determined later.

LEMMA 2.1. Consider the $(m_2 + 1)$ vector $\hat{v}$ of the form

$$(2.9) \quad \hat{v} = \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_{m_2-1} \\ 1 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_{m_2-1} \\ v_{m_2} \end{bmatrix}$$

which satisfies the $(m_2 - 1)$ equations

$$(2.10) \quad \gamma v_{k-1} + \hat{\alpha} v_k + \beta v_{k+1} = 0, \quad k = 1, 2, \dots, (m_2 - 1).$$

Let

$$(2.11) \quad v = \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_{m_2-1} \end{bmatrix}, \quad w = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_{m_2-1} \\ 1 \end{bmatrix}.$$

Then

$$(2.12) \quad C_0 v = \begin{bmatrix} \sigma \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad C_0 w = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \sigma \end{bmatrix},$$

where

$$(2.13) \quad \sigma = \hat{\alpha} + \beta v_1 + \gamma v_{m_2-1}.$$

Moreover,

$$(2.14) \quad 0 < v_j \leq 1.$$

Proof. Direct verification yields (2.12) and (2.13). Since the Peclet condition (1.11) holds and β, γ are positive, the bound (2.14) follows from a standard maximum principle (convexity) argument. $\square$

We seek an eigenvector U of (2.7) of the form

$$(2.15) \quad U = xv + yw.$$

Then

$$(2.16) \quad C_0(xv + yw) = \begin{bmatrix} x\sigma \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ y\sigma \end{bmatrix}$$

and

$$(2.17) \quad Q_0(xv + yw) = \begin{bmatrix} (\gamma v_{m_2-1})x & + & \gamma y \\ & 0 & \\ & \cdot & \\ & \cdot & \\ & 0 & \\ \beta x & + & (\beta v_1)y \end{bmatrix}.$$

LEMMA 2.2. *Let M be the matrix*

$$(2.18) \quad M = \begin{bmatrix} \gamma v_{m_2-1} & 0 & \cdots & 0 & \gamma \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ \beta & 0 & \cdots & 0 & \beta v_1 \end{bmatrix}.$$

Let $[x, 0, \dots, 0, y]^T$ be an eigenvector of M with associated nonzero eigenvalues μ . Then

$$(2.19) \quad U = (xv + yw)$$

satisfies

$$(2.20) \quad \left(\frac{\mu}{\sigma}\right) C_0(xv + yw) = \frac{\mu}{\sigma} C_0 U = Q_0 U.$$

Thus, U is an eigenvector and $\mu/\sigma = \rho$ is an eigenvalue of the eigenvalue problem (2.7).

Proof. From (2.16) and (2.17) we have

$$(2.21) \quad \frac{\mu}{\sigma} C_0(xv + yw) = \mu \begin{bmatrix} x \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ y \end{bmatrix}$$

and

$$(2.22) \quad Q_0(xv + yw) = M \begin{bmatrix} x \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ y \end{bmatrix} = \mu \begin{bmatrix} x \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ y \end{bmatrix}.$$

Therefore the lemma is proven. $\square$

THEOREM 2.3. *Let*

$$(2.23) \quad R = (\gamma v_{m_2-1} + \beta v_1).$$

Then the nonzero eigenvalues of M are given by

$$(2.24) \quad \mu = \frac{R \pm \sqrt{R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}}{2}$$

and the eigenvalues ρ of (2.7) are given by

$$(2.25) \quad \rho = \frac{R \pm \sqrt{R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}}{2(-2 + R - 2D_0)}$$

and the value “0” ($m_2 - 2$) times.

Proof. The theorem is proven by a direct computation. $\square$

LEMMA 2.4. Let D_0 be a number of order 1. Consider ρ_+ given by

$$(2.26) \quad \rho_+ = \frac{R + \sqrt{R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}}{2(-2 + R - 2D_0)}.$$

Then $\rho_+ < 0$ and

$$(2.27) \quad |\rho_+| \leq \frac{1 + \sqrt{2}}{2D_0}.$$

Proof. The lemma is proven by observing the following:

$$0 \leq R \leq 2, \quad 0 \leq 4\gamma\beta(1 - v_1 v_{m_2-1}) \leq 4,$$

and

$$|-2 + R - 2D_0| \geq 2D_0. \quad \square$$

3. The difference equations. In this section we study the eigenvalues ρ_+ and ρ_- and their dependence on the parameter D_0 . This involves a rather technical discussion of the properties of the vector $\hat{v}$ described by (2.9) and (2.10). This discussion uses differentiation with respect to $(2D_0)$.

As before, we assume (1.11). Since h_2 and ϵ are always positive, we have

$$(3.1) \quad \frac{|b|h_2}{2\epsilon} < 1.$$

As is well known [H] the $\{v_j; j = 1, 2, \dots, (m_2 - 1)\}$ are linear combinations of the roots S_1, S_2 of the quadratic equation

$$(3.2) \quad \beta S^2 + \hat{\alpha} S + \gamma = 0.$$

LEMMA 3.1. Under the assumption (1.11) the roots of (3.2)

$$(3.3) \quad S_1 = \frac{-\hat{\alpha} + \sqrt{\hat{\alpha}^2 - 4\gamma\beta}}{2\beta}, \quad S_2 = \frac{-\hat{\alpha} - \sqrt{\hat{\alpha}^2 - 4\gamma\beta}}{2\beta}$$

are both positive. If $D_0 > 0$, then

$$(3.4) \quad 0 < S_2 < 1 < S_1.$$

Finally, the solution of (2.10) subject to the boundary condition $v_0 = v_{m_2} = 1$ is given by

$$(3.5) \quad v_k = \frac{(S_1^{m_2} - 1)S_2^k + (1 - S_2^{m_2})S_1^k}{S_1^{m_2} - S_2^{m_2}}.$$

Proof. The formulae (3.3) are elementary. Since

$$\hat{\alpha}^2 - 4\gamma\beta > 4 - 4 \left(1 - \frac{(bh_2)^2}{4\epsilon^2} \right) = \left(\frac{bh_2}{\epsilon} \right)^2 \geq 0,$$

the roots S_1 and S_2 are distinct and positive. Furthermore,

$$(3.6) \quad S_1 \geq \frac{2(1 + D_0) + \left| \frac{bh_2}{\epsilon} \right|}{2 + \left| \frac{bh_2}{\epsilon} \right|} \geq 1$$

and strictly greater than one if $D_0 > 0$.

If $b \leq 0$, then

$$(3.7) \quad S_1 S_2 = \frac{\gamma}{\beta} = \frac{1 - \left| \frac{bh_2}{2\epsilon} \right|}{1 + \left| \frac{bh_2}{2\epsilon} \right|} \leq 1.$$

Since $S_1 > 1$, we see that $S_2 < 1$.

On the other hand, if $b > 0$, we observe that

$$(3.8) \quad \theta_1 = \frac{1}{S_2}, \quad \theta_2 = \frac{1}{S_1}$$

are the roots of the equation $\gamma\theta^2 + \hat{\alpha}\theta + \beta = 0$. The previous argument shows that

$$\theta_1 = \frac{1}{S_2} > 1, \quad \theta_2 = \frac{1}{S_1} < 1.$$

Thus, we have established (3.4). Finally, the formula (3.5) is verified by evaluation of v_0 and v_{m_2} . $\square$

Let “.” represent differentiation with respect to $(2D_0)$. That is, for any quantity m ,

$$(3.9) \quad \frac{\partial}{\partial(2D_0)} m := \dot{m}.$$

LEMMA 3.2. For $D_0 > 0$ we have

$$(3.10) \quad \dot{S}_1 = \left[1 + \frac{-\hat{\alpha}}{\sqrt{\hat{\alpha}^2 - 4\gamma\beta}} \right] / 2\beta = \left[\frac{S_1}{\sqrt{\hat{\alpha}^2 - 4\gamma\beta}} \right] > 0,$$

$$(3.11) \quad \dot{S}_2 = \left[1 - \frac{-\hat{\alpha}}{\sqrt{\hat{\alpha}^2 - 4\gamma\beta}} \right] / 2\beta = \left[-\frac{S_2}{\sqrt{\hat{\alpha}^2 - 4\gamma\beta}} \right] < 0.$$

Moreover,

$$(3.12) \quad \dot{v}_1 < 0, \quad \dot{v}_{m_2-1} < 0.$$

Proof. The formulae (3.10), (3.11) are established by a simple computation. Let us consider $v_1 = v_1(D_0)$. We have

$$(3.13) \quad v_1 = \frac{(S_1^{m_2} - 1)S_2 + (1 - S_2^{m_2})S_1}{S_1^{m_2} - S_2^{m_2}}.$$

A computation yields

$$(3.14) \quad \dot{v}_1 = Z_1 + Z_2,$$

where

$$(3.15) \quad Z_1 = \frac{\dot{S}_1(1 - S_2^{m_2})\{(S_1^{m_2} - S_2^{m_2}) - m_2(S_1 - S_2)S_1^{m_2-1}\}}{(S_1^{m_2} - S_2^{m_2})^2},$$

$$(3.16) \quad Z_2 = \frac{\dot{S}_2(S_1^{m_2} - 1)\{(S_1^{m_2} - S_2^{m_2}) - m_2(S_1 - S_2)S_2^{m_2-1}\}}{(S_1^{m_2} - S_2^{m_2})^2}.$$

We have

$$S_1^{m_2} - S_2^{m_2} = S_1^{m_2} \left(1 - \frac{S_2}{S_1}\right) \sum_{k=0}^{m_2-1} \left(\frac{S_2}{S_1}\right)^k.$$

Thus,

$$(S_1^{m_2} - S_2^{m_2}) < m_2(S_1 - S_2)S_1^{m_2-1}.$$

Since $\dot{S}_1 > 0$ and $0 < S_2 < 1$, we see that

$$(3.17) \quad Z_1 < 0.$$

As for Z_2 , we have

$$S_1^{m_2} - S_2^{m_2} = S_2^{m_2-1}(S_1 - S_2) \sum_{k=0}^{m_2-1} \left(\frac{S_1}{S_2}\right)^k.$$

Thus,

$$(S_1^{m_2} - S_2^{m_2}) > m_2 S_2^{m_2-1}(S_1 - S_2).$$

Since $\dot{S}_2 < 0$ and $S_1 > 1$, we see that

$$(3.18) \quad Z_2 < 0.$$

Therefore we have

$$(3.19) \quad \dot{v}_1 < 0.$$

The proof that $\dot{v}_{m_2-1} < 0$ follows from the observation that

$$\begin{aligned} v_{m_2-1} &= \frac{(\theta_1^{m_2} - 1)\theta_2 + (1 - \theta_2^{m_2})\theta_1}{\theta_1^{m_2} - \theta_2^{m_2}}, \\ \theta_1 &= \frac{-\hat{\alpha} + \sqrt{\hat{\alpha}^2 - 4\gamma\beta}}{2\gamma} = \frac{1}{S_2}, \\ \theta_2 &= \frac{-\hat{\alpha} - \sqrt{\hat{\alpha}^2 - 4\gamma\beta}}{2\gamma} = \frac{1}{S_1}, \end{aligned}$$

and the previous argument applies. $\square$

LEMMA 3.3. *Let R and ρ_+ be defined as in the previous section. Then*

$$(3.20) \quad \dot{R} < 0, \quad -\dot{\rho}_+ < 0.$$

Proof. We have $\dot{R} < 0$ because of Lemma 3.2. And after some algebra we see that

$$(3.21) \quad -\dot{\rho}_+ \leq \frac{2[2R\dot{R} - 2\gamma\beta(\dot{v}_1 v_{m_2-1} + v_1 \dot{v}_{m_2-1})][2 + 2D_0 - R]}{[2(2 + 2D_0 - R)]^2 \sqrt{R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}} - 2 \frac{(1 - \dot{R})[R + \sqrt{R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}]}{[2(2 + 2D_0 - R)]^2}.$$

The second term of (3.21) is negative. Hence we need prove only that

$$2R\dot{R} - 2\gamma\beta(\dot{v}_1 v_{m_2-1} + v_1 \dot{v}_{m_2-1}) < 0.$$

However, after some algebra we see that this reduces to

$$(3.22) \quad 2(\gamma^2 v_{m_2-1} \dot{v}_{m_2-1} + \beta^2 v_1 \dot{v}_1) < 0. \quad \square$$

Remark 3.4. Since $\rho_+ < 0$, this implies that

$$(3.23) \quad |\rho_+| \text{ decreases as } D_0 \text{ increases.}$$

We believe that

$$|\rho_-| \text{ decreases as } D_0 \text{ increases,}$$

but we cannot prove it.

Remark 3.5. This lemma, and (3.23) in particular, enables us to study the largest eigenvalues in Case 2 by studying the smallest eigenvalues τ_j which are $O(h^2)$. In those cases we can estimate v_1 and v_{m_2-1} by studying a particular boundary value problem for an ordinary differential equation. We shall see this in section 4.

While the next theorem is valid in both cases, it is used primarily in Case 1, where it yields the asymptotic distribution of the eigenvalues.

THEOREM 3.6. *Assume that*

$$(3.24) \quad D_0 = D_0(h_2) \geq D_1 > 0,$$

where D_1 is a constant. Further assume that

$$(3.25) \quad D_0(h_2) \rightarrow \overline{D}_0 \quad \text{as } h_2 \rightarrow 0.$$

Observe that as $h_2 \rightarrow 0$ the quantities γ and β have limits $\gamma^\infty, \beta^\infty$. In Case 1, where (1.26) holds, we have

$$(3.26) \quad \beta \equiv 1 - \frac{b}{2}G_2 = \beta^\infty, \quad \gamma \equiv 1 + \frac{h}{2}G_2 = \gamma^\infty.$$

In Case 2, where $\epsilon \equiv 1$, we have

$$(3.27) \quad \beta \rightarrow 1 = \beta^\infty, \quad \gamma \rightarrow 1 = \gamma^\infty \quad \text{as } h_1, h_2 \rightarrow 0.$$

Furthermore,

$$(3.28) \quad S_1 \rightarrow S_1^\infty = \frac{(1 + \overline{D}_0) + \sqrt{(1 + \overline{D}_0)^2 - \gamma^\infty \beta^\infty}}{\beta^\infty},$$

$$(3.29) \quad S_2 \rightarrow S_2^\infty = \frac{(1 + \overline{D}_0) - \sqrt{(1 + \overline{D}_0)^2 - \gamma^\infty \beta^\infty}}{\beta^\infty}.$$

Then as $m_2 \rightarrow \infty$ (i.e., $h_2 \rightarrow 0$) we have

$$(3.30) \quad -\rho_+ = |\rho_+| \rightarrow -\rho_+^\infty = \frac{(1 + \overline{D}_0) + \sqrt{\gamma^\infty \beta^\infty} - Z}{2Z},$$

$$(3.31) \quad \rho_- = |\rho_-| \rightarrow \rho_-^\infty = \frac{Z + \sqrt{\gamma^\infty \beta^\infty} - (1 + \overline{D}_0)}{2Z},$$

where

$$(3.32) \quad Z = \sqrt{(1 + \overline{D}_0)^2 - \gamma^\infty \beta^\infty}.$$

Proof. Under the hypotheses (3.24) and (3.25) the remarks before (3.30) and (3.31) are obvious. We proceed to prove (3.30), (3.31).

Since

$$(3.33) \quad v_1 = \frac{(S_1^{m_2} - 1)S_2 + (1 - S_2^{m_2})S_1}{S_1^{m_2} - S_2^{m_2}}$$

and

$$(3.34) \quad v_{m_2-1} = \frac{(1 - S_1^{-m_2})S_2^{m_2-1} + (1 - S_2^{m_2})S_1^{-1}}{1 - \left(\frac{S_2}{S_1}\right)^{m_2}},$$

we see that as $m_2 \rightarrow \infty$ ($h_2 \rightarrow 0$) we have

$$(3.35) \quad v_1 \rightarrow v_1^\infty = S_2^\infty,$$

$$(3.36) \quad v_{m_2-1} \rightarrow v_{m_2-1}^\infty = \frac{1}{S_1^\infty}.$$

Hence

$$(3.37) \quad R \rightarrow R^\infty = \frac{\gamma^\infty}{S_1^\infty} + \beta^\infty S_2^\infty.$$

However,

$$(3.38) \quad \frac{\gamma^\infty}{S_1^\infty} = \beta^\infty S_2^\infty.$$

Hence

$$(3.39) \quad -\rho_+^\infty = \frac{R^\infty + 2\sqrt{\gamma^\infty \beta^\infty}}{2[2(1 - \beta^\infty S_2^\infty) + 2\overline{D}_0]},$$

$$(3.40) \quad \rho_-^\infty = \frac{2\sqrt{\gamma^\infty \beta^\infty} - R^\infty}{2[2(1 - \rho^\infty S_2^\infty) + 2\overline{D}_0]}.$$

The theorem now follows from algebraic manipulation using the fact that

$$\beta^\infty S_2^\infty = (1 + \overline{D}_0) - \sqrt{(1 + \overline{D}_0)^2 - \gamma^\infty \beta^\infty}. \quad \square$$

COROLLARY 3.7.

$$(3.41) \quad -\dot{\rho}_+^\infty < 0, \quad \dot{\rho}_-^\infty < 0,$$

and

$$(3.42) \quad \rho_-^\infty \leq 1/2.$$

4. The case $\epsilon = 1$ in one dimension. In this section we consider the case $\epsilon \equiv 1$ and $h_2 = \frac{1}{m_2+1} \rightarrow 0$. At first we consider the case where

$$2D_0 = \hat{d}h_2^2$$

and $\hat{d}$ is a fixed constant of modest size. In this case the solutions of the equations

$$(4.1) \quad \frac{1}{h_2^2} [\gamma v_{k-1} + \hat{\alpha} v_k + \beta v_{k+1}] = 0, \quad k = 1, 2, \dots, (m_2 - 1),$$

$$(4.2) \quad v_0 = v_{m_2} = 1$$

approximate the function $u_h(x)$ which satisfies

$$(4.3) \quad -u_h'' + bu_h' + \hat{d}u_h = 0, \quad 0 \leq x \leq 1 - h_2,$$

$$(4.4) \quad u_h(0) = u_h(1 - h_2) = 1.$$

We use this fact to obtain reasonable estimates for ρ_+, ρ_- , and

$$(4.5) \quad \lambda_+ = 1 - \rho_-, \quad \lambda_- = 1 - \rho_+.$$

The solution of (4.3) and (4.4) is

$$(4.6) \quad u_h(x) = \frac{(e^{M_1(1-h_2)} - 1)e^{M_2x} + (1 - e^{M_2(1-h_2)})e^{M_1x}}{e^{M_1(1-h_2)} - e^{M_2(1-h_2)}},$$

where

$$(4.7) \quad M_1 = \frac{b + \sqrt{b^2 + 4\hat{d}}}{2}, \quad M_2 = \frac{b - \sqrt{b^2 + 4\hat{d}}}{2}.$$

Therefore

$$(4.8) \quad u_h'(0) = \frac{M_2(e^{M_1(1-h_2)} - 1) + M_1(1 - e^{M_2(1-h_2)})}{e^{M_1(1-h_2)} - e^{M_2(1-h_2)}}$$

and

$$(4.9) \quad u'_h(1-h_2) = \frac{M_2(e^{M_1(1-h_2)} - 1)e^{M_2(1-h_2)} + M_1(1 - e^{M_2(1-h_2)})e^{M_1(1-h_2)}}{e^{M_1(1-h_2)} - e^{M_2(1-h_2)}}.$$

We observe that

$$u'_h(0) < 0, \quad u'_h(1-h_2) > 0.$$

Moreover, there is a d_0 and an $H > 0$ such that for $\hat{d} \geq d_0$ and $0 < h_2 < H$ we have

$$|u'(0)| > \frac{|b|}{2}, \quad |u'(1)| > \frac{|b|}{2}.$$

Hence, for $\hat{d} \geq d_0$ we have

$$(4.10) \quad \beta v_1 \approx \left(1 - \frac{bh_2}{2}\right) (1 + u'_h(0)h_2) < 1$$

and

$$(4.11) \quad \gamma v_{m_2-1} \approx \left(1 + \frac{bh_2}{2}\right) (1 - u'_h(1-h_2)h_2) < 1.$$

Since v_1 and v_{m_2-1} decrease with increasing D_0 , once (4.10) and (4.11) hold for some $D_0 > 0$, they hold for all larger D_0 . These inequalities will be important in the latter part of this section, where we discuss the case of large D_0 and we cannot use (4.6) to approximate v_1 and v_{m_2-1} .

We recall that

$$\sigma = \hat{\alpha} + \beta v_1 + \gamma v_{m_2-1}.$$

Since

$$(4.12) \quad v_1 = u_h(h_2) + O(h_2^3), \quad v_{m_2-1} = u_h(1-2h_2) + O(h_2^3),$$

we see that

$$(4.13) \quad \begin{aligned} \sigma = & \left[\beta u_h(h_2) - \left[1 + \frac{\hat{d}h_2^2}{2} \right] u_h(0) \right] \\ & + \left[\gamma u_h(1-2h_2) - \left[1 + \frac{\hat{d}h_2^2}{2} \right] u_h(1-h_2) \right] + O(h_2^3). \end{aligned}$$

A careful computation using the differential equation shows that

$$(4.14) \quad \sigma = s_h h_2 + O(h_2^3),$$

where

$$(4.15) \quad s_h = u'_h(0) - u'_h(1-h_2).$$

Therefore, after a lengthy computation we see that

$$(4.16) \quad s_h = s_0 [1 + K_0 h_2 + K_1 h_2],$$

where

$$(4.17) \quad s_0 = -\frac{e^{M_1} + e^{M_2} - 1 - e^b}{e^{M_1} - e^{M_2}} \sqrt{b^2 + 4\hat{d}} = u'_0(0) - u'_0(1),$$

$$(4.18) \quad K_0 = \frac{be^b - (M_1e^{M_1} + M_2e^{M_2})}{(e^{M_1} - 1)(1 - e^{M_2})},$$

and

$$(4.19) \quad K_1 = \frac{M_1e^{M_1} - M_2e^{M_2}}{e^{M_1} - e^{M_2}}.$$

We also require the quantity

$$(4.20) \quad W = [u'_0(0) + u'_0(1)]^2 - 2b[u'_0(0) + u'_0(1)].$$

We are now prepared to state and prove the main theorem for this case.

THEOREM 4.1. *For $2D_0 = \hat{d}h_2^2$ and h_2 small we have*

$$(4.21) \quad \rho_+ = \frac{1}{2} + \frac{2}{\sigma} + O(h_2)$$

and

$$(4.22) \quad \rho_- = \frac{1}{2} + O(h_2).$$

Therefore

$$(4.23) \quad \lambda_+ = 1 - \rho_+ = \frac{1}{2} + \frac{2}{|s_h|h_2} + O(h_2),$$

or

$$(4.24) \quad \lambda_+ = \frac{1}{2} + \frac{2(K_0 + K_1)}{s_0} + \frac{2}{|s_0|h_2} + O(h_2),$$

and

$$(4.25) \quad \lambda_- = 1 - \rho_- = \frac{1}{2} + \frac{h_2^2}{|s_h|} \left(\frac{\hat{d}}{2} - \frac{W}{8} \right) + O(h_2) = \frac{1}{2} + O(h_2).$$

Proof. Since (see (2.26))

$$|\rho_+| = \frac{R + \sqrt{(\beta v_1 + \gamma v_{m_2-1})^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}}{2|\sigma|},$$

therefore

$$(4.26) \quad |\rho_+| = \frac{R + \sqrt{(\beta v_1 - \gamma v_{m_2-1})^2 + 4 - (bh_2)^2}}{2|\sigma|}.$$

Straightforward computations yield

$$(4.27) \quad R = 2 + \sigma + \hat{d}h_2^2$$

and

$$(4.28) \quad (\beta v_1 - \gamma v_{m_2-1})^2 = h_2^2 \{W + b^2\} + O(h_2^2).$$

Hence

$$\rho_+ \approx \frac{2 + \sigma + \hat{d}h_2^2 + \sqrt{4 + Wh_2^2 + O(h_2^2)}}{2\sigma}.$$

Thus,

$$(4.29) \quad \rho_+ \approx \frac{2}{\sigma} + \frac{1}{2} + O\left(\frac{h_2^2}{\sigma}\right).$$

Hence we have proven (4.21). A similar calculation yields (4.22). Finally, (4.24) follows from (4.29), (4.14), and (4.16). $\square$

We now turn to those cases where $2D_0$ is not necessarily small. We recall that for $d \geq d_0$ we have (4.10) and (4.11).

THEOREM 4.2. *Let*

$$(4.30) \quad \gamma v_{m_2-1} < 1, \quad \beta v_1 < 1.$$

Consider ρ_- given by

$$(4.31) \quad \rho_- = \frac{R - \sqrt{R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1})}}{2(-2 + R - 2D_0)}.$$

Then

$$(4.32) \quad |\rho_-| \leq \frac{\sqrt{5} - 1}{2} \approx .618.$$

Proof. Let

$$(4.33) \quad 1 - \gamma v_{m_2-1} = p, \quad 1 - \beta v_1 = r.$$

Observe that

$$(4.34) \quad R^2 + 4\gamma\beta(1 - v_1 v_{m_2-1}) = (\gamma v_{m_2-1} - \beta v_1)^2 + 4\gamma\beta$$

and

$$(4.35) \quad \gamma v_{m_2-1} - \beta v_1 = r - p.$$

Let

$$(4.36) \quad \theta = \frac{\sqrt{5} - 1}{2}.$$

The assertion (4.32) is equivalent to the assertion

$$\frac{\sqrt{(r-p)^2 + 4\gamma\beta} - R}{2[(2-R) + 2D_0]} \leq \theta.$$

Now $2 - R = (p + r)$, so this assertion is equivalent to the statement

$$\sqrt{(r-p)^2 + 4\gamma\beta} \leq 2\theta[2 - R + 2D_0] + R$$

or

$$\sqrt{(r-p)^2 + 4\gamma\beta} \leq (2\theta - 1)(p+q) + 2(1 + 2\theta D_0).$$

Since both sides of this inequality are positive, this inequality is equivalent to

$$(r-p)^2 + 4\gamma\beta = (2\theta - 1)^2(p+r)^2 + 4(2\theta - 1)(1 + 2\theta D_0)(p+r) + 4(1 + 2\theta D_0)^2.$$

This inequality is equivalent to

$$4\gamma\beta \leq [(2\theta - 1)^2 - 1](p^2 + r^2) + 2[(2\theta - 1)^2 + 1]pr + 4(2\theta - 1)(1 + 2\theta D_0)(p+r) + 4(1 + 2\theta D_0)^2.$$

Since $|p|$ and $|r|$ are each less than one and $4\gamma\beta \leq 4$, it is sufficient to prove that

$$4 \leq [(2\theta - 1)^2 + 4(2\theta - 1) - 1][p^2 + r^2] + 4.$$

However, $2\theta - 1 = (\sqrt{5} - 2)$ is a root of

$$m^2 + 4m - 1.$$

Hence the inequality is proven. $\square$

Remark 4.3. Notice that while this estimate is sufficient for our purposes, we have made no effort to make a sharp estimate.

5. Two dimensions. We now return to the basic eigenvalue problem (1.23). From the discussion following (1.22) we must determine the eigenvalues τ_j of T_1 . However, these are known. We have the following lemma.

LEMMA 5.1. *Consider the matrix T_1 given by (1.13) with A, B, C given by (1.6) and (1.7). The eigenvalues of T_1 are*

$$(5.1) \quad \tau_j = A + 2\sqrt{BC} \cos \pi j h_1.$$

Proof. This is a well-known result; see [P]. $\square$

Our first basic theorem is for the case of the convection-diffusion equation studied in [LH]. However, we must remark that due to differences in notation we have not attempted to show the exact equivalence of these formulae and those of [LH].

Unfortunately, the description of these eigenvalues requires quite a bit of notation. Our basic result is the following theorem.

THEOREM 5.2. *Let $|a| > 0$. Let $0 < \epsilon \ll 1$, and let*

$$(5.2) \quad \delta = \sqrt{1 - \left(\frac{aG_1}{2}\right)^2} = \varphi^2 \sqrt{BC},$$

$$(5.3) \quad D_m = \varphi^2(1 - \delta),$$

$$(5.4) \quad D_M = \varphi^2(1 + \delta),$$

$$(5.5) \quad X_m = 1 + D_m,$$

$$(5.6) \quad X_M = 1 + D_M,$$

$$(5.7) \quad \delta_1 = \sqrt{1 - \left(\frac{bG_2}{2}\right)^2},$$

$$(5.8) \quad Y_m = \sqrt{X_m^2 - \delta_1^2},$$

$$(5.9) \quad Y_M = \sqrt{X_M^2 - \delta_1^2},$$

$$(5.10) \quad c_1 = 1 + \frac{X_M + \delta_1 - Y_M}{2Y_M},$$

$$(5.11) \quad d_1 = 1 + \frac{X_m + \delta_1 - Y_m}{2Y_m},$$

$$(5.12) \quad a_1 = 1 - \frac{Y_M + \delta_1 - X_M}{2Y_M},$$

$$(5.13) \quad b_1 = 1 - \frac{Y_m + \delta_1 - X_m}{2Y_m}.$$

Then the eigenvalues λ_j of (1.23) subject to the conditions (1.26) fall into three groups: (i) There are $m_1 m_2 - 2m_1$ eigenvalues which are exactly one. (ii) There are m_1 eigenvalues which lie in an interval $(\bar{a}_1(h), \bar{b}_1(h))$ and

$$(5.14) \quad \bar{a}_1(h) \rightarrow a_1 \geq \frac{1}{2} \text{ as } h_1, h_2 \rightarrow 0$$

and

$$(5.15) \quad \bar{b}_1(h) \rightarrow b_1 \leq 1 \text{ as } h_1, h_2 \rightarrow 0.$$

(iii) Finally, there are m_1 eigenvalues which lie in a finite interval $(\bar{c}_1(h), \bar{d}_1(h))$ and

$$(5.16) \quad \bar{c}_1(h) \rightarrow c_1 \geq 1 \text{ as } h_1, h_2 \rightarrow 0$$

and

$$(5.17) \quad \bar{d}_1(h) \rightarrow d_1 < \infty \text{ as } h_1, h_2 \rightarrow 0.$$

Proof. In this case the eigenvalues τ_j are given by

$$(5.18) \quad \tau_j = 2\varphi^2(1 - \delta \cos \pi j h_1).$$

Therefore the argument in the introduction shows that we are led to study the problems of sections 2 and 3 with

$$D_0 = D_0(j) = \varphi^2(1 - \delta \cos \pi j h_1) + \frac{dh_2^2}{2\epsilon}.$$

And for all $j = 1, 2, \dots, m_1$ we have

$$(5.19) \quad D_m \leq D_0(j) \leq D_M.$$

We now apply Theorem 3.6 with $D_0 = D_m$ and $D_0 = D_M$, together with (3.41) and (3.42), to see that the limiting eigenvalues all fall within the indicated intervals. $\square$

We now turn to the uniformly elliptic case with $\epsilon \equiv 1$.

THEOREM 5.3. *Let $\epsilon \equiv 1$. Let j be a fixed integer, and set*

$$(5.20) \quad \hat{d}(j) = d - \frac{\tau_j}{h_2^2}.$$

Let

$$(5.21) \quad M_1 = M_1(j) = \frac{b + \sqrt{b^2 + 4\hat{d}(j)}}{2}, \quad M_2 = M_2(j) = \frac{b - \sqrt{b^2 + 4\hat{d}(j)}}{2}$$

and

$$(5.22) \quad K_0(j) = \frac{be^b - [M_1(j)e^{M_1(j)} + M_2(j)e^{M_2(j)}]}{(e^{M_1(j)} - 1)(1 - e^{M_2(j)})},$$

$$(5.23) \quad K_1(j) = \frac{M_1(j)e^{M_1(j)} - M_2(j)e^{M_2(j)}}{e^{M_1(j)} - e^{M_2(j)}}.$$

Let

$$(5.24) \quad \bar{s} = \bar{s}(j) = -\frac{e^{M_1} + e^{M_2} - 1 - e^b}{e^{M_1} - e^{M_2}} \sqrt{b + 4\hat{d}(j)}.$$

Let

$$\lambda_1 > \lambda_2 > \dots > \lambda_{m_1}$$

be the m_1 largest eigenvalues of (1.23). Then as $h_1, h_2 \rightarrow 0$ we have

$$(5.25) \quad \lambda_j \approx \frac{1}{2} + \frac{2[K_0(j) + K_1(j)]}{\bar{s}(j)} + \frac{1}{h_2} \frac{2}{|s(j)|}.$$

Furthermore, for $1 \leq j < s \leq m_1$ we have

$$(5.26) \quad 1 \leq \lambda_s \leq \lambda_j.$$

Finally, for $m_1/2 \leq j \leq m_1$ we have

$$(5.27) \quad 1 \leq \lambda_j \leq 1 + \frac{1 + \sqrt{2}}{2\varphi^2}.$$

Proof. For each j we have the situation described in Theorem 4.1 with $\hat{d}$ given by $\hat{d}(j)$ in (5.20). Hence (5.25) follows from Theorem 4.1. The estimate (5.26) follows from Lemma 3.3. Finally, the estimate (5.27) follows from the fact that

$$|\tau_j| \geq 2\varphi^2, \quad \frac{m_1}{2} < j \leq m_1,$$

and from (2.27). $\square$

Remark 5.4. We note that

$$(5.28) \quad \frac{\tau_j}{h_2^2} \rightarrow -\left(\frac{a^2}{4} + (\pi j)^2\right) = -\mu_j.$$

And the quantity μ_j is precisely the j th eigenvalue of the operator

$$L_a := -\left(\frac{d}{du}\right)^2 + a\frac{d}{du}.$$

Furthermore, for very large j

$$\bar{s}(j) \rightarrow \sqrt{b^2 + 4\mu_j} \approx 2(j\pi).$$

Hence

$$\lambda_j \approx K_2 + \frac{1}{h_2} \frac{1}{(j\pi)},$$

where K_2 is a constant. Thus, for j modestly large the coefficient of $\frac{1}{h_2}$ is quite small. And (5.26) and (5.27) assure us that it is even smaller for larger values of j .

THEOREM 5.5. *There are m_1 eigenvalues $\lambda(h)$ of (1.23) which satisfy*

$$(5.29) \quad .38 < \frac{3 - \sqrt{5}}{2} \leq \lambda \leq 1$$

as $h_1, h_2 \rightarrow 0$.

Proof. For any fixed j we have the results of Theorem 4.2 which yields

$$\lambda_- \approx \frac{1}{2} - h_2 \bar{S},$$

where $\bar{S}$ is a constant depending on b and j .

On the other hand, if j is fixed and so large that (4.30) holds, we have (4.32) which implies that (5.29) holds for all j . $\square$

6. Numerical tests. In this section we provide a series of numerical examples to illustrate the theories developed in the previous sections. For this, we take the following examples by recalling (1.4) and (1.11).

Example 1. Consider

$$-\epsilon \Delta u + 2u_x + u_y.$$

We fix $\frac{h}{\epsilon} = \frac{3}{4}$, where $h = h_1 = h_2$ so that $\epsilon = \frac{4}{3}h$. The non-one eigenvalues are in either $[0.7136, .84762]$ or $[1.2186, 1.67316]$, which can be predicted by Theorem 5.2. The two groups of eigenvalues are listed in Tables 6.1 and 6.2.

Example 2. Consider

$$-\epsilon \Delta u + 3u_x + 3u_y + u.$$

We fix

$$\frac{h}{\epsilon} = \frac{1}{6}, \quad \text{where } h = h_1 = h_2$$

TABLE 6.1
Two groups of eigenvalues for $-\epsilon\Delta u + 2u_x + u_y$.

Mesh size	Eigenvalues	Eigenvalues
$h = \frac{1}{12}, \quad \epsilon = \frac{1}{9}$	0.71835511876813	1.22163948354662
	0.73076629515934	1.22913287556744
	0.74816845617320	1.24219275206102
	$\vdots$	$\vdots$
	0.83684051876625	1.50835544156721
	0.84287352767439	1.58621137822330
$h = \frac{1}{20}, \quad \epsilon = \frac{1}{15}$	0.84643374383331	1.65101906814222
	0.71482281489471	1.22007306632850
	0.71976536398126	1.22272233383939
	0.72742605877215	1.22721010848919
	$\vdots$	$\vdots$
	0.84377959972260	1.59928967271870
	0.84591431356486	1.63764397287832
	0.84718698934437	1.66385884311555

TABLE 6.2
Two groups of eigenvalues for $-\epsilon\Delta u + 2u_x + u_y$.

Mesh size	Eigenvalues	Eigenvalues
$h = \frac{1}{36}, \quad \epsilon = \frac{1}{27}$	0.71363243881612	1.21946709926894
	0.71520966610960	1.22027908908089
	0.71777733892708	1.22163902770585
	$\vdots$	$\vdots$
	0.84643342008918	1.64796148998121
	0.84708778274694	1.66165790078332
	0.84747937636160	1.67023428901938

so that

$$\epsilon = 6h.$$

The non-one eigenvalues are in either $[0.58861, .8563394]$ or $[1.2014, 3.23054]$, which can be predicted by Theorem 5.2. The two groups of eigenvalues are listed in Tables 6.3 and 6.4.

Example 3. Consider

$$-\Delta u + u.$$

The two groups of extreme eigenvalues are listed in Table 6.5. The range of these non-one eigenvalues can be predicted by Theorems 5.3 and 5.5.

Example 4. Consider

$$-\Delta u + 2u_x + u_y.$$

TABLE 6.3
Two groups of eigenvalues for $-\epsilon\Delta u + 3u_x + 3u_y + u$.

Mesh size	Eigenvalues	Eigenvalues
$h = \frac{1}{12}, \quad \epsilon = \frac{1}{2}$	0.62466691796258	1.20374405122117
	0.65546439052744	1.21266616467924
	0.69692692594231	1.22864225975214
	$\vdots$	$\vdots$
	0.84310400457523	1.77404662541921
	0.85079543914318	2.15993704327438
	0.85524282838158	2.98996959219607
	<hr/>	
	0.60537755820952	1.20214214350094
	0.61998545186220	1.20527177800520
$h = \frac{1}{20}, \quad \epsilon = \frac{3}{10}$	0.64351646258473	1.21062284753986
	$\vdots$	$\vdots$
	0.85180405592886	2.24806343511870
	0.85447327937503	2.63972089973388
	0.85605326117245	3.17985080512051

TABLE 6.4
Two groups of eigenvalues for $-\epsilon\Delta u + 3u_x + 3u_y + u$.

Mesh size	Eigenvalues	Eigenvalues
$h = \frac{1}{36}, \quad \epsilon = \frac{1}{6}$	0.59440675231608	1.20158915529747
	0.60043847405159	1.20254607834540
	0.61030104435121	1.20415353499653
	$\vdots$	$\vdots$
	0.85503620670062	2.77188289202506
	0.85584854532074	3.01565361766728
	0.85633390013561	3.22519300391089

TABLE 6.5
Intervals of non-one eigenvalues for $-\Delta u + u$.

h_1	h_2	Interval of eigenvalues	Interval of eigenvalues
$\frac{1}{10}$	$\frac{1}{10}$	[0.59004667366385, 0.85158439347714]	[1.21106725151139, 3.92860278756037]
	$\frac{1}{20}$	[0.54471305741227, 0.72162412759633]	[1.62803608757833, 7.17580980406249]
	$\frac{1}{40}$	[0.52225190888377, 0.62000575794387]	[2.58323341635383, 13.72238351656002]
	$\frac{1}{80}$	[0.51109595032917, 0.56134263742518]	[4.57546877591445, 26.84016379297525]
	$\frac{1}{160}$	[0.50553997118222, 0.53084591888113]	[8.60479998696234, 53.08770653678894]
$\frac{1}{20}$	$\frac{1}{20}$	[0.54480311242263, 0.85306283371772]	[1.20808925812872, 7.15212321759623]
	$\frac{1}{40}$	[0.52229798679507, 0.72311100784202]	[1.62051844692941, 13.67584277325651]
	$\frac{1}{80}$	[0.51111919462630, 0.62095166870805]	[2.56694130531961, 26.74789578807177]
$\frac{1}{40}$	$\frac{1}{40}$	[0.52230953204310, 0.85343085930757]	[1.20735192871894, 13.66424738691479]
	$\frac{1}{80}$	[0.51112501865209, 0.72348284346403]	[1.61865410393456, 26.72490726700744]

TABLE 6.6
Two groups of eigenvalues for $-\Delta u + 2u_x + u_y$.

Mesh size	Eigenvalues	Eigenvalues	Predicted eigenvalues
$h_1 = \frac{1}{10}, \quad h_2 = \frac{1}{20}$	0.54542861516161	1.62987429948952	
	$\vdots$	$\vdots$	
	0.68750237545680	2.26976641446903	2.1979
	0.70340882342242	2.74846688963437	2.6923
	0.71461453038915	3.74970109457185	3.7099
	0.72126356537295	7.16806444851755	7.1365
$h_1 = \frac{1}{10}, \quad h_2 = \frac{1}{40}$	0.52260538394637	2.58744503148715	
	$\vdots$	$\vdots$	
	0.59911298302300	3.93229722429224	3.8958
	0.60866013977106	4.91275919905165	4.8844
	0.61557719684619	6.93558127427726	6.9157
	0.61976363553775	13.70450516030257	13.6890
$h_1 = \frac{1}{10}, \quad h_2 = \frac{1}{80}$	0.51127161201154	4.58419759593318	
	$\vdots$	$\vdots$	
	0.55030179229328	7.30991697075110	7.2916
	0.55531702253128	9.28283232359111	9.2686
	0.55898081759268	13.33712479068527	13.3272
	0.56121153499154	26.80162392976290	26.7939
$h_1 = \frac{1}{10}, \quad h_2 = \frac{1}{160}$	0.50562753543833	8.62241607624211	
	$\vdots$	$\vdots$	
	0.52524671556309	14.09235001577338	14.0832
	0.52778614033191	18.04414211391962	18.0370
	0.52964529138036	26.15516009926207	26.1502
	0.53077901958173	53.00764397304166	53.0038

TABLE 6.7
Growth rates of the first five largest eigenvalues for $-\Delta u + 2u_x + u_y$.

jth largest eigenvalue	1st	2nd	3rd	4th	5th
Computed rates	0.32682	0.15929	0.10821	0.08313	0.06857
	0.32743	0.16004	0.10925	0.08444	0.07012
	0.32756	0.16023	0.10952	0.08478	0.07053
Predicted rate	0.3276	0.1603	0.1096	0.0849	0.0707

We list a sequence of eigenvalues in Table 6.6. We also list several of the predicted growth rates of the first five largest eigenvalues with their computed growth rates in Table 6.7 when $h_1 = \frac{1}{10}$ and h_2 takes $\frac{1}{20}$, $\frac{1}{40}$, $\frac{1}{80}$, and $\frac{1}{160}$. These predictions are based on (4.24), and the computed rates are calculated using Table 6.6. For example, for $h_1 = \frac{1}{10}$ fixed we assume these larger eigenvalues can be written as

$$X_j + Y_j \frac{1}{h_2} = \lambda(j, h_2).$$

Then, a first approximation to Y_j is given by

$$Y_j = \frac{\lambda(j, \frac{1}{40}) - \lambda(j, \frac{1}{20})}{20}.$$

Appendix. Consider the case where $0 < \epsilon \ll 1$ and

$$(A.1) \quad h_1/\epsilon = G_1, \quad h_2/\epsilon = G_2.$$

For definitiveness we assume that

$$(A.2) \quad a < 0, \quad b < 0.$$

Thus the boundary layers in (1.1) occur on $\partial\Omega_b$ given by

$$(A.3) \quad \partial\Omega_b := \{(x, 0); 0 < x < 1\} \cup \{(0, y); 0 < y < 1\}.$$

Let $\partial\Omega_H = \partial\Omega \setminus \partial\Omega_b$ be the hyperbolic boundary, i.e.,

$$(A.4) \quad \partial\Omega_H = \{(x, 1); 0 < x < 1\} \cup \{(1, y); 0 < y < 1\}.$$

Let $u = \{u_{k,j}\}$ be the solution of (1.5) with zero Dirichlet boundary values. Let $\Phi(x, y)$ be the solution of

$$(A.5) \quad a\Phi_x + b\Phi_y = f$$

with boundary condition

$$(A.6) \quad \Phi = 0 \quad \text{on} \quad \partial\Omega_H.$$

The basic result of this appendix is the following theorem.

THEOREM A.1. *Let $0 < \delta < 1$. Let*

$$(A.7) \quad \Omega_\delta := \{(x, y) : \delta < x, y < 1\}.$$

Then, for every such δ , if $(x_k, y_j) \in \Omega_\delta$, then

$$(A.8) \quad \text{Max } |u_{k,j} - \Phi(x_k, y_j)| \rightarrow 0 \quad \text{as } h_1, h_2 \rightarrow 0.$$

The proof follows from the argument below.

Let

$$(A.9) \quad u = u^I + u^{II},$$

where

$$(A.10) \quad L_h u^I = f \quad \text{in } \Omega,$$

$$(A.11) \quad u^I = \Phi \quad \text{on } \partial\Omega,$$

and

$$(A.12) \quad L_h u^{II} = 0 \quad \text{in } \Omega,$$

$$(A.13) \quad u^{II} = -\Phi \text{ on } \partial\Omega.$$

We observe that (A.13) means

$$(A.14) \quad u^{II} = 0 \text{ on } \partial\Omega_H, \quad u^{II} = -\Phi \text{ on } \partial\Omega_b.$$

LEMMA A.2. *There is a constant $K_3 > 0$, depending only on a , such that*

$$(A.15) \quad \|u\|_\infty \leq K_3 \|f\|_\infty.$$

Proof. Let

$$(A.16) \quad \theta = 1 + \frac{h_1}{a} \approx e^{-\frac{h_1}{|a|}},$$

and set

$$(A.17) \quad u_{k,j} = \theta^k v_{k,j}.$$

Then $v = \{v_{k,j}\}$ satisfies the equation

$$(A.18) \quad -\frac{\epsilon}{h_2^2} \left\{ \frac{C}{\theta} v_{k-1,j} - 2\varphi^2 v_{k,j} + B\theta v_{k+1,j} + \gamma v_{k,j-1} + \alpha v_{k,j} + \beta v_{k,j+1} \right\} = \theta^{-k} f_{k,j}.$$

We approximate $\frac{1}{\theta}$ by $1 - \frac{h_1}{a}$. Let

$$(A.19) \quad C_1 = C/\theta, \quad B_1 = B\theta,$$

$$(A.20) \quad g = \left(1 - \frac{h_1^2}{2\epsilon} \right) = 1 - \frac{1}{2} G_1 h_1,$$

$$(A.21) \quad P = \left(\frac{ah_1}{2\epsilon} - \frac{h_1}{a} \right) / g = \left(\frac{aG_1}{2} - \frac{h_1}{a} \right) / g.$$

Then an algebraic computation shows that

$$(A.22) \quad C_1 = \varphi^2 g(1 + P), \quad B_1 = \varphi^2 g(1 - P),$$

$$(A.23) \quad -2\varphi^2 = -2\varphi^2 g - \frac{h_1^2}{\epsilon} \varphi^2.$$

Since

$$\frac{h_1^2}{\epsilon} \varphi^2 = \frac{h_2^2}{\epsilon},$$

we see that v satisfies

$$(A.24) \quad -\frac{\epsilon}{h_2^2} \{ \varphi^2 g[(1 + P)v_{k-1,j} - 2v_{k,j} + (1 - P)v_{k+1,j}] + [\gamma v_{k,j-1} + \alpha v_{k,j} + \beta v_{k,j+1}] \} + v_{k,j} = \theta^{-k} f_{k,j}.$$

An easy application of maximum principle arguments shows that

$$(A.25) \quad \|v\|_\infty \leq \theta^{-m_1} \|f\|_\infty \approx e^{\frac{1}{|a|}} \|f\|_\infty.$$

From (A.17) we see that

$$(A.26) \quad \|u\|_\infty \leq \|v\|_\infty \leq K_3 \|f\|_\infty,$$

and the lemma is proven. $\square$

LEMMA A.3. *Let $W > 0$ be a constant. Let $w = \{w_{k,j}\}$ satisfy*

$$(A.27) \quad L_h w = 0,$$

$$(A.28) \quad w = 0 \text{ on } \partial\Omega_H,$$

$$(A.29) \quad |w| \leq W \text{ on } \partial\Omega_b.$$

Let

$$(A.30) \quad \theta_1 = \sqrt{C/B}, \quad \theta_2 = \sqrt{\gamma/\beta}.$$

Note. Since $a < 0, b < 0$, we have

$$(A.31) \quad \theta_1 < 1, \quad \theta_2 < 1.$$

Then

$$(A.32) \quad |w_{k,j}| \leq \theta_1^k \theta_2^j |W|.$$

Proof. Let

$$(A.33) \quad w_{k,j} = \theta_1^k \theta_2^j v_{k,j}.$$

Then v satisfies

$$(A.34) \quad -\frac{\epsilon}{h_2^2} \{\bar{C}v_{k-1,j} - 2\varphi^2 v_{k,j} + \bar{C}v_{k+1,j} - \bar{\gamma}v_{k,j-1} + \alpha v_{k,j} + \bar{\gamma}v_{k,j+1}\} = 0.$$

And, since $w = 0$ on $\partial\Omega_H$, we have

$$(A.35) \quad v = w \text{ on } \partial\Omega.$$

Here

$$(A.36) \quad \bar{C} = \sqrt{CB} = \varphi^2 \sqrt{1 - \left(\frac{aG_1}{2}\right)^2} \leq \varphi^2,$$

$$(A.37) \quad \bar{\gamma} = \sqrt{\gamma\beta} = \sqrt{1 - \left(\frac{bG_2}{2}\right)^2} \leq 1.$$

An application of the maximum principle now shows that v takes on its maximum on $\partial\Omega$. Hence

$$\|v\|_\infty \leq W$$

and

$$(A.38) \quad |w_{k,j}| \leq \theta_1^k \theta_2^j W. \quad \square$$

Proof of Theorem A.1. To complete the proof of Theorem A.1 we must show that

$$(A.39) \quad \|u^I - \Phi\|_\infty \rightarrow 0 \quad \text{as } h_1, h_2 \rightarrow 0.$$

For the sake of completeness we consider two cases. First we deal with the easy case where

$$\Phi \in C^4(\bar{\Omega}).$$

Then we deal with the general case.

THEOREM A.4. *Let $\Phi \in C^4(\bar{\Omega})$. Then there is a constant K such that*

$$(A.40) \quad \|u^I - \Phi\|_\infty \leq K[h_1^2 + h_2^2].$$

Proof. In this case we see that

$$\|L_h u^I - L_h \Phi\|_\infty \leq K(h_1^2 + h_2^2).$$

Hence the theorem follows from Lemma A.2. $\square$

THEOREM A.5. *Let $f \in (C\bar{\Omega})$. Then*

$$(A.41) \quad \|\Phi - u^I\|_\infty \rightarrow 0 \quad \text{as } h_2 = \epsilon/G_2 \rightarrow 0.$$

Proof. Since $f \in C(\bar{\Omega})$, we know that $\Phi \in C^1(\bar{\Omega})$. Standard “mollifier” arguments (see [N]) show that for each ϵ, h_1, h_2 with $\epsilon/h_s = G_s$, a fixed number, there exists a function $\Phi_\epsilon(x, y)$ and a number $\eta_\epsilon > 0$ such that

$$(A.42) \quad \|\Delta \Phi_\epsilon\| \leq \epsilon^{-1/2},$$

$$(A.43) \quad \|\Phi - \Phi_\epsilon\|_\infty + \left\| \frac{\partial}{\partial x} [\Phi - \Phi_\epsilon] \right\|_\infty + \left\| \frac{\partial}{\partial y} [\Phi - \Phi_\epsilon] \right\|_\infty \leq \eta_\epsilon,$$

$$(A.44) \quad \eta_\epsilon \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Therefore

$$(A.45) \quad L_h \Phi_\epsilon = f_\epsilon,$$

where

$$(A.46) \quad \|f - f_\epsilon\|_\infty \leq [|a| + |b| + |c|]\eta_\epsilon + \epsilon^{1/2}.$$

Hence, since

$$\|L_h(u^I - \Phi_\epsilon)\|_\infty \leq \|f - f_\epsilon\|,$$

Lemma A.2 yields

$$(A.47) \quad \|u^I - \Phi_\epsilon\|_\infty \leq K[|a| + |b| + |c|]\eta_\epsilon + \epsilon^{1/2}.$$

Finally,

$$\|u^I - \Phi\|_\infty \leq \|u^I - \Phi_\epsilon\|_\infty + \|\Phi_\epsilon - \Phi\|_\infty.$$

Therefore

$$\|u^I - \Phi\|_\infty \leq K[|a| + |b| + |c|]\eta_\epsilon + \eta_\epsilon + \epsilon^{1/2},$$

and the theorem is proven. $\square$

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