REFLEXIVE GRAPHS WITH NEAR-UNANIMITY BUT NO SEMILATTICE POLYMORPHISMS

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Abstract. We show that every generator, in a certain set of generators for the variety of reflexive near unanimity graphs, admits a semilattice polymorphism. We then find a retract of a product of such graphs (paths, in fact) that has no semilattice polymorphism. This verifies for reflexive graphs that the variety of graphs with semilattice polymorphisms does not contain the variety of graphs with near-unanimity, or even 3-ary near-unanimity polymorphisms.

1. Introduction

For relational structures such as graphs, the existence of relation preserving operations, or polymorphisms, satisfying various identities has been of great interest recently due to its relation to the complexity of the decision problem of homomorphism to the given structure. We refer the reader to [6] for a general discussion of such topics, to [9] for a discussion of the results on general digraphs, or to [5] and [8] for more concise discussion directly related to the present paper.

In this paper we look at near-unanimity (NU), and semilattice (SL) polymorphisms on reflexive graphs. For context also talk of totally symmetric idempotent (TSI) polymorphisms. The necessary definitions of these are given in the next section.

It is a trivial fact that any structure with an SL polymorphism has a TSI polymorphism, and it is known, see [11], that the converse is not generally true. Moreover, there are structures admitting SL (and so TSI) polymorphisms, but not NU polymorphisms, and vice versa.

When one restricts ones scope to reflexive graphs though, things change. It is known, see for example [13], that any graph having an NU polymorphism also has TSI polymorphisms of all arities. Moreover it was shown in [5] that any reflexive graph with a NU polymorphism has a symmetric NU polymorphism. It is natural to ask if the existence of an NU polymorphism on a reflexive graph might imply the existence of an SL polymorphism, or vice-versa. Indeed, it was asked in [11] and again in [9] if there are posets (another subfamily of reflexive digraphs) that are NU but not SL. For the rest of the paper, all graph are reflexive and symmetric.

The class, NU, of graphs admitting NU polymorphisms has been well studied; see, for example, [1], [3], [5], [10], and [12]. In [3] for example, it was verified that the class NU is a variety, i.e., is closed under products and retractions, as is the class $k$-NU of graphs admitting $k$-ary NU polymorphisms, for all $k \geq 3$. The variety

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$(k + 1)$-NU contains $k$-NU for all $k$. It was also shown that every chordal graph is in $k$-NU for some $k$, but also that for every $k$ there chordal graphs in $(k + 1)$-NU but not $k$-NU. In [5] an explicit description of the generators of the variety $k$-NU was given for all $k \geq 3$.

The class SL of graphs admitting SL polymorphisms, on the other hand, has not been so extensively studied. It has only been looked at recently in [8] and in a more specialised context in [14]. In [8], we showed that SL contains the class of chordal graphs. We also verified that the class SL is not closed under retraction, so though it is closed under products, it is not a variety. This shows that it is different from the classes TSI and NU. We also found graphs in SL that are not in NU, and asked, as [9] did for posets, whether or not every reflexive graph in NU must also be in SL. In this paper, we answer this question in the negative.

In Proposition 3.2, we observe that the generators of the variety $k$-NU, found in [5], all have SL polymorphisms. This is, of course, a first step towards showing that $k$-NU is in SL. Our second result however, Theorem 4.1 answering the question above, shows that this is not true. We find a retraction of a product of paths (the generators of 3-NU) which does not have SL. This shows that 3-NU, and so NU, is not contained in SL.

Proposition 3.2 is not simply a ploy for building tension before surprising the reader with Theorem 4.1. It also yields some alternate proofs of known facts, which we discuss briefly now, and raises some questions that we talk of in Section 5.

It was observed in [4] that every NU structure $H$ is the retraction of some universal structure $U_{TSI}(H)$ which can easily be shown to admit an SL polymorphism. So every NU graph is a retract of an SL graph. This follows also from Proposition 3.2. The graph $U_{TSI}(H)$ is large as its vertices all subsets of vertices of $H$; our result generally embeds an NU graph as a retract of a much smaller SL graph.

In [3] it was shown that there are chordal graphs in $k$-NU but not $(k - 1)$-NU for all $k \geq 4$. As chordal graphs were shown to have SL polymorphisms in [8], it follows that there are SL graphs that are in $k$-NU but not $(k - 1)$-NU. Corollary 3.3 points out how this also follows from Proposition 3.2 but the examples it provides are far from chordal, and the proof is much different.

In Section 2 we introduce the required definitions. In Section 3 we introduce the generators of $k$-NU from [5] and prove Proposition 3.2. In Section 4 we prove Theorem 4.1. Finally, in Section 5 we ask some questions.

2. Basics

2.1. Semilattices. In this subsection we recall some standard definitions related to semilattices.

A *semilattice* on a set $V$ can alternately be described as a an ordering $\leq$ such that for every pair $u, v \in V$ there is a unique greatest lower bound denoted $u \land v$; or as a 2-ary function $\land : V \times V \to V : u \times v \mapsto u \land v$ on $V$ that is idempotent (i.e., $u \land u = u$), symmetric, and associative. We thus use ‘$\land$’ and ‘$\leq$’ interchangeably, and may refer to either of them as a semilattice ordering.

It is well known, and easily verified, that the two definitions are related through the identity

$$u \leq v \iff u \land v = u.$$
As our semilattices are finite, the existence of a lower bound for every pair of elements extends by
\[ \bigwedge S = s_1 \wedge \cdots \wedge s_d. \]
to subsets \( S \subseteq V \).

The \textit{width} of an semilattice is the maximum number of pairwise incomparable elements. An element \( v \) \textit{covers} or is a \textit{cover} of an element \( u \) if \( v \geq u \) and there is no \( x \) such that \( v > x > u \). Given a semilattice \( \wedge_1 \) on \( V_1 \) and a semilattice \( \wedge_2 \) on \( V_1 \), the \textit{product semilattice} \( \wedge = \wedge_1 \times \wedge_2 \) defined by

\[ (u_1, u_2) \wedge (v_1, v_2) = (u_1 \wedge_1 v_1, u_2 \wedge_2 v_2) \]
is a semilattice on \( V_1 \times V_2 \).

2.2. \textbf{Semilattice polymorphisms.} We denote the adjacency of two vertices \( u \) and \( v \) of a graph by \( u \sim v \). A \((k\text{-ary})\) \textit{polymorphism} \( F : G^k \rightarrow G \) of a graph \( G \) is function \( f : V(G)^k \rightarrow V(G) \), on the vertex set \( V(G) \), which satisfies the following for all choices of \( u_i, v_i \in V(G) \).

\[ u_i \sim v_i \text{ for all } i \in \{1, \ldots, k\} \Rightarrow f(u_1, \ldots, u_k) \sim f(v_1, \ldots, v_k) \]

A semilattice \( \wedge : V(G) \times V(G) \rightarrow V(G) \) is compatible with \( G \), or is a \textit{semilattice (SL) polymorphism} on \( G \), if it is a polymorphism. It is easily seen that a semilattice \( \wedge \) on \( V(G) \) is a polymorphism of a reflexive graph \( G \) if and only if it satisfies the equation

\[ a \sim a', b \sim b' \Rightarrow (a \wedge b) \sim (a' \wedge b') \tag{1} \]

A reflexive graph \( G \) is an \textit{SL graph} if it admits a semilattice polymorphism, and \( \text{SL} \) is the class of all SL graphs. The \textit{(categorical) product} of two graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \times G_2 \) with vertices \( V(G_1) \times V(G_2) \) such that \( (a_1, b_1) \sim (a_2, b_2) \) if \( a_1 \sim a_2 \) and \( b_1 \sim b_2 \). A \textit{retraction} of a graph \( G \) is a homomorphism \( r : G \rightarrow G' \) to a subgraph \( G' \) which is the identity on \( G' \).

As we mentioned above, it was verified in [8] that \( \text{SL} \) is closed under taking products by verifying the following standard fact; though it was shown that \( \text{SL} \) is not closed under retractions.

\textbf{Fact 2.1.} If \( \wedge_1 \) is a \text{SL} polymorphism of \( G_1 \) and \( \wedge_2 \) is an \text{SL} polymorphism of \( G_2 \) then the product semilattice \( \wedge \) of \( \wedge_1 \) and \( \wedge_2 \) is an \text{SL} polymorphism of \( G_1 \times G_2 \).

We will frequently use products of paths. Let \( P_\ell \) denote the path of length \( \ell \) having vertex set \( \{0, \ell\} \), where \( a \sim b \) if \( |a - b| \leq 1 \). Figure 1 shows the product \( \mathcal{P} = P_3 \times P_3 \) of two 3-paths, and a typical retract \( R \) of \( \mathcal{P} \). Though the graph is reflexive, we have omitted all loops from the figure. We will do so on all figures.

Given a semilattice \( (V, \wedge) \), a sub-semilattice consists of a subset \( V' \subseteq V \) that is closed under \( \wedge \):

\[ a, b \in V' \Rightarrow a \wedge b \in V'. \]

A subset \( S \) of the vertices of a graph \( G \) is \textit{conservative} (sometimes called a subalgebra) if for every idempotent polymorphism \( \phi : G^d \rightarrow G \) of \( G \), \( s_1, \ldots, s_d \in S \) implies that \( \phi(s_1, \ldots, s_d) \in S \). In particular, an \text{SL} polymorphism of an graph \( G \) induces a sub-semilattice on any conservative set. So the subgraph of any \text{SL} graph induced by any conservative set is also an \text{SL} graph. It is well known that the \textit{i-th} \textit{distance neighbourhood} \( N^i(v) \) of a reflexive graph \( G \), consisting of all vertices that are distance at most \( i \) from a vertex \( v \), are conservative. It is also known that the intersection of conservative sets is conservative.

That is to say, we have the following.
Figure 1. The product $\mathcal{P}$ of 3-paths and a retract $R$. (Loops omitted.)

Figure 2. The graph $K^0(T)$ when $T = K_{1,3}$, the reflexive claw. All vertices are shown, some edges are hidden: two vertices are adjacent if they are in the same unit cube.

**Fact 2.2.** Let $G$ be a reflexive SL graph, then following sets induce SL subgraphs of $G$.

(i) The set $N^i(v)$ for any vertex $v \in G$, and any integer $i \geq 0$.

(ii) Intersections of the above sets.

As the only SL on a two element set is a totally ordered set, the following useful fact is immediate from the above fact.

**Fact 2.3.** If an edge $(u, v)$ of an SL graph $G$ is the intersection of distance neighbourhoods of vertices of $G$, then either $u \geq v$ or $v \geq u$ with respect to any compatible SL.

2.3. NU polymorphisms. A $k$-ary polymorphism $f : G^d \to G$ is near-unanimity ($k$-NU) if

$$f(v_1, \ldots, v_k) = a$$

when at least $k - 1$ of the $v_i$ are $a$. A 3-NU polymorphism is also known as a majority polymorphism.

There are many characterisations of the class NU. We introduce here the description from [5].
A reflexive graph \( T \) towards is a retract of the product of the graphs \( K \) and \( U \). Let \( D \) be the partition of its vertices into colour classes, and let \( \text{partition} \ z \) where \( \text{partition} x \) and \( \text{partition} y \) the leaf of the edge \( e_3 \). The semilattice \( \wedge_z \) is the transitive closure of the shown digraph on \( \mathbb{K}^0(T) \).

**Definition 3.1.** Let \( T \) be a tree with \( k \) leaves and \( m \) edges \( e_1, \ldots, e_m \). Let \( U \) and \( D \) be the partition of its vertices into colour classes, and let \( U^* \) and \( D^* \) be the subsets of \( U \) and \( D \) respectively, of vertices of degree at least 2. Define a graph \( \mathbb{K}^0(T) \) as follows: its vertices are the tuples \((x_1, \ldots, x_m)\) such that

(i) \( x_i \in \{0, 1, 2\} \) for every \( 1 \leq i \leq m \);
(ii) for each \( u \in U^* \), \( x_i = 2 \) for at least one edge \( e_i \) incident with \( u \); and
(iii) for each \( d \in D^* \), \( x_i = 0 \) for exactly one edge \( e_i \) incident with \( d \).

Tuples \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\) are adjacent if \(|x_i - y_i| \leq 1\) for all \( i \).

The following was a main result of [5]

**Theorem 2.5.** A reflexive graph \( G \) admits a \( k\)-NU polymorphism if and only if it is a retract of the product of the graphs \( \mathbb{K}^0(T_i) \), for a finite family of trees \( T_1, \ldots, T_d \) each having at most \( k-1 \) leaves.

3. The generators of \( k\)-NU have SL

We define an SL on \( \mathbb{K}^0(T) \).

**Definition 3.2.** Let \( T \) be a tree with \( k \) leaves and \( m \) edges \( e_1, \ldots, e_m \), having vertex partition \( U \) and \( D \) as in Definition 2.4. Choosing a root \( z \) of \( T \) orient the edges of \( T \) towards \( z \), (so that any vertex of \( T \) has at most one incident edge oriented away from it).

Define \( \wedge = \wedge_z : \mathbb{K}^0(T) \times \mathbb{K}^0(T) \to \mathbb{K}^0(T) \) as follows: for \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_m) \in \mathbb{K}^0(T) \), let

\[(x_1, \ldots, x_m) \wedge (y_1, \ldots, y_m) = (z_1, \ldots, z_m)\]

where \( z_i = \max(x_i, y_i) \) if \( e_i \) is oriented towards \( U \), and is \( \min(x_i, y_i) \) if \( e_i \) is oriented towards \( D \). (See Figure 3 for an example.)

**Proposition 3.2.** The map \( \wedge_z : \mathbb{K}^0(T) \times \mathbb{K}^0(T) \to \mathbb{K}^0(T) \) defined in Definition 3.2 is a semilattice polymorphism on \( \mathbb{K}^0(T) \).

**Proof.** To see that this function is onto \( \mathbb{K}^0(T) \) we must show for \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\) in \( \mathbb{K}^0(T) \) that

\[(z_1, \ldots, z_m) = (x_1, \ldots, x_m) \wedge_z (y_1, \ldots, y_m)\]

is also in \( \mathbb{K}^0(T) \). This requires showing that for any \( d \in D^* \), there is at least one each edge \( e_i \) incident to \( d \) such that \( z_i = 0 \); and for any \( u \in U^* \) there is an incident
edge $e_i$ such that $z_i = 2$. We show the former, the proof of the latter is essentially the same.

Let $d \in D^*$ have incident edges $e_1, \ldots, e_c$. At most one, say $e_1$, is directed away from $d$. If there is some $i \neq 1$ such that $x_i = 0$ or $y_i = 0$, then $z_i = \min(x_i, y_i) = 0$. Otherwise, both $x_1 = 0$ and $y_1 = 0$ and so $z_1 = \max(x_1, y_1) = 0$. So $(z_1, \ldots, z_m) \in \mathbb{K}^0(T)$, as needed.

To see that $\kappa_x$ it is a homomorphism, assume that $x_i ~ x_i'$ and $y_i ~ y_i'$ for all $i$, where $x, x', y$ and $y'$ are in $\mathbb{K}^0(T)$. Then $|x_i - x_i'| \leq 1$ and $|y_i - y_i'| \leq 1$. As both $|\min(x_i, y_i) - \min(x_i', y_i')|$ and $|\max(x_i, y_i) - \max(x_i', y_i')|$ are clearly at most 1, we get that $|z_i - z_i'| \leq 1$ for all $i$, and so $(z_1, \ldots, z_m) \sim (z_1', \ldots, z_m')$.

The homomorphism is symmetric and idempotent and associative, as it is in each coordinate. □

In [5] it was shown that for a tree $T$ with $k - 1$ leaves, $\mathbb{K}^0(T)$ admits $k$-NU polymorphisms but not $(k - 1)$-NU polymorphisms. Thus we get the following.

**Corollary 3.3.** For all $k \geq 4$ there are graphs in $\text{SL} \cap (k - \text{NU})$ that are not in $(k - 1) - \text{NU}$.

This is already known, it follows from [3] and [5]: but the examples one gets from these papers are chordal or products of chordal graphs. The above examples are far from this.

In [5] we defined retracts $\mathbb{K}(T)$ of the $\mathbb{K}^0(T)$ which also served as generators of the variety $k$-NU. One can show that the $SL$-polymorphism defined above survives the retraction from $\mathbb{K}^0(T)$ to $\mathbb{K}(T)$, so these smaller generators are also in SL. The proof of this is basic, but is too messy for what it gains us; we chose to omit it from the paper.

## 4. Retract of product of 3 paths without SL

In this section we prove the following theorem.

**Theorem 4.1.** There exists a reflexive graph admitting a 3-NU polymorphism, but no $SL$ polymorphism.

Our proof is constructive. For a graph $G$ with a compatible $SL \wedge$, we say an edge $u \sim v$ is oriented $u \rightarrow v$ by $\wedge$ if $u > v$ in the SL. The following simple observation is key.

**Lemma 4.2.** Given an SL on a graph there can be no bad 2-path: an induced path of edges oriented away from each other: $u \leftarrow v \rightarrow w$.

Proof. Indeed, if $u \leftarrow v \rightarrow w$, then $u = u \wedge v \sim v \wedge w = w$. This contradicts the fact that the 2-path $u \sim v \sim w$ is induced. □

The main idea, is that in light of the above observation, very few different SL polymorphisms are possible on a product of paths $P$, and that they, upto some skewing, are much like the product of SLs on the component paths. With respect to the minimal vertex of an assumed SL, we can split the product of paths up into orthants, on which the SL is of a very restrictive form. If one of these orthants is big enough, we can ‘kill’ the SL on that orthant with a retraction. For a product of long enough paths, we will make several ‘killing’ retractions in such a way that

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1See appendix of this preprint.
wherever the minimum vertex is for an SL, one of the killing retractions is contained well inside an orthant, and so kills the SL on that orthant.

Before we get to the actually constructive proof, we set up for it by defining our ‘killing’ retractions.

4.1. **Setup for the proof.** Let \( \mathcal{P} = \mathcal{P}_\ell^3 \) be a product of 3 paths of length \( \ell \), so vertices of \( \mathcal{P} \) are triples \( v = (v_1, v_2, v_3) \in [0, \ell]^3 \). For \( i \in [3] \), let \( e_i \) denote the vector with a 1 in the \( i \)th coordinate, and 0 elsewhere, so that for a vertex \( v, v + e_i \) is the vertex we get from it by increasing the \( i \)-coordinate by one.

A vertex \( v \) of \( \mathcal{P} \) is an *positive (or negative) i-boundary vertex* if \( v_i = \ell \) (or \( v_i = 0 \)). An edge of \( \mathcal{P} \) of the form \( \{v, v + e_i\} \) is *i-square*. It is *inner i-square* if it does not contain any \( j \)-boundary vertices for \( j \neq i \). These will be important because of the following simple observation.

**Lemma 4.3.** Any inner square edge of \( \mathcal{P} \) is oriented.

**Proof.** Clearly we may assume our inner square edge is of the form \( \{u, u + e_1\} \). In this case, as it is an inner square edge, the set

\[
S = \{u, u + e_1, u \pm e_2, u \pm e_3\}
\]

is contained in \( \mathcal{P} \). As \( \cap_{s \in S} N(s) = \{u, u + e_1\} \), we have by Fact 2.3 that \( \{u, u + e_1\} \) is oriented.

For a subgraph \( G \) of \( \mathcal{P} \), the square edges are *consistently oriented* if for all \( i \), there exists \( d_i = \pm 1 \) such that all \( i \)-square edges of \( G \) are oriented \( v \rightarrow v + d_ie_i \). If \( d_i = 1 \) they are *positively oriented*, if \( d_i = -1 \) they are negatively oriented.

For a square edge \( \{v, v + de_i\} \) let \( C(v; d, i) \) be the *cone* of vertices that are closer to \( v + de_i \) than to \( v \). That is, let

\[
C(v; d, i) = \left\{ v + \sum_{i=1}^{3} a_i e_i \in V(\mathcal{P}) : \forall j \neq i, 0 \leq |a_j| < da_i \right\}.
\]

The notation is chosen so that \( \mathcal{P} \setminus C(v; d, i) \) is the graph we get from removing from \( \mathcal{P} \) the ‘cone above \( v \) in the positive \( i \)-direction if \( d = +1 \) and in the negative \( i \)-direction if \( d = -1 \).’ This graph is in fact a retract of \( \mathcal{P} \), as one can easily check that the following map \( r \), which ‘pushes \( C(v; d, i) \) in the negative (positive if \( d = -1 \)) \( i \)-direction’, is a homomorphism: for \( x \in \mathcal{P} \setminus C(v; d, i) \), let \( r(x) = x \); and for \( x \in C(v; d, i) \) let \( r(x) = x - dm_x e_i \) for the smallest \( m_x > 0 \) such that \( x - dm_x e_i \) is not in \( C(v; d, i) \).

The retract \( R \) in Figure 1 can be viewed as is a 2-dimensional version of the construction \( \mathcal{P} \setminus C(v; d, i) \). Specifically it would be \( \mathcal{P} \setminus C((2,1); -1,1) \) as we have removed the cone of vertices closer to \((1,1)\) than to \((2,1)\).

Figure 2 shows two different depictions of \( B = \mathcal{P}_2 \setminus C((1,1,1); +1,1) \). The first depiction shows the subgraphs induced by the 1-layers of \( \mathcal{P} \), the \( i \)th layer being the set of vertices \( v \) such that \( v_i = i \). We have only shown the edges between 1-layers that involve the vertex \((1,0,2)\). The second depiction, in which the \( i \)-coordinate increases towards you, is more suggestive of how \( B \) is as subgraph of a product of paths achieved by removing a cone. Many edges are hidden, but any ‘unit cube’ in this picture induces a clique of \( B \).

The following will be our main obstruction to SL polymorphisms.
From a boundary vertex). Let $C \in \mathbb{C}^2$, $x = 4.2$.

For each proof.

(See Figure 4 for the case $d = 1$ and $i = 1$.) There is no SL on $B$ with consistently oriented square edges in which the i-square edges are negatively oriented.

Proof. To simplify notation, we assume that $d = 1$ and $i = 1$. Towards contradiction assume that $\wedge$ is such an SL. Without loss of generality we may assume that the $j$-square edges are negatively oriented for all $j \in [3]$; so $x \leq y$ if $x_j \leq y_j$ for all $j$.

We now show that there is no valid value of $(2, 2, 1) \wedge (2, 1, 2)$. Indeed, let $x = (2, 2, 1) \times (2, 1, 2)$. Then $x \leq (2, 2, 1)$ and $x \leq (2, 1, 2)$ and as $(2, 2, 1) \sim (2, 1, 2)$, we have that $(2, 2, 1) \sim x \sim (2, 1, 2)$. By the latter, $x$ can only be one of the seven vertices in $\{1, 2\}^3 \times (2, 1, 2)$; and by the former, it cannot be $(2, 2, 1)$. By Lemma 4, $x$ cannot be one of $(2, 2, 1), (2, 1, 2), (1, 2, 2), (1, 1, 2), (1, 2, 1)$, or $(1, 2, 1)$. As for each of these vertices, being less than $(2, 2, 1)$ and $(2, 1, 2)$, would put $(2, 2, 1)$ or $(2, 1, 2)$ as the middle vertex of a bad 2-path. Finally, $x$ cannot be $(1, 1, 1)$. Indeed, assume it is. As $(2, 0, 0)$ is below both $(2, 2, 1)$ and $(2, 1, 2)$, it must be below $x$. But then $(0, 1, 1) \sim (1, 1, 1) \sim (2, 0, 0)$ is a bad 2-path.

4.2. Graph with 3-NU but no SL. Let $\mathcal{P} = P_{17}^2$. Consider the sets

$V_i^{-1} = \{v \in \mathcal{P} \mid v_i = 2 \text{ and } v_j \in \{4, 9, 13\} \text{ for } j \neq i\}$

$V_i^{+1} = \{v \in \mathcal{P} \mid v_i = 15 \text{ and } v_j \in \{4, 9, 13\} \text{ for } j \neq i\}$,

and let $\mathcal{V}$ be their union, (containing a total of 54 vertices, each two square edges in from a boundary vertex). Let $C(v) = C(v; d, i)$ for each $v \in V_i^{+1}$, and let

$R = \mathcal{P} \setminus \bigcup_{v \in \mathcal{V}} C(v)$.

Figure 5 shows part of the graph $R$. The dimples in the face of the cube are the 54 different cones $C(v)$ removed from $\mathcal{P}$.

Lemma 4.5. The retract $R$ of $\mathcal{P} = P_{17}^3$ defined above has no SL polymorphisms.

Proof. For each $v \in \mathcal{V}$ let $B(v)$ be the neighbourhood of $v$ in $R$. It is easy to check that the vertices in $\mathcal{V}$ are far enough apart that each subgraph $B(v)$ is isomorphic

\begin{figure}
\centering
\includegraphics{figure4}
\caption{Two depictions of graph $B = \mathcal{P} \setminus C((2, 1, 1); +1, 1)$}
\end{figure}
Figure 5. The retract $R = \mathcal{P} \setminus (\cup C(v))$ from Lemma 4.5

...to the graph $B$ from Lemma 4.4. Applying the lemma we have that for each $v \in \mathcal{V}_i^d$, there is no SL such that the $i$-square edges are negatively (positively if $d = -1$) oriented, and the $j$-square edges are consistently oriented for all $j \neq i$. We now assume that $R$ has an SL polymorphism $\land$, and show that under $\land$, there is some $v \in \mathcal{V}$ that contradicts this.

**Claim 4.6.** Any inner $i$-square edge of $R$ is oriented.

**Proof.** By Fact 2.3 it is enough to show that any such edge, which we may assume to be $\{x, x + e_1\}$, is the intersection of distance neighbourhoods.

This is easy if all neighbours of $x$ and $x + e_1$ in $\mathcal{P}$ are in $R$, as then $\{x, x + e_1\} = \cap_{s \in S} N(s)$ where $S = \{x, x + e_1, x + e_2, x + e_3\}$. If $x + de_j$ is not in $R$ for some $d = \pm 1$ and some $j = 2$ or 3, then we can replace it in the set $S$. Indeed, it is enough to consider the case that $x + de_2$ is not in $R$. In this case, the proof above works by replacing $x + de_2$ in $S$ with any vertex in

$$\{x + e_1 + de_2, x + de_2 \pm e_3, x + e_1 + de_2 \pm e_3\}.$$  

So we must show that at least one of these vertices are in $R$.

If none of them are, then as the subgraphs $B(v)$ of $\mathcal{P}$ are distance at least 3 apart, all of these vertices are in $C(v)$ for some single $v \in \mathcal{V}$.

They, along with $x + de_2$ make up copy of $P_2 \times P_3$ and all have the same 2-coordinate $x_2 + 1$. The only way they can fit into a single $C(v)$ is if $v$ is $\mathcal{V}_i^1$, and if one of $x$ and $x + e_1$ is $v$. But then $v \notin R$ contradicts the fact that both $x$ and $x + e_1$ are in $R$. \hfill $\bigcirc$

Now, for a vertex $v \in \mathcal{P}$ let the $i$-line of $v$ be the set

$$L_i(v) = \{u \in v \mid v_j = u_j \text{ for } j \neq i\}$$
of vertices that differ from it in at most the $i^{th}$ coordinate.

Observe that for any $i$-line $L = L_i(v)$ of inner square edges, which we know are oriented by the above claim, Lemma 4.2 gives us that $L$ contains a center vertex $c_L$ towards which all edges must be oriented.

**Claim 4.7.** Let $u$ and $v$ be adjacent, then for any $i$, the centers of $L_i(u)$ and $L_i(v)$ are adjacent.

**Proof.** Assume that the centers $a = c_{L_i(u)}$ and $b = c_{L_i(v)}$ are not adjacent. Then as they are in the $i$-lines of adjacent vertices, their $i$-coordinates must differ by more than one. We may assume that $a_i < b_i - 1$. Let $a'$ be the vertex in $L_i(u)$ with $a'_i = b_i - 1$. Then $a' \sim b$ and $a' - e_i \sim b - e_i$, but

$$a' - e_i = a' \land a' - e_i \sim b \land b - e_i = b,$$

which contradicts the fact that $a' \not\sim b$.

The inner $i$-floor is graph induced by the set of centers of the inner $i$-lines. By the claim we have that it is a product of two paths. A simple consequence of the claim is that if $x$ and $x'$ are in the $i$-floor then

$$|x_i - x'_i| \leq |x_j - x'_j| \quad \text{for all } j \neq i \tag{2}$$

Equation (2) now implies that the inner 1-floor cannot intersect $B(v)$ and $B(v')$ for $v$ in $V_1^1$ and $v'$ in $V_1^2$. Indeed the maximum distance $|x_j - x'_j|$ between the $j \neq 1$ coordinates of vertices $x \in B(v)$ and $x' \in B(v')$ 13 - 4 + 2 = 12 while the distance $|x_1 - x'_1|$ between their 1 coordinates is at least 17 - 4 = 13. So we may assume that the 1-floor does not intersect $B(v)$ for any $v \in V_1^1$.

Similarly equation (2) gives us that the the inner 2-floor can intersect $B(v)$ for at most three $v \in V_1^3$. We show that for any $m \in \{4, 9, 13\}$, it can intersect $B(v)$ for at most one $v$ with $v_1 = 15$ and $v_3 = m$. Indeed, if it were to contain $x \in B(v)$ and $x' \in B(v')$ for $v \neq v'$ where $v_1 = 15 = v'_1$ and $v_3 = m = v_3$, then $|x_3 - x'_3| \leq 2$ and as $v \neq v'$, $|x_2 - x'_2| \geq (9 - 1) - (4 + 1) = 3$.

So there is some $v \in V_1^1$ such that $B(v)$ is not intersected by any $i$-floor. This means that its square edges are consistently oriented, and in particular its 1-square edges are negatively oriented. This is the contradiction of Lemma 4.4 we were looking for.

As the graph $R$ is a retract of a product of paths, it follows by [7] that it admits a 3-NU polymorphism, so this completes the proof of Theorem 4.1.

5. Questions and Discussion

5.1. Getting an NU from an SL. Given a meet-semilattice $\leq$, various subsets $S$ of the vertices may also have least upper bound $\bigvee S$. It was observed in [24] that if the $k$-ary function

$$\phi(v_1, \ldots, v_{k+1}) = \bigvee_{i=1}^{k+1} \bigwedge_{j \neq i} v_j \tag{3}$$

is well-defined, then it is near-unanimity. It was further shown that in this case the function has many strong properties such as symmetry and so-called super-associativity.

Let $T$ be a polyad a tree that has exactly one vertex $z$ of degree greater than 2. It was shown in [5] that $\mathbb{K}^0(T)$ can alternately be described as follows.
Where the leaves of $T$ are $\ell_1, \ldots, \ell_k$, and leaf $\ell_i$ has distance $d_i$ from $z$, the vertices of $K^0(T)$ are the $k$-tuples $(x_1, \ldots, x_k)$ such that $0 \leq x_i \leq d_i + 1$ for all $i$, and at least one of the $x_i$ is 0. Again, two vertices are adjacent if the differ by at most 1 in each coordinate.

Now let $\wedge$ be the semilattice polymorphism defined on $K^0(T)$ in Definition 4.1 with respect to this high-degree vertex $z$. It can easily be shown that $\wedge$ is simply the coordinatewise min function with respect to this description of $K^0(T)$.

The coordinatewise maximum function $\vee$ does not necessarily map pairs in $K^0(T)$ to $K^0(T)$, and it is easy to see that for any set of $k$ vertices $v_1, \ldots, v_{k+1}$ of $K^0(T)$ the function $\phi$ defined by (3) is the coordinatewise 'min-but-one' function which takes a set $\{x_1, \ldots, x_{k+1}\}$ to $z_2$ where $\{x_1, \ldots, x_{k+1}\} = \{z_1, \ldots, z_{k+1}\}$ and $z_1 \leq z_2 \leq \cdots \leq z_{k+1}$. This is a well-defined function on $K^0(T)$ as by the pigeonhole principle, there is some index $\alpha$ such that at least two of the $v_i$ have a 0 in the $\alpha^{th}$ coordinate. So $\phi$ takes that coordinate to 0.

This function is exactly the $(k+1)$-NU defined on $K^0(T)$ in [5]. For other trees the NU function defined on $K^0(T)$ is symmetric, but does not seem to come from a semilattice function by (3), at least not via the obvious definition of $\vee$.

**Question 5.1.** Given a tree $T$, can one find a semilattice $\wedge$ on $K^0(T)$ and a (natural) function $\vee$ such that the function $\phi$ of (3) is well defined.

**5.2. Retracts of Grids.** The construction for Theorem 4.1 is certainly bigger than it needs to be, having $18^3 - 540$ vertices. It is probably not hard to refine the construction to make it a little smaller, but a much smaller example would be interesting. In particular, our construction is a retraction of a product of three paths.

We spent considerable time trying to prove the results with a retract of a product of two paths, but it proved to be quite a stubborn problem.

**Question 5.2.** Does every retract of a product of two paths admit an SL polymorphism?

As there are generators of $k$-NU for $k \geq 5$ that retract to a product three paths, so there are retracts of such generators that omit SL. However the following question is also unclear: a positive answer implies a positive answer to the above question.

**Question 5.3.** Does every retract of $K(T)$, where $T$ is a tree with three leaves, admit an SL polymorphism?

Observe that trees with three leaves have only one vertex of degree greater than 2, so one can use the definition of $K(T)$ from the previous subsection.

**Appendix A. SL polymorphism on reduced duals**

In [5], we had a smaller version $K(T)$ of the generator $K^0(T)$ that was defined by replacing the “at least one” in conditions (ii) and (iii) of Definition 2.4 with “exactly one”. This smaller dual was emphasised in that paper, as one looks for minimal generators of a variety. It was shown that the following map is a retraction of $K^0(T)$ to $K(T)$.

**Definition A.1.** For $T$, $K^0(T)$ and $K(T)$ as in Definition 2.4 let the map $r : K^0(T) \to K(T)$ be defined as follows. For any vertex $x = (x_1, \ldots, x_m) \in K^0(T)$, let $r(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ where
Theorem A.2. Let the retraction $K$ be a semilattice polymorphism on $\land$ from the fact that $\phi$ is a homomorphism. That it is symmetric and idempotent is immediate and only if $\land r (\beta d) \land \land r (\beta d) = \land r (\beta d)$, and so this yields associativity.

We now show that the SL homomorphism on $K^0(T)$ from Definition A.1, the composition $r \circ \land : K(T) \times K(T) \rightarrow K^0(T) \rightarrow K(T)$ is a semilattice polymorphism on $K(T)$.

Proof. That $r \circ \land$ is a polymorphism is immediate as the composition of homomorphisms is a homomorphism. That it is symmetric and idempotent is immediate from the fact that $\land$ is symmetric and idempotent, and that $r$ is a retraction to (so the identity on) $K(T)$. What has to be shown is that $r \circ \land$ is associative. We show that $a := r(r(x \land y) \land z) = r(x \land y \land z) = b$.

The fact that $r(x \land r(y \land z)) = r(x \land y \land z)$ follows by the symmetry of the operation $\land$, and so this yields associativity.

As $a_i$ and $b_i$ are in $\{0, 1, 2\}$ it is clearly enough to show for each $i$ that $a_i = 0$ if and only if $b_i = 0$ and $a_i = 2$ if and only if $b_i = 2$. Further, for $d \in D^*$ (or similarly for $u \in U^*$), with incident edge $e_i$, showing $a_i = 0$ only if $b_i = 0$ accomplishes the same thing, as we already know $a_i = 0$ for exactly one $e_i$ incident to $d$, and $b_j = 0$ for exactly one $e_j$ incident to $d$.

By the symmetry of the construction with respect to $U$ and $D$, it is there for enough to show the following.

Claim A.3. For a leaf $d \in D$, where $e_i$ is the edge incident to $d$, $a_i = 0$ if and only if $b_i = 0$. For each vertex in $d \in D^*$, and each edge $e_i$ incident to $d$, $a_i = 0$ implies $b_i = 0$.

We complete our proof by proving the claim.

For the first part, let $d$ be a leaf in $D$ and $e_i$ be the incident edge. As $d$ is not in $D^*$ any change $r$ makes to the $i$th coordinate is from 2 to 1. So $a_i = 0$ if and only if $\min(x_i, y_i, z_i) = 0$ if and only if $b_i = 0$.

Now, for the second statement of the lemma, fix $d \in D^*$ and let $e_1, \ldots, e_c$ be the edges adjacent to $d$ in $T$. Let $\alpha$ be the unique index in $[c]$ such that $u_\alpha = 0$. Similarly let $\beta$ and $\gamma$ be the indexes in $[c]$ such that $v_\beta = w_\gamma = 0$.

For $i \in [c]$, if $e_i$ is directed up, then $a_i = 0$ only if $\alpha = \beta = \gamma = i$; and in this case we clearly have that $b_i = 0$. So assume that $e_i$ is directed down.

Then $a_i = 0$ means that either

(i) $\min\{\alpha, \beta, \gamma\} = i$, or

(ii) $e_m$ is directed up where $m = \min\{\alpha, \beta, \gamma\}$, and $i = \min\{\alpha, \beta, \gamma\} - \{m\}$.

In the first case, clearly $b_i = 0$. In the second case, we have that $\alpha, \beta$ and $\gamma$ are not all the same. So at least one of $u_m, v_m$ and $w_m$ are non-zero. If it is $u_m$ or $v_m$ then $(r(u \land v))_m$ is non-zero, (as $e_m$ is directed up); if it is $w_m$ then $(r(u \land v) \land w)_m$...
is non-zero. Either way $b_m$ is non-zero. At the same time at least one of $u, v$ and $w$ are zero. We split into two cases.

If it is $w_i$ that is zero, then $u_m$ is non-zero, so $(r(u \land v) \land w)_m$ is non-zero while $(r(u \land v) \land w)_i$ is zero. As by the choice of $i$ there is no other $j < i$ such that $u_j, v_j$ or $w_j$ is zero, we have that $(r(u \land v) \land w)_j$ is non-zero for all $j < i$. So applying $r$, $b_i = 0$.

On the other hand, if it is $u_i$ or $v_i$ that is zero, then $u_m$ or $v_m$ is non-zero, so $(u \land v)_m$ is non-zero while $(u \land v)_i$ is zero. As above, the choice of $i$ assures there is no $j < i$ at which $(u \land v)_j$ is zero, so $(r(u \land v))_m$ is non-zero and $(r(u \land v))_i$ is zero. So $(r(u \land v) \land w)_i$ is zero and $(r(u \land v) \land w)_j$ is non-zero for $j < i$, which means the same holds after applying $r$. Thus $b_i = 0$.

This completes the proof of the claim, and so of the Theorem. □

References