

# RECOLOURING REFLEXIVE DIGRAPHS

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ABSTRACT. Given digraphs  $G$  and  $H$ , the colouring graph  $\mathbf{Col}(G, H)$  has as its vertices all homomorphism of  $G$  to  $H$ . There is an arc  $\phi \rightarrow \phi'$  between two homomorphisms if they differ on exactly one vertex  $v$ , and if  $v$  has a loop we also require  $\phi(v) \rightarrow \phi'(v)$ . The *recolouring problem* asks if there is a path in  $\mathbf{Col}(G, H)$  between given homomorphisms  $\phi$  and  $\psi$ . We examine this problem in the case where  $G$  is a digraph and  $H$  is a reflexive, digraph cycle. Except in the case that  $H$  contains certain 4-cycles, we give necessary and sufficient conditions for the existence of a path between homomorphisms. Further we give a certifying polynomial algorithm to find such a path if it exists.

## 1. INTRODUCTION

There are several constructions in the literature, defining a digraph or complex on the set  $\text{Hom}(G, H)$  of homomorphisms from a digraph  $G$  to a digraph  $H$ , for applying topological ideas to graph theoretical problems. For example, several recent papers on graph recolouring may be viewed as determining the connectedness, the existence of paths between given vertices, or the hamiltonicity of one such graph, the *colour graph*  $\mathbf{Col}(G, H)$  (defined below) for graphs  $G$  and  $H$ .

In this paper, we focus on the problem of finding a path between two given homomorphisms for digraphs  $G$  and  $H$  in which loops are allowed in  $G$  and required in  $H$ . Our main result is a polynomial time algorithm for this recolouring problem when  $H$  is a cycle not containing certain disallowed four cycles.

Generalizing the definition of  $\mathbf{Col}(G, H)$  to digraphs can be done in several ways, but they are all accounted for in the related, but more general construction known as the Hom-graph,  $\mathbf{Hom}(G, H)$ . It turns out that for undirected graphs there is a path from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, H)$  if and only if there is a path in  $\mathbf{Hom}(G, H)$ . This does not hold for digraphs. In this work we elude to some of the differences between these two graphs, but our focus is on  $\mathbf{Col}(G, H)$ . A further study of connectivity in  $\mathbf{Hom}(G, H)$  appears in a companion paper [2].

We consider finite digraphs. We allow loops and symmetric arc pairs (simply called symmetric edges). A digraph is *reflexive* (resp. *irreflexive*) if all vertices (resp. no vertex) have a loop. A digraph is *oriented* if it has no symmetric edges, i.e. at most one of  $(u, v)$  or  $(v, u)$  can be an arc for all vertices  $u, v$ . If  $(u, v)$  is an arc of a digraph  $G$  we write  $u \rightarrow v$ . Varying from standard notation, we refer to

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an **ordered** pair  $uv$  of vertices of  $G$  as an *edge*, and write  $u \sim v$  if either  $u \rightarrow v$  or  $u \leftarrow v$  (or both). So  $u \sim v$  if  $v \sim u$ . In the case  $u \rightarrow v$  and  $u \leftarrow v$ , we often write  $u \leftrightarrow v$ .

A *walk*  $W$  from  $w_1$  to  $w_\ell$  is a sequence of consecutively adjacent vertices. When the orientation of the edges is not important, we simply write

$$W = w_1 \sim w_2 \sim \cdots \sim w_\ell.$$

A walk  $w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_\ell$  is *directed* and a walk  $w_1 \leftrightarrow w_2 \leftrightarrow \cdots \leftrightarrow w_\ell$  is *symmetric*.

A *homomorphism*  $\phi$  from a digraph  $G$  to a digraph  $H$ , denoted  $\phi : G \rightarrow H$ , is a vertex mapping  $\phi : V(G) \rightarrow V(H)$  such  $\phi(u) \rightarrow \phi(v)$  in  $H$  whenever  $u \rightarrow v$  in  $G$ . Thus a vertex in  $G$  with a loop must map to a vertex in  $H$  with a loop. If  $u \leftrightarrow v$ , then  $\phi(u) \leftrightarrow \phi(v)$ .

**1.1. The Colour graph.** For (undirected, irreflexive) graphs  $G$  and  $H$ , the *colour graph*  $\mathbf{Col}(G, H)$  is the graph whose vertex set is the set  $\text{Hom}(G, H)$  of homomorphisms from  $G$  to  $H$  and in which there is an edge  $\phi\psi$  between homomorphisms  $\phi, \psi \in \text{Hom}(G, H)$  if they differ on exactly one vertex of  $G$ .

As a homomorphism from  $G$  to  $H$  is known as an  $H$ -colouring, a walk between two homomorphisms in  $\mathbf{Col}(G, H)$  is classically known as an  *$H$ -recolouring sequence*. Such a walk corresponds to the process of changing one  $H$ -colouring of  $G$  to another, by changing the image of one vertex at a time, and always remaining an  $H$ -colouring. When  $H = K_k$ , homomorphisms are simply  $k$ -colourings, and walks in  $\mathbf{Col}(G, K_k)$  are  *$k$ -recolouring sequences*.

The colour graph has been studied by many authors. Of particular interest are the papers, [1, 7, 9, 10] which focus on  $\mathbf{Col}(G, K_k)$ . In [7, 8], they study conditions under which  $\mathbf{Col}(G, K_k)$  is connected, known as the  *$k$ -mixing problem*. In [10], Gray codes of  $k$ -colourings are studied; these correspond to Hamilton cycles in  $\mathbf{Col}(G, K_k)$ . The decision problem  $\text{RECOL}(K_k)$ , takes as input a graph  $G$  and two  $k$ -colourings of  $G$ . The task is to decide if there is a walk between two colourings in  $\mathbf{Col}(G, K_k)$ . The problem is polynomial time solvable (in the size of  $G$ ) for  $k \leq 3$  [9] and PSPACE-complete for  $k \geq 4$  [1]. In fact, the problem remains PSPACE-complete for  $4 \leq k \leq 6$  when the input is restricted to bipartite, planar graphs  $G$ .

For circular colourings, the mixing problem, i.e is  $\mathbf{Col}(G, G_{p,q})$  connected?, is studied in [4, 5]. In [3] it is shown that  $\text{RECOL}(G_{p,q})$  is polynomial time solvable if  $p/q < 4$  and is PSPACE-complete if  $p/q \geq 4$ .

More recently, Wrochna [14], showed that  $\text{RECOL}(H)$  is polynomial time solvable when  $H$  is  $C_4$ -free.

To extend the definition of  $\mathbf{Col}(G, H)$  to more general structures we observe that its definition for graphs coincides with a subgraph of a better known construction, the Hom-graph. For digraphs  $G$  and  $H$  in which loops are allowed, the *Hom-graph*  $\mathbf{Hom}(G, H)$  is the digraph on the vertex set  $\text{Hom}(G, H)$ , in which  $\phi \rightarrow \phi'$  is an arc of  $\text{Hom}(G, H)$  for homomorphisms  $\phi, \phi' \in \text{Hom}(G, H)$ , if  $\phi(u) \rightarrow \phi'(v)$  in  $H$  whenever  $u \rightarrow v$  in  $G$ .

The *weight* of an arc  $(\phi, \phi')$  of  $\mathbf{Hom}(G, H)$  is the number of vertices on which  $\phi$  and  $\phi'$  differ. Let  $\mathbf{Hom}_i(G, H)$  be the spanning subgraph of  $\mathbf{Hom}(G, H)$  consisting of arcs of weight at most  $i$ . Viewing a graph as an *irreflexive symmetric digraph*—one without loops and in which  $u \rightarrow v$  is an arc if and only if  $v \rightarrow u$  is— it is not hard to see, for graphs  $G$  and  $H$ , that  $\mathbf{Col}(G, H)$  is just  $\mathbf{Hom}_1(G, H)$ . We thus generalize the definition of  $\mathbf{Col}$  by letting  $\mathbf{Col}(G, H) = \mathbf{Hom}_1(G, H)$  for any

digraphs  $G$  and  $H$ . We highlight a subtlety here: when the vertices of  $G$  have loops it is possible to have two homomorphisms  $\phi, \psi \in \text{Hom}(G, H)$  that differ on a single vertex but are not adjacent in  $\mathbf{Hom}_1(G, H)$ . Consider the irreflexive symmetric cycle  $C_4 : 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 1$ . Let  $C_4^+$  denote the reflexive cycle obtained by adding a loop to each vertex of  $C_4$ . Let  $\phi(i) = i, i = 1, 2, 3, 4$  and  $\psi(i) = i, i = 1, 2, 3, \phi(4) = 2$ . It is easy to check  $\phi \sim \psi$  in  $\mathbf{Hom}_1(C_4, C_4^+)$  but  $\phi \not\sim \psi$  in  $\mathbf{Hom}_1(C_4^+, C_4^+)$ . In general, if  $\phi \sim \psi$  in  $\text{Hom}(G, H)$ , then for each vertex  $v$  in  $G$  with a loop, we must have  $\phi(v) \sim \psi(v)$  in  $H$ . To reiterate, our choice is to define  $\mathbf{Col}(G, H) = \mathbf{Hom}_1(G, H)$ .

**1.2.  $H$ -Recolouring and Variations.** We generalize the problem of  $H$ -Recolouring to testing for the existence of a walk in  $\mathbf{Col}(G, H)$ . Recall we define a walk from  $\phi_1$  to  $\phi_n$  as a sequence of vertices  $\phi_1 \sim \phi_2 \sim \dots \sim \phi_n$ . In particular the orientation of the arc between  $\phi_i$  and  $\phi_{i+1}$  does not matter.

Our main interest is the following problem. Let  $H$  be a fixed digraph.

RECOL( $H$ )

Instance:  $(G, \phi, \psi)$ : a digraph graph  $G$ , and two homomorphisms  $\phi, \psi \in \text{Hom}(G, H)$ .  
 Question: Is there a walk from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, H)$ ?

When  $H$  is reflexive, it is common to study  $\mathbf{Hom}(G, H)$  where  $G$  is also reflexive. While loops on  $G$  have no effect on the set  $\text{Hom}(G, H)$ , as noted above, they do effect the arcs of  $\mathbf{Hom}(G, H)$  and  $\mathbf{Col}(G, H)$ . Thus it is natural to consider RECOL( $H$ ) for only reflexive instances:  $(G, \phi, \psi)$ . We let  $\text{RECOL}_r(H)$  denote the problem RECOL( $H$ ) restricted to reflexive instances.

There are a couple more very natural variations on RECOL( $H$ ) and  $\text{RECOL}_r(H)$ . One can ask when there is a directed or symmetric walk from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, H)$ . These variants were in fact where our investigations began, but it turns out that the results we get about them follow, with only small work, from our proofs of about general walks; so we deal with them only briefly in Section 7.

For undirected, irreflexive graphs, an edge in  $\mathbf{Hom}(G, H)$  of weight exceeding one, can be decomposed into a sequence of weight one edges, see for example [5]. Thus the connectedness of  $\mathbf{Hom}(G, H)$  and  $\mathbf{Col}(G, H)$  are equivalent. This does not hold when  $H$  contains non-symmetric edges. Thus, another natural question is asking if there is a walk between  $\phi$  and  $\psi$  in  $\mathbf{Hom}(G, H)$ , or in  $\mathbf{Hom}_i(G, H)$  for some other  $i$ . This adds considerable complication, and is studied in our companion paper [2].

**1.3. Observations from Related Literature.** In both [9] and [3], the authors determine necessary and sufficient conditions for the existence of a recolouring sequence from  $\phi$  to  $\psi$  for a given graph  $G$ , where  $\phi, \psi$  are 3-colourings in the former paper and  $(p, q)$ -colourings,  $p/q < 4$  in the latter. In both papers there are three obstructions identified.

First, a vertex  $v$  of  $G$  is *fixed* under  $\phi$  if there is no recolouring sequence starting at  $\phi$  that allows the image of  $v$  to change from  $\phi(v)$ . In particular,  $\phi(v) = \psi(v)$  for all fixed  $v$  is clearly a necessary condition for the existence of a recolouring sequence from  $\phi$  to  $\psi$ .

Second given a cycle  $C$  of  $G$ , [9] and [3] define a weight with respect to the colouring. The recolouring process cannot change the weight of a cycle; thus, if  $\phi$  recolours to  $\psi$ , all cycles in  $G$  must have the same weight under both colourings.

The third obstruction is that if  $P$  is a path in  $G$  whose end points are fixed under  $\phi$ , then the weight of  $P$  cannot change under any recolouring sequence. Thus, if  $\phi$  can be recoloured to  $\psi$ , the path  $P$  must have the same weight under  $\phi$  and  $\psi$ .

It turns out these three necessary conditions are also sufficient for the existence of a recolouring sequence from  $\phi$  to  $\psi$ . Moreover, there is a polynomial time algorithm that either finds the recolouring sequence or discovers one of the three obstructions.

In a striking generalization of these papers, Wrochna [13, 14] examines the  $\text{RECOL}(H)$  problem for  $C_4$ -free graphs  $H$ . The obstruction of fixed vertices persists in this setting. Wrochna identifies another obstruction, the lack of what he calls *topologically valid* walks, which combines the second and third obstruction of the above mentioned papers.

Though we are unaware of any papers that address the problem  $\text{RECOL}(H)$  explicitly for reflexive graphs  $H$ , there is applicable literature. In [11], the authors investigate the relation between the existence of an  $NU$ -operation on  $H$  and the connectedness of  $\mathbf{Hom}(G, H)$  for various graphs  $G$ . They show, in particular, that if a reflexive graph  $H$  is dismantlable, then  $\mathbf{Hom}(G, H)$  is connected for all reflexive graphs  $G$ .

So  $\text{RECOL}_r(H)$  is trivial for all dismantlable reflexive symmetric graphs. This includes such reflexive graphs as chordal graphs, and retracts of products of paths. In particular, it tells us that  $\text{RECOL}_r(H)$  is trivial for any reflexive clique  $H$ , as is  $\text{RECOL}(H)$ , by observing that for a clique  $H$ , loops on  $G$  have no effect on arcs of  $\mathbf{Col}(G, H)$ .

We record one part of this.

**Fact 1.1.** *For  $n \geq 1$ , the symmetric reflexive clique  $K_n$  is dismantlable, so  $\text{RECOL}(K_n)$  is trivial.*

Building on ideas from [11], and [12] which extends many of the results of [11] to digraphs, we define, in [6] several notions of dismantlability for reflexive digraphs corresponding to several notions of connectedness considered in [12]. In particular, a *weak dismantling retraction* of a reflexive digraph  $H$  is any non-identity map  $\phi$  adjacent to the identity map in  $\mathbf{Col}(H, H)$ . We say it yields a *dismantling retraction* of  $H$  onto the subgraph  $\phi(H)$ . A reflexive digraph  $H$  *weak dismantles* to a subgraph  $H'$  if there sequence of dismantling retractions reducing  $H$  to  $H'$ . In [6] we observe, that if a reflexive digraph  $H$  weak dismantles to  $H'$  then for all  $G$ ,  $\mathbf{Hom}(G, H)$  and  $\mathbf{Hom}(G, H')$  have the same number of weak components. It follows that if  $H$  weak dismantles to a single vertex, then  $\text{RECOL}(H)$  is trivial. This holds for such reflexive digraphs such as transitive tournaments or graphs on 3 vertices containing a transitive triangle. We record a portion of this.

**Fact 1.2.** *If  $H$  is a reflexive digraph on 3 vertices containing the transitive triangle  $T_3$ , then it is weakly dismantlable, so  $\text{RECOL}(H)$  is trivial.*

The fact that  $\text{RECOL}(H)$  is trivial in this case, which is all that we need for this paper, is easily shown without talk of dismantlability.

**1.4. Our Main Results.** A cycle  $C = v_1 \sim v_2 \sim \dots \sim v_c \sim v_1$  is a closed walk in which each vertex is distinct. An edge  $v_i v_{i+1}$  of  $C$  is *forward* if  $v_i \rightarrow v_{i+1}$  and  $v_i \not\leftarrow v_{i+1}$ , *backward* if  $v_i \leftarrow v_{i+1}$  and  $v_i \not\rightarrow v_{i+1}$ , and *symmetric* if  $v_i \leftrightarrow v_{i+1}$ . A cycle is *symmetric* if every edge is symmetric, *oriented* if no edge is symmetric, and *directed* if oriented and every edge is forward or every edge is backward.

Henceforth,  $B = 1 \sim 2 \sim \dots \sim b \sim 1$  shall be a reflexive cycle on the integers modulo  $b$ . (As a slight non-standard choice of notation we use  $1, \dots, b$  rather than  $0, \dots, b-1$  for the integers modulo  $b$ .)

Facts 1.1 and 1.2 say that that  $\text{RECOL}(B)$  is trivial when  $b = 1$  or  $2$ , or  $b = 3$  and  $B$  contains  $T_3$ . Topologically, we view  $B$  in these cases as a contractible cycle. On the other hand, when  $B$  has girth at least 4, or is a directed 3-cycle, we view it as a non-contractible cycle for reflexive graphs, in the sense that a map that wraps a reflexive cycle around it cannot be transformed to a constant map. This is formally defined in Section 2, and in Corollary 2.4, we show that the ‘wind’ of a reflexive cycle  $C$  to such a cycle  $B$  is invariant over components of  $\mathbf{Col}(C, B)$ . This invariance of wind allows us to solve  $\text{RECOL}_r(B)$  in polynomial time.

When we consider the full problem  $\text{RECOL}(B)$ , there are several reflexive 4-cycles  $B$  we cannot deal with. Specifically, the *algebraic girth* of an oriented cycle is the number of forward edges minus the number of backwards edges. If  $C$  is not reflexive, the invariance of wind fails when mapping to a cycle  $B$  containing an oriented 4-cycle of algebraic girth 0. We cannot prove that  $\text{RECOL}(B)$  is tractible in this case. In fact, in [13] it is shown that for the symmetric 4-cycle  $C_4$ ,  $\text{RECOL}(C_4)$  is PSPACE-complete for symmetric instances, and thus PSPACE-complete in general. Though we do not have a classification for all 4-cycles, we expect a dichotomy holds.

Removing the trivial cases of contractible  $B$ , and those difficult 4-cycle cases when the invariance of wind fails, we restrict our focus in the rest of the paper to the following setup.

**Definition 1.3.** Let  $\mathcal{FGB}$  be the family of pairs  $(G, B)$  where  $B$  is a reflexive digraph cycle of girth  $b \geq 3$ , not containing a transitive triangle and  $G$  is a digraph on  $n$  vertices, such that either

- $G$  is reflexive; or
- $B$  contains no oriented 4-cycle of algebraic girth 0.

In our work we have two obstructions to the existence of a walk from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$ . We use the concept of a *topologically valid difference function*. Defined in Definition 4.1, this is similar to the topologically valid set of paths from [14]. The notion of fixed vertices [3, 9, 14] is generalized to the idea of *range* in Section 5. Our requirement is that our difference function  $\delta$  *respects ranges*, defined in Definition 5.5. Our main result is the following.

**Theorem 6.1.** *Let  $(G, B) \in \mathcal{FGB}$  and  $\phi, \psi : G \rightarrow B$  be two homomorphisms. Then there is a walk from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$  if and only if there exists a topologically valid difference function  $\delta$  for  $\phi$  and  $\psi$  that respects ranges. Furthermore, there is a polynomial time algorithm that certifies the existence of such a walk or the non-existence of such a function  $\delta$ .*

From this, and the discussion preceding, we get the following immediately.

**Corollary 1.4.** *For a reflexive digraph cycle  $B$  the problem  $\text{RECOL}_r(B)$  is polynomial time solvable, and  $\text{RECOL}(B)$  is polynomial time solvable unless  $B$  contains a 4-cycle of algebraic girth 0.*

Further results specializing Theorem 6.1 to directed and symmetric settings are given in Sections 7 and 8.

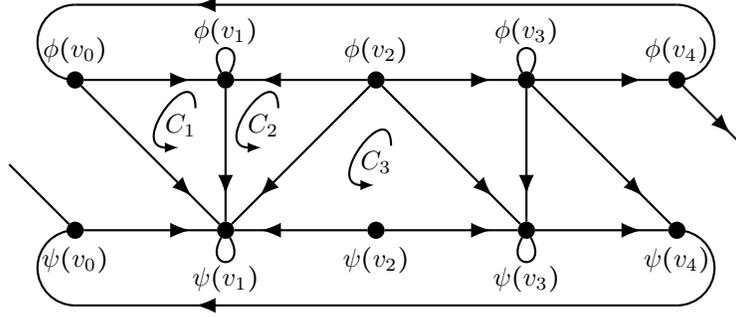


FIGURE 1. The homomorphism  $\tau : C_5 \times \vec{K}_2^+ \rightarrow B$  corresponding to the arc  $\phi \rightarrow \psi$  in  $\mathbf{Hom}(C_5 \times \vec{K}_2^+, B)$ .

## 2. NET-LENGTH AND THE INVARIANCE OF WIND

The *net-length* of an edge  $uv$  of  $B$  is  $\text{nl}(uv) = v - u$ . Note the orientation of  $uv$  does not affect net-length. We reduce modulo  $b$  so that  $\text{nl}(uv)$  is 1, 0, or  $-1$ . The edge  $uv$  is *increasing*, *folded*, or *decreasing* respectively for these three values. The net-length of a walk  $W = w_0 \sim w_1 \sim \dots \sim w_\ell$  of  $B$  is defined as  $\sum_{i=1}^{\ell} \text{nl}(w_{i-1}w_i)$ , i.e. the number of increasing edges minus the number of decreasing edges. For a closed walk  $C$  we clearly have that  $\text{nl}(C) = b \cdot w(C)$  for some integer  $w(C)$ . We call  $w(C)$  the *wind* of  $C$ . For a path  $P$  in  $G$  and a homomorphism  $\phi : G \rightarrow B$ ,  $\phi(P)$  is a walk of  $B$  so we let  $\text{nl}_\phi(P) = \text{nl}(\phi(P))$ . For a cycle  $C$  in  $G$  we let  $w_\phi(C) = w(\phi(C))$ .

The basic topological notion that the wind of a cycle around a hole is invariant under homeomorphism has an analogue, which is considerably easier to prove, in our discrete setting. For  $(G, B) \in \mathcal{FGB}$  we will show that the wind  $w_\phi(C)$  of a cycle  $C$  in  $G$  is constant as  $\phi$  varies over any component of  $\mathbf{Hom}(G, B)$ . We refer to this notion informally as the ‘invariance of wind’.

To prove the invariance of wind, we slightly extend our definition of net-length and use ideas of homology theory.

A *chain*  $X$  of  $B$  is any multiset of edges of  $B$ . The definition of net-length extends additively from edges to chains:  $\text{nl}(X) = \sum_{uv \in X} \text{nl}(uv)$ ; so  $\text{nl}(X \cup Y) = \text{nl}(X) + \text{nl}(Y)$ . If a chain  $X$  contains both an edge  $uv$  and its reverse edge  $vu$ , then we *reduce*  $X$  by removing them both. Let  $X_r$  be the chain we get by reducing  $X$  until it can be reduced no more. Clearly  $\text{nl}(X) = \text{nl}(X_r)$ .

Let  $T_3$  be the transitive triangle  $1 \rightarrow 2 \rightarrow 3 \leftarrow 1$ . The following is clear.

**Claim 2.1.** *Let  $B$  be a digraph cycle of girth at least 3 that does not contain a transitive triangle. For any homomorphism  $\phi : T_3 \rightarrow B$  we have  $w_\phi(T_3) = 0$ .  $\square$*

The following is almost as clear, but we give a proof anyway.

**Claim 2.2.** *Let  $B$  be a reflexive digraph cycle of girth at least 3 that does not contain a transitive triangle or a 4-cycle of algebraic girth 0. For any homomorphism  $\phi : C_4 \rightarrow B$  where  $C_4$  is a 4-cycle of algebraic girth 0, we have  $w_\phi(C_4) = 0$ .*

*Proof.* First observe that there are two different oriented 4-cycles of algebraic girth 0, but for either of them if we fold exactly one arc, we get a  $T_3$ . If  $\phi$  folds an arc then, it maps through a map  $T_3 \rightarrow B$ , and the result follows from the previous claim. So we may assume that  $\phi : C_4 \rightarrow B$  identifies 1 and 3. But then  $\phi(12)$  is the reverse of  $\phi(23)$  and  $\phi(34)$  is the reverse of  $\phi(41)$ . So  $\phi(C_4)_r$  is empty.  $\square$

**Theorem 2.3.** *Let  $C$  be a digraph cycle and  $B$  be a digraph cycle not containing a transitive triangle or a 4-cycle of algebraic girth 0. For any edge  $\phi \sim \psi$  of  $\mathbf{Hom}(C, B)$ , we have  $w_\phi(C) = w_\psi(C)$ .*

*Proof.* Let  $C$  be the cycle  $v_1 \sim v_2 \sim \dots \sim v_c \sim v_1$ . Let  $\vec{K}_2^+$  denoted the reflexive  $K_2$  on  $\{0, 1\}$  with the edge 01 oriented  $0 \rightarrow 1$ . The mapping  $\tau : C \times \vec{K}_2^+ \rightarrow B$  is defined by:

$$\tau(v_i, j) = \begin{cases} \phi(v_i) & \text{if } j = 0 \\ \psi(v_i) & \text{if } j = 1 \end{cases}$$

Following the notation in Figure 1, we can express  $C \times \vec{K}_2^+$  as a union of cycles  $C_1, C_2, \dots, C_m$ . Each cycle is either a transitive triangle or a  $C_4$  of algebraic girth 0. Since  $w_\tau(C_i) = 0$  for each  $i$ , their union has net-length 0. However, using the orientations in Figure 1, it is easy to see the edges of  $\tau(C \times \vec{K}_2^+)$  reduce to the edges of  $\phi(C) \cup \psi(C)$  such that

$$0 = \text{nl}_\tau(C \times \vec{K}_2^+) = \text{nl}_\phi(C) - \text{nl}_\psi(C)$$

as required.  $\square$

**Corollary 2.4.** *Let  $C$  be a digraph cycle and  $(C, B)$  be in  $\mathcal{FGB}$ . For any arc  $\phi \rightarrow \psi$  of  $\mathbf{Hom}(C, B)$ , we have  $w_\phi(C) = w_\psi(C)$ .*

*Proof.* If  $C$  is not reflexive, then  $B$  is such that Theorem 2.3 applies, and so gives us the result. If  $C$  is reflexive, then the product  $C \times \vec{K}_2^+$  is a union of transitive triangles which reduces to  $\phi(C) - \psi(C)$ . So the result follows by Claim 2.1.  $\square$

### 3. THE NO-JUMPING LEMMA

One simple but essential reduction in our proof is the basic idea that as we move from a homomorphism  $\phi$  to an adjacent homomorphism  $\psi$ , the image of a vertex only moves up or down by 1; or at least one can decompose the edge  $\phi \sim \psi$  into a path of maps where vertices move by 1 between successive maps.

This fails when mapping to a 4-cycle with algebraic girth 0; this contributes to the difficulty of this case. The image of  $v$  may jump across the cycle from say 1 to 3 without being able to move through 2 or 4.

**Lemma 3.1** (No-jumping Lemma). *When  $(G, B) \in \mathcal{FGB}$ , if  $\phi \rightarrow \psi$  in  $\mathbf{Col}(G, B)$  then there is a directed path in  $\mathbf{Col}(G, H)$  from  $\phi$  to  $\psi$  in which between any consecutive maps, the moving vertex only goes up or down by one.*

*Proof.* Let  $v$  be the vertex that changes image in moving from  $\phi$  to  $\psi$ . If  $v$  is an irreflexive, isolated vertex, the result is trivial. So  $v$  has an arc.

Let  $\phi(v) = x$ . Any neighbour  $u$  of  $v$  has  $\phi(u) \in \{x-1, x, x+1\}$ . If some neighbour  $u$  has  $\phi(u) = x$ , then  $\psi(v) \in \{x-1, x+1\}$  as required since  $\phi(u) = \psi(u) \sim \psi(v)$  in  $B$ . This includes the case that  $v$  has a loop. So we may assume  $v$  has a neighbour, and that all neighbours  $u$  have  $\phi(u) \in \{x-1, x+1\}$ .

First suppose  $B$  has girth at least five. If there exist  $u_1, u_2 \in N(v)$  such that  $\phi(u_1) = x-1$  and  $\phi(u_2) = x+1$ , then by the girth condition,  $\psi(v) = x$ . Hence, consider the case that  $\phi(u) = x+1$  for all neighbours  $u$  of  $v$ . (The case for  $x-1$  is analogous.) If  $\psi(v) \in \{x, x+1\}$ , then we are done. If  $\psi(v) = x+2$ , then  $\phi \rightarrow \phi_1 \rightarrow \psi$  in  $\mathbf{Col}(G, B)$  where  $\psi_1(v) = x+1$ .

If  $B$  has girth 3, then  $B$  must be the directed 3-cycle  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$  by the definition of  $\mathcal{FGB}$ . Consider a vertex  $v$  with colour 1. If  $v$  has neighbours coloured

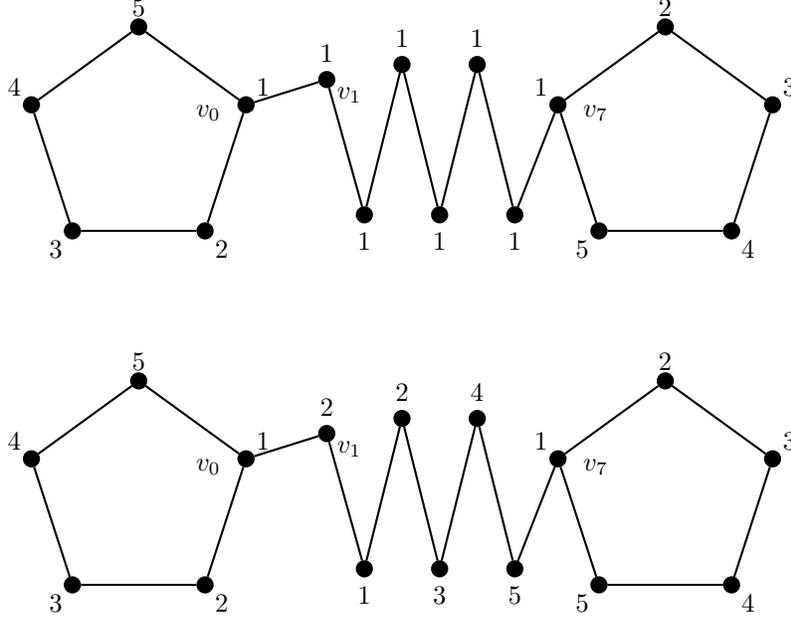


FIGURE 2. Two maps  $\phi, \psi$  of a graph  $G$  to a reflexive symmetric  $C_5$

2 and 3, then  $\psi(v) = 1$ . Hence  $v$  only has neighbours coloured with 2 or with 3 (or neither). In the former case  $v$  can move up and in the latter it can move down. Notice the well defined notion of up and down is lost when  $B$  contains  $T_3$ .

Thus consider the case that  $B$  is a 4-cycle. The proof above holds for  $B$  except for the case that there is a pair  $\phi(u_1) = x - 1$  and  $\phi(u_2) = x + 1$ . In  $C_4$ , both  $x$  and  $x + 2$  are neighbours of  $\{x - 1, x + 1\}$ . However, if  $\psi(v) = x + 2$ , it is easy to see  $x - 1 \sim x \sim x + 1 \sim x + 2 \sim x - 1$  contains a  $C_4$  of algebraic girth 0.  $\square$

#### 4. NET-CHANGE FUNCTIONS AND TOPOLOGICAL VALIDITY

The ideas developed here build on the work of [13, 14]. Given a path  $\mathcal{P}$  from  $\phi$  to  $\psi$  in  $\mathbf{Hom}(G, B)$ , we have, using Lemma 3.1, that every vertex  $v \in V(G)$  traces a walk  $\mathcal{P}(v)$  from  $\phi(v)$  to  $\psi(v)$  in  $B$ . This defines a function  $\delta_{\mathcal{P}}(v) = \text{nl}(\mathcal{P}(v))$  which measures the net-change of the image of  $v$  when going from  $\phi$  to  $\psi$  along  $\mathcal{P}$ .

Of course, we must have that  $\delta_{\mathcal{P}}(v) \equiv \psi(v) - \phi(v) \pmod{b}$ , but there is a stronger continuity condition that is closely related to the invariance of wind.

**Definition 4.1.** A function  $\delta : V(G) \rightarrow \mathbb{Z}$  is *topologically valid* for  $\phi$  and  $\psi$  in  $\mathbf{Hom}(G, B)$  if for every vertex  $v$  of  $G$  we have

$$\delta(v) \equiv \psi(v) - \phi(v) \pmod{b}, \quad (1)$$

and for every two vertices  $v, w \in V(G)$ , and every walk  $W$  from  $v$  to  $w$ , we have

$$\delta(w) = \text{nl}_{\psi}(W) + \delta(v) - \text{nl}_{\phi}(W). \quad (2)$$

**Example 4.2.** As an example of how the difference of the wind of a cycle under two different maps can be captured by the notion of topological validity, consider  $\gamma$  in Figure 4 (in Section 5), and  $\gamma'$ , the constant map that takes all vertices of  $C$  to

1. We have  $\gamma(v_5) - \gamma'(v_5) = 2$ . Thus under any path  $\mathcal{P}$  from  $\gamma$  to  $\gamma'$  in  $\mathbf{Hom}(C, B)$ ,  $\delta_{\mathcal{P}}(v_5) \equiv 2 \pmod{4}$ . However, such a  $\delta_{\mathcal{P}}$  is not topologically valid as certified by  $W_1 = v_5 \sim v_4 \sim v_3 \sim v_2 \sim v_1$  and  $W_2 = v_5 \sim v_6 \sim v_1$ :  $\text{nl}_{\gamma'}(W_1) - \text{nl}_{\gamma}(W_1) = -2$  while  $\text{nl}_{\gamma'}(W_2) - \text{nl}_{\gamma}(W_2) = 2$ . Essentially  $W_1$  requires  $v_5$  to move down two colours and  $W_2$  requires  $v_5$  move up two colours.

There are two simple reductions that make it easier to verify that a function  $\delta$  is topologically valid for  $\phi$  and  $\psi$ . When  $G$  is connected, it is easy to see that (1) holds for every vertex if and only if it holds for some vertex  $v$ . Indeed assume it holds for  $v$  and that  $v \sim u$ . Then we have

$$\begin{aligned} \delta(u) &= \text{nl}_{\psi}(vu) + \delta(v) - \text{nl}_{\phi}(vu) \\ &\equiv (\psi(u) - \psi(v)) + (\psi(v) - \phi(v)) - (\phi(u) - \phi(v)) \pmod{b} \\ &\equiv \psi(u) - \phi(u) \pmod{b}. \end{aligned}$$

Similarly, when  $U$  is a  $wu$ -walk and  $W$  is a  $vw$ -walk we have

$$\begin{aligned} \delta(u) &= \text{nl}_{\psi}(U) + \delta(w) - \text{nl}_{\phi}(U) \\ &= \text{nl}_{\psi}(U) + (\text{nl}_{\psi}(W) + \delta(v) - \text{nl}_{\phi}(W)) - \text{nl}_{\phi}(U) \\ &= \text{nl}_{\psi}(W \cdot U) + \delta(v) - \text{nl}_{\phi}(W \cdot U). \end{aligned}$$

Thus (2) is preserved under concatenation of walks, so (2) holds if and only if it holds whenever  $W$  is an edge.

**Lemma 4.3.** *For connected  $G$ ,  $(G, B) \in \mathcal{FGB}$ , and  $\phi$  and  $\psi$  in  $\mathbf{Hom}(G, B)$ , there is a topologically valid function  $\delta$  for  $\phi$  and  $\psi$  if and only if  $w_{\phi}(C) = w_{\psi}(C)$  all cycles  $C$  in  $G$ . Moreover, if such a function  $\delta$  exists, it is unique up to a constant shift, and its existence can be determined in polynomial time.*

*Proof.* Suppose  $\delta$  is topologically valid function on  $G$ . Then a cycle  $C$  of  $G$  is a walk  $W$  from some  $w$  to  $w$ . We have that

$$w_{\phi}(C) = \text{nl}_{\phi}(W) = \delta(w) - \delta(w) + \text{nl}_{\psi}(W) = \text{nl}_{\psi}(W) = w_{\psi}(C).$$

Conversely, suppose  $w_{\phi}(C) = w_{\psi}(C)$  all cycles  $C$  in  $G$ . Clearly we can define a function  $\delta$  that is topologically valid for  $\phi$  and  $\psi$  restricted to a spanning tree  $T$  of  $G$  in polynomial time. We simply set  $\delta(v_0) = \psi(v_0) - \phi(v_0)$  for some vertex  $v_0$ , and then extend  $\delta$  recursively by setting  $\delta(u) = \text{nl}_{\psi}(vu) + \delta(v) - \text{nl}_{\phi}(vu)$  for any edge  $vu$  in  $T$  for which  $\delta(v)$  is defined but  $\delta(u)$  is not. Since (1) holds on  $v_0$ , and (2) is valid on all edges of  $T$ , by the remarks before the lemma,  $\delta$  is topologically valid on  $T$ .

If  $\delta'$  is any other topologically valid function on  $T$ , then it is easy to see  $\delta'(u) - \delta'(v_0) = \delta(u) - \delta(v_0)$ , i.e.  $\delta$  is unique up to a constant shift.

Now, consider an arc  $uv$  not in  $T$  and the path  $vTu$  from  $v$  to  $u$  in  $T$ .

$$\begin{aligned} 0 &= w_{\psi}(vTu + uv) - w_{\phi}(vTu + uv) \\ &= \text{nl}_{\psi}(vTu) + \text{nl}_{\psi}(uv) - \text{nl}_{\phi}(vTu) - \text{nl}_{\phi}(uv) \\ &= \delta(u) - \delta(v) + \text{nl}_{\psi}(uv) - \text{nl}_{\phi}(uv) \end{aligned}$$

as required.  $\square$

**Example 4.4.** Using the examples in Figure 2 and setting  $\delta(v_0) = 0$ , the resulting function  $\delta$  assigns 0 to the vertices of the five cycle containing  $v_0$ , 5 to the vertices

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**Input:** A digraph  $G$  and two homomorphisms  $\phi, \psi : G \rightarrow B$   
**Output:** A topologically valid  $\delta$  for  $\phi$  and  $\psi$  or a cycle  $C$  such that  $w_\phi(C) \neq w_\psi(C)$ .

1. Let  $v_0, v_1, \dots, v_{n-1}$  be a BFS ordering of  $V(G)$  and  $T$  the spanning tree.
2.  $\delta(v_0) \leftarrow \psi(v_0) - \phi(v_0)$
3. **For**  $i = 1$  to  $n - 1$  **do**
  - 3.1.  $u \leftarrow \text{parent}(v_i)$
  - 3.2.  $\delta(v_i) \leftarrow \text{nl}_\psi(uv_i) + \delta(u) - \text{nl}_\phi(uv_i)$
- od**
4. **For each**  $uv \notin T$  **do**
  - 4.1. **If**  $\delta(v) \neq \text{nl}_\psi(uv) + \delta(u) - \text{nl}_\phi(uv)$  **then**
    - 4.1.1.  $C \leftarrow vTu + uv$
    - 4.1.2. **Print** “ $w_\phi(C) \neq w_\psi(C)$ ”
    - 4.1.3. **Return**  $C$
    - 4.1.4. **STOP**
- od**

---

FIGURE 3. Recolouring Algorithm Phase 1

of the cycle containing  $v_7$ , and values to the path of 1, 0, 1, 2, 3, 4. If instead one sets  $\delta'(v_0) = -5$ , the result is  $\delta$  shifted down by 5. Since  $\delta(v_0) \equiv \psi(v_0) \pmod{b}$ , in general these constant shifts are multiples of  $b$ .

We also remark that the  $\delta$  constructed on  $T$  in the proof of Lemma 4.3 is either topologically valid for  $G$  or we discover a cycle  $vTu + uv$  for which the wind is different under  $\psi$  and  $\phi$ . This proves that Phase 1 of our algorithm as shown in Figure 3 correctly returns the required output. Building the BFS tree requires  $O(|V(G)| + |E(G)|)$  time, Step 3 requires  $O(|V(G)|)$  and Step 4 requires  $O(|E(G)|)$ . Thus Phase 1 runs in time  $O(n^2)$ .

## 5. FIXED CYCLES AND RANGE

**5.1. The Auxillary Graphs.** Given a map  $\gamma \in \text{Hom}(G, B)$ , we say a vertex  $u$  *moves up* (respectively *moves down*) if there is a map  $\gamma'$  adjacent to  $\gamma$  in  $\text{Col}(G, B)$  such that  $\gamma'(u) = \gamma(u) + 1$  (respectively  $\gamma'(u) = \gamma(u) - 1$ ).

It is not hard to see for a vertex  $u$  under a homomorphism  $\gamma$ , that  $u$  can move up, if and only if there is no vertex  $v$  adjacent to  $u$  satisfying either of the following conditions:

- $\gamma(v) = \gamma(u) - 1$ , or
  - $\gamma(v) = \gamma(u)$  and  $v \mapsto \gamma(u), u \mapsto \gamma(u) + 1$  is not a homomorphism.
- (3)

This leads to the following definition.

**Definition 5.1.** For  $\gamma \in \text{Hom}(G, B)$  let the *auxillary graph*  $\mathbb{A}_\gamma^+$  have vertex set  $V(G)$  and for  $u \sim v$  in  $G$  let  $u \rightarrow v$  in  $\mathbb{A}_\gamma^+$  if  $u$  and  $v$  satisfy (3).

One can interpret  $u \rightarrow v$  in  $\mathbb{A}_\gamma^+$  as meaning that for  $u$  to move up,  $v$  must move up first. Thus the only vertices of  $G$  that can be moved up are those that have no out arcs in  $\mathbb{A}_\gamma^+$ , i.e. sinks. If in  $\mathbb{A}_\gamma^+$ ,  $u$  is in a directed cycle, or a directed path to a

directed cycle, then there is no sequence of up moves that will allow in  $u$  to move up. (Some vertex in the directed cycle would have to move first.) In fact, we show below that there is no sequence of moves (up or down) that will allow  $u$  to move up from the value  $\gamma(u)$ .

**Example 5.2.** Consider Figure 4 showing a map  $\gamma$  of an oriented 6-cycle  $C$  to an oriented 4-cycle  $B$ . Both cycles are reflexive, though the loops are not shown. The graph  $\mathbb{A}_\gamma^+$  is shown in Figure 5. Notice  $v_2$  is the only sink in  $\mathbb{A}_\gamma^+$  and hence the only vertex that can move up to form a new homomorphism  $\gamma'$ . In turn,  $v_3$  is the only sink in  $\mathbb{A}_{\gamma'}^+$ . Continuing one obtains  $\mathbb{A}_{\gamma'}^+$  which is a directed cycle. At this point no vertex can move up any further.

Analogously one can define  $\mathbb{A}_\gamma^-$  to determine which vertices of  $G$  can move down. Similar ideas are used in [3, 9], but the symmetry of edges in undirected graphs allow both  $\mathbb{A}^+$  and  $\mathbb{A}^-$  to be coded in a single digraph.

We finish this section with a description of the arcs in  $\mathbb{A}_\gamma^+$ , when  $G = C = v_1 \sim v_2 \sim \dots \sim v_c$  is a cycle, which we use in the proofs below. The following is immediate from the definition of  $\mathbb{A}_\gamma^+$ .

**Claim 5.3.** *Suppose  $v_i \sim v_{i+1}$  is an edge in  $C$  and  $\gamma : C \rightarrow B$ . Then:*

1.  $\gamma(v_i) = \gamma(v_{i+1}) - 1$ , implies  $v_i \leftarrow v_{i+1}$  in  $\mathbb{A}_\gamma^+$
2.  $\gamma(v_i) = \gamma(v_{i+1}) + 1$  implies  $v_i \rightarrow v_{i+1}$  in  $\mathbb{A}_\gamma^+$ ,
3.  $\gamma(v_i) = \gamma(v_{i+1})$  and  $\gamma(v_i) \leftrightarrow \gamma(v_i) + 1$  in  $B$  implies  $v_i \not\sim v_{i+1}$  in  $\mathbb{A}_\gamma^+$ ,
4.  $\gamma(v_i) = \gamma(v_{i+1})$  and  $\gamma(v_i) \sim \gamma(v_i) + 1$  is oriented in  $B$  implies
  - (i)  $v_i \sim v_{i+1}$  is oriented in  $\mathbb{A}_\gamma^+$  when  $v_i \sim v_{i+1}$  is oriented in  $C$ , and
  - (ii)  $v_i \leftrightarrow v_{i+1}$  in  $\mathbb{A}_\gamma^+$  when  $v_i \leftrightarrow v_{i+1}$  in  $C$ .

By the claim when,  $\mathbb{A}_\gamma^+$  contains no symmetric arcs when  $C$  is oriented. Furthermore, if  $B$  is also oriented, then  $\mathbb{A}_\gamma^+$  will be an oriented cycle of length  $|V(C)|$ . If  $B$  contains some symmetric arcs, then  $\mathbb{A}_\gamma^+$  may be either an oriented cycle or a collection of oriented paths.

**5.2. Range of a vertex.** For  $\phi \in \text{Hom}(G, B)$ , let  $\text{range}_\phi(v)$  be the set of values  $\delta_{\mathcal{P}}(v) = \text{nl}(\mathcal{P}(v))$  over walks  $\mathcal{P}$  in  $\text{Col}(G, B)$  starting at  $\phi$ . By Lemma 3.1 this is an interval of integers which contains 0. The point of considering the range of a vertex is because of the following condition that is clearly necessary for the existence of a path between  $\phi$  and  $\psi$ .

**Fact 5.4.** *If there is a path  $\mathcal{P}$  from  $\phi$  to  $\psi$  in  $\text{Col}(G, B)$ , then  $\delta_{\mathcal{P}}(v) \in \text{range}_\phi(v)$  for every vertex  $v$  of  $G$ .  $\square$*

This leads to the following definition.

**Definition 5.5.** A difference function  $\delta$  that is topologically valid for  $\phi$  and  $\psi$  in  $\text{Hom}(G, B)$  respects ranges if

$$\delta(v) \in \text{range}_\phi(v) \text{ for all } v \in V(G). \quad (4)$$

For the existence of a path from  $\phi$  to  $\psi$  in  $\text{Hom}(G, B)$  we need that there is a difference function  $\delta$  that is topologically valid for  $\phi$  and  $\psi$ , which respects ranges.

In this section, we develop the tools to compute  $\text{range}_\phi(v)$ . We isolate what it is that can restrict  $\text{range}_\phi(v)$ , for some  $v$ . For a cycle  $C$  in  $G$ , let  $\phi|_C$  be the restriction of  $\phi$  to  $C$ , and  $\text{range}_{\phi|_C}(v)$  be the the range computed in  $\text{Hom}(C, B)$ .

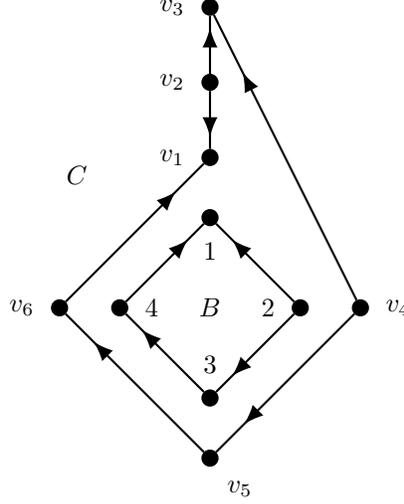


FIGURE 4. A map  $\gamma : C \rightarrow B$  under which vertices have different ranges.

It is clear that  $\text{range}_\phi(v)$  is contained in  $\text{range}_{\phi|_C}(v)$ . If  $\text{range}_{\phi|_C}(v)$  is other than  $(-\infty, \infty)$  for one vertex  $v$  of  $C$ , then it is for all vertices of  $C$ ; in this case, we call  $C$  *tight* under  $\phi$ . We show below that each vertex in  $G$  has finite range if and only if some cycle  $C$  in  $G$  is tight. So determining range comes down to finding tight cycles.

**Example 5.6.** Referring again to Figure 4 and using  $\mathbb{A}_\gamma^+$  and  $\mathbb{A}_\gamma^-$ , we see that  $\text{range}_\gamma(v_1) = \{-1, 0\}$  and  $\text{range}_\gamma(v_2) = \{-1, 0, 1\}$ . For a given positive integer  $r$ , one can easily come up with examples of oriented cycles  $C$  and  $B$ , with  $\phi : C \rightarrow B$ , such that  $\text{range}_\phi(v) = \{0\}$  for some vertex  $v$  and that  $\text{range}_\phi(u)$  has size exceeding  $r$  for some vertex  $u$ . This is similar to the undirected cases studied in [3, 9] where a vertex  $v$  such that  $\text{range}_\phi(v) = \{0\}$  is called *fixed*. By contrast, in our setting we can also obtain examples where no vertex is fixed, but the cycle is tight. This does not occur in the undirected case.

Computing  $\text{range}_\phi(v)$  is quite simple. We greedily move vertices up until either we know all vertices have infinite range, or we have found  $\max \text{range}_\phi(v)$  for each  $v$ . Similarly, we find  $\min \text{range}_\phi(v)$  for each  $v$  by greedily moving vertices down. The following results show that recolouring steps commute in a way that allows a greedy algorithm to succeed.

We start with some notation. For an edge  $\gamma \sim \gamma'$  in  $\mathbf{Col}(G, B)$  the maps  $\gamma$  and  $\gamma'$  differ on a single vertex  $v_i$ , which by Lemma 3.1 we may assume either moves up or moves down by 1. Representing the arc by  $\uparrow v_i$  if  $v_i$  moves up, or by  $\downarrow v_i$  if  $v_i$  moves down, we can write  $\gamma'$  as  $\gamma + \uparrow v_i$  or  $\gamma + \downarrow v_i$ , and we can represent a path  $\mathcal{P}$  from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$  by a sequence of symbols  $\square v_1, \square v_2, \dots, \square v_\ell$  where each  $\square$  is either  $\uparrow$  or  $\downarrow$ .

**Claim 5.7.** *Up and down moves in a path can be commuted.*

*Proof.* Towards contradiction, say  $\dots + \downarrow w + \uparrow u + \dots$  occurs in a path, but we cannot replace it with  $\dots + \uparrow u + \downarrow w + \dots$ ; that is,  $u$  cannot move up until  $w$  has moved

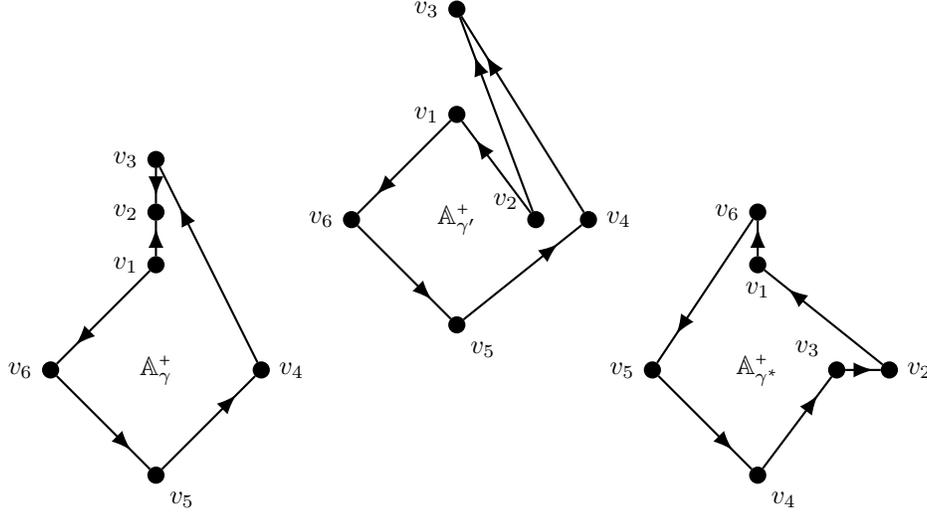


FIGURE 5. The graph  $\mathbb{A}_\gamma^+$ . The graph  $\mathbb{A}_{\gamma'}^+$ , obtained by moving  $v_2$  up. After a series of 5 moves the graph  $\mathbb{A}_{\gamma^*}^+$  is obtained where no vertex can move up.

down. Let  $\gamma$  be the mapping before  $\downarrow w$ . As  $u$  cannot move up,  $u$  is not a sink in  $\mathbb{A}_\gamma^+$ , but it is a sink in  $\mathbb{A}_{\gamma+\downarrow w}^+$ . This implies  $u \rightarrow w$  in  $\mathbb{A}_\gamma^+$  but not in  $\mathbb{A}_{\gamma+\downarrow w}^+$  (only arcs incident with  $w$  change). This implies  $\gamma(w) \leq \gamma(u)$  contrary to the fact that  $w \mapsto \gamma(w) - 1$  and  $u \mapsto \gamma(u) + 1$  is a homomorphism. (In the event that  $B$  contains edges  $\gamma(w) \sim \gamma(u)$ ,  $(\gamma(w) - 1) \sim \gamma(u)$ , and  $(\gamma(w) - 1) \sim (\gamma(u) + 1)$ , it is easy to prove  $B$  contains a transitive triangle, which we exclude.)  $\square$

**Corollary 5.8.** *If there is a path from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$ , then there is a unimodal path: one in which all up moves occur before all down moves. Moreover, it can be assumed that no vertex moves both up and down.*

*Proof.* Assume that there is a path from  $\phi$  to  $\psi$  such that  $v$  moves up and down. Using Claim 5.7 we can commute moves to get a path in which all down moves come first. Then we can commute moves again so that all up moves come first. In doing so we commuted  $\uparrow v$  with  $\downarrow v$ . At this time we can cancel them, removing them both from the path.  $\square$

**Corollary 5.9.** *If there is a path  $\mathcal{P}$  from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$  in which  $\text{nl}(\mathcal{P}(v_i)) \geq 1$  and  $v_i$  is a sink of  $\mathbb{A}_\phi^+$ , then there is a unimodal path in which  $\uparrow v_i$  is the first move.*

*Proof.* Make  $\mathcal{P}$  unimodal. As  $\text{nl}(\mathcal{P}(v_i)) \geq 1$ ,  $\mathcal{P}$  contains an arc  $\uparrow v_i$ . Simply append  $\uparrow v_i \downarrow v_i$  to the front of the path. (This is allowed as  $v_i$  is a sink.) Then commute the  $\downarrow v_i$  through the up moves following until it cancels with the later  $\uparrow v_i$ .  $\square$

With this, we get the following result which allows us to certify a bound on the range of a vertex.

**Lemma 5.10.** *Let  $\mathcal{U}$  be a walk from  $\phi$  to  $\gamma^+$  in  $\mathbf{Hom}(G, B)$ . If  $\mathbb{A}_{\gamma^+}^+$  contains no sinks, then  $\text{nl}(\mathcal{U}(v)) = \max \text{range}_\phi(v)$  for all  $v \in V(G)$ .*

*Proof.* Suppose to the contrary that  $\mathbb{A}_{\gamma^+}^+$  has no sinks but there is a walk  $\mathcal{P}$  and a vertex  $v$  such that  $\text{nl}(\mathcal{P}(v)) > \text{nl}(\mathcal{U}(v))$ . Then there is a walk  $\mathcal{W} = -\mathcal{U} + \mathcal{P}$  starting at  $\gamma^+$  with  $\delta_{\mathcal{W}}(v) > 0$ . By Corollary 5.8 there is thus a walk from  $\gamma^+$  that starts with an up move. This contradicts the fact that  $\mathbb{A}_{\gamma^+}^+$  contains no sinks.  $\square$

This justifies our greedy algorithm for determining  $\text{max range}_{\phi}(v)$  for all vertices—while we can, choose sinks of  $\mathbb{A}^+$  and move them up. It works now unless  $\text{max range}_{\phi}(v)$  is unbounded or is too large for this to complete in polynomial time. Our next tool bounds what  $\text{max range}_{\phi}(v)$  can be if it is finite. We do this first when  $G$  is a cycle, then extend it to full generality.

**Lemma 5.11.** *Let  $C$  be a cycle and let  $\phi$  be in  $\text{Hom}(C, B)$ . Either  $\text{range}_{\phi}(v) = (-\infty, \infty)$  for every vertex  $v$  of  $C$ , or  $|\text{range}_{\phi}(v)| < b$  for some vertex  $v$ .*

*Proof.* First assume  $C$  has a symmetric edge. If  $B$  is symmetric, then  $\text{range}_{\phi}(v) = \{0\}$  for every  $v$  if  $C$  is tight, or  $\text{range}_{\phi}(v) = (-\infty, \infty)$  for every  $v$  otherwise. To see this note if all edges in  $C$  under  $\phi$  are described by Case 1 from Claim 5.3, then  $\mathbb{A}_{\phi}^+$  is a directed cycle; the same holds if all edges belong to Case 2. If there is a mix of Case 1 and 2 edges or a non-edge (Case 3), then  $\mathbb{A}_{\phi}^+$  will have a sink  $v$ , and the resulting  $\mathbb{A}_{\phi+\uparrow v}^+$  will also have a sink.

If  $B$  is not symmetric, then there is a vertex  $v$  of  $C$  incident with a symmetric edge. It cannot move past a non-symmetric edge of  $B$ , so has  $|\text{range}_{\phi}(v)| < b$ .

Now we consider the case that  $C$  is oriented, and show the stronger statement that  $\text{range}_{\phi}(v) = (-\infty, \infty)$  for every vertex  $v$  of  $C$  or  $|\text{range}_{\phi}(v)| < b$  for every vertex  $v$  of  $C$ .

Let  $C = v_1 \sim v_2 \sim \dots \sim v_c \sim v_1$ . Suppose some vertex has range of width at least  $b$ . Then we may assume there is a path  $\mathcal{P}$  from  $\phi''$  to  $\phi'$  such that  $\text{nl}(\mathcal{P}(v_c)) = b$  and  $\text{nl}(\mathcal{P}(v_i)) \leq b$  for all  $i$  where  $\phi''$  and  $\phi'$  are in the same component of  $\mathbf{Col}(C, B)$  as  $\phi$ . Note it may not be the case the  $v_c$  can move up  $b$  from  $\phi(v)$ , hence we use  $\phi''$  and  $\phi'$  for  $\mathcal{P}$ . If  $\text{nl}(\mathcal{P}(v_i)) = b$  for all  $i$ , then we are done, as  $\mathcal{P}$  is then a closed walk in  $\mathbf{Col}(C, B)$  and by traversing it repeatedly the ranges of all vertices are unbounded.

Assume  $\text{nl}(\mathcal{P}(v_1)) < b$  and let  $k > 1$  be the minimum index such that  $\text{nl}(\mathcal{P}(v_k)) = b$ ; thus,  $\text{nl}(\mathcal{P}(v_j)) < b$  for  $j = 1, 2, \dots, k-1$ . We claim we can move some such  $v_j$  up. By repeated application of this claim, we obtain a new path  $\mathcal{P}'$  such that  $\text{nl}(\mathcal{P}'(v_i)) = b$  for all  $i$ , so as observed above, we are done.

Consider the subgraph induced by  $\{v_c, v_1, \dots, v_k\}$  in  $\mathbb{A}_{\phi'}^+$ . First observe that  $v_1 \not\leftarrow v_c$  in  $\mathbb{A}_{\phi'}^+$ . Indeed either  $\text{nl}(\mathcal{P}(v_1)) = b-2$  or  $\text{nl}(\mathcal{P}(v_1)) = b-1$ . In the former case,  $\phi'(v_1) = \phi'(v_c) - 1$ , and we clearly have that  $v_1 \leftarrow v_c$  in  $\mathbb{A}_{\phi'}^+$ . In the latter case it follows because moving  $v_1$  up returns  $v_1$  and  $v_c$  to the colours they have under  $\phi''$ . This is clearly an allowed assignment. Similarly we get that  $v_{k-1} \not\leftarrow v_k$  in  $\mathbb{A}_{\phi'}^+$ . Since  $C$  being oriented means  $\mathbb{A}_{\phi'}^+$  has no symmetric edges, this implies that  $v_j$  is a sink in  $\mathbb{A}_{\phi'}^+$  for some  $j \in \{1, \dots, k-1\}$ . This  $v_j$  can move up.  $\square$

**Corollary 5.12.** *Let  $G$  be a connected graph on  $n$  vertices and  $\phi$  be in  $\text{Hom}(G, B)$ . Either  $|\text{range}_{\phi}(v)| < b$  for some vertex  $v$  and so  $|\text{range}_{\phi}(v)| < b + 2n - 2$  for all  $v$  in  $G$ , or  $\text{range}_{\phi}(v) = (-\infty, \infty)$  for all  $v$  in  $G$ .*

*Proof.* Clearly  $|\text{range}_{\phi}(v)| < b$  implies  $|\text{range}_{\phi}(u)| < b + 2n - 2$  for any other vertex  $u$  in  $G$ , as  $G$  being connected means there is a path of length at most  $n-1$  between

$u$  and  $v$ . To prove the result then, we assume that  $\max \text{range}_\phi(v)$  is finite for some  $v$ , and so for all  $v$  as  $G$  is connected, and show that  $|\text{range}_\phi(u)| < b$  for some  $u$ .

Indeed, if  $\max \text{range}_\phi(v)$  is finite for all  $v$ , then there is a path  $\mathcal{U}$  from  $\phi$  to a map  $\gamma^+$  such that no vertex can move up from  $\gamma^+$ . So  $\mathbb{A}_{\gamma^+}^+$  has no sinks, and so contains a directed cycle  $C$ .

Restricting every map in  $\mathcal{U}$  to the subgraph  $C$ , we get a path in  $\mathbf{Hom}(C, G)$  from  $\phi|_C$  to  $\gamma^+|_C$  such that  $\mathbb{A}_{\gamma^+|_C}^+ = (\mathbb{A}_{\gamma^+}^+)|_C$  has no sinks. So by Lemma 5.10 we have  $\max \text{range}_\phi(v) \leq \max \text{range}_{\phi|_C}(v) = \text{nl}(\mathcal{U}(v))$  is finite for all  $v$  in  $C$ . By Lemma 5.11 we get that  $|\text{range}_\phi(v)| < b$  for some  $v$  in  $C$ .  $\square$

Note that from this corollary it follows that if there is a topologically valid difference function from  $\phi$  to  $\psi$  that respects ranges, then this is unique, or all ranges are unbounded and so every shift by a constant multiple of  $b$  also respects ranges.

At this point we observe that through the greedy process of choosing a sink in  $\mathbb{A}^+$  and moving it up, we will either move some vertex  $b + 2n - 2$  at which point we can conclude by Corollary 5.12 that all ranges are infinite, or we will construct a walk  $\mathcal{U}$  from  $\phi$  to some  $\gamma^+$  such that  $\mathbb{A}_{\gamma^+}^+$  is sink free, and so by Lemma 5.10,  $\max \text{range}_\phi(v) = \text{nl}(\mathcal{U}(v))$  for all  $v$ . Similarly, one can compute  $\min \text{range}_\phi(v)$  for all  $v$ , by greedily moving vertices down to construct a walk  $\mathcal{D}$  from  $\phi$  to  $\gamma^-$  where  $\mathbb{A}_{\gamma^-}$  is sink free. This is Phase 2 of our algorithm; see Figure 6. As each vertex moves at most  $b + 2n - 2$  and updating  $\mathbb{A}^+$  takes time  $O(n)$  this algorithm completes in time  $O(n^3)$ . The output is either a  $\delta$  shifted to respect ranges, or a certificate that no such  $\delta$  exists. The following corollary provides the obstruction that certifies that no such  $\delta$  can exist.

**Corollary 5.13.** *Suppose  $\mathcal{U}$  and  $\mathcal{D}$  are walks in  $\mathbf{Col}(G, B)$  from  $\phi$  to  $\gamma^+$  and  $\gamma^-$  respectively where  $\mathbb{A}_{\gamma^+}^+$  and  $\mathbb{A}_{\gamma^-}$  are sink free auxiliary graphs. Let  $\delta$  be a topologically valid function for  $\phi$  and  $\psi$ . Then*

$$\delta(v) - \delta(u) \leq \text{nl}(\mathcal{U}(v)) - \text{nl}(\mathcal{D}(u)) \leq \max \text{range}_\phi(v) - \min \text{range}_\phi(v)$$

for all vertices  $v$  and  $u$  in  $V(G)$ .

*Proof.* Let  $\mathcal{P}$  be a path from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$  and let  $\delta_{\mathcal{P}}(v) = \text{nl}(\mathcal{P}(v))$  for all  $v$ . Then  $\delta_{\mathcal{P}}$  is topologically valid and by Lemma 4.3,  $\delta_{\mathcal{P}}$  is a shift of  $\delta$ . Consequently, for all vertices  $v, u \in V(G)$ ,  $\delta(v) - \delta(u) = \delta_{\mathcal{P}}(v) - \delta_{\mathcal{P}}(u)$ . By Lemma 5.10,  $\text{nl}(\mathcal{P}(v)) \leq \text{nl}(\mathcal{U}(v))$ . By similar reasoning  $\text{nl}(\mathcal{P}(u)) \geq \text{nl}(\mathcal{D}(u))$ . Thus,

$$\delta(v) - \delta(u) = \delta_{\mathcal{P}}(v) - \delta_{\mathcal{P}}(u) = \text{nl}(\mathcal{P}(v)) - \text{nl}(\mathcal{P}(u)) \leq \text{nl}(\mathcal{U}(v)) - \text{nl}(\mathcal{D}(u))$$

as required.  $\square$

## 6. PROOF OF MAIN THEOREM

In this section we prove the following theorem.

**Theorem 6.1.** *Let  $(G, B) \in \mathcal{FGB}$  and  $\phi, \psi : G \rightarrow B$  be two homomorphisms. Then there is a walk from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$  if and only if there exists a topologically valid difference function  $\delta$  that respects ranges. Furthermore, there is a polynomial time algorithm that certifies the existence of such a walk or the non-existence of such a function  $\delta$ .*

---

**Input:** A digraph  $G$ , two homomorphisms  $\phi, \psi : G \rightarrow B$ , and a topologically valid  $\delta$  for  $\phi$  and  $\psi$ .

**Output:** A shift of  $\delta$  so that  $\delta(v) \in \text{range}_\phi(v)$  for all  $v$  or a pair of vertices  $u$  and  $v$  such that  $\delta(v) - \delta(u) > \max \text{range}_\phi(v) - \min \text{range}_\phi(u)$ .

1.  $\text{TopRange}(v) \leftarrow 0$  for each  $v \in V(G)$ ;  $\gamma \leftarrow \phi$ .
  2. **Construct**  $\mathbb{A}_\gamma^+$ .
  3. **Let**  $S = \{u \mid u \text{ is a sink in } \mathbb{A}_\gamma^+\}$
  4. **While**  $(\exists u \in S)$  **do**
    - 4.1.  $\gamma(u) \leftarrow \gamma(u) + 1$ .
    - 4.2.  $\text{TopRange}(u) \leftarrow \text{TopRange}(u) + 1$ . *Move u up.*
    - 4.3. **If**  $(\text{TopRange}(u) > b + 2n - 2)$  **then**  $\text{range}_\phi(v) = (-\infty, \infty), \forall v$ 
      - 4.1. **Break** *Move to Phase 3*
    - 4.2. Update  $\mathbb{A}_\gamma^+$  and  $S$
  5. **Let**  $k = \min\{i : \delta(u) - i \cdot b \leq \text{TopRange}(u)\}$ .
  6.  $\delta(u) \leftarrow \delta(u) - k \cdot b$ . **Let**  $u_0$  be a vertex where  $\delta(u_0) + b > \text{TopRange}(u_0)$ .  
Reduce  $\delta$  to make demand feasible for each  $u$ . Save  $u_0$
  7.  $\text{BottomRange}(v) \leftarrow 0$  for each  $v \in V(G)$ ;  $\gamma \leftarrow \phi$ .
  8. **Construct**  $\mathbb{A}_\gamma^-$ .
  9. **Let**  $S = \{u \mid u \text{ is a sink in } \mathbb{A}_\gamma^-\}$
  10. **While**  $(\exists u \in S)$  **do**
    - 10.1  $\gamma(u) \leftarrow \gamma(u) - 1$ .
    - 10.2  $\text{BottomRange}(u) \leftarrow \text{BottomRange}(u) - 1$ . *Move u down.*
    - 10.3 Update  $\mathbb{A}_\gamma^-$  and  $S$
  11. **If** for all  $v$ ,  $\delta(v) \geq \text{BottomRange}(v)$  **then Return YES** and  $\delta$ ; **otherwise let**  $\delta(v_0) < \text{BottomRange}(v_0)$ , **Return NO** and  $\mathcal{U}(u_0), \mathcal{D}(v_0), \delta$ .
- 

FIGURE 6. Recolouring Algorithm Phase 2

Phase 3 of the algorithm (Figure 7) constructs the desired walk from  $\phi$  to  $\psi$  given the topologically valid difference function  $\delta$  (from Phase 1) that has been shifted so that  $\delta(v) \in \text{range}_\phi(v)$  (in Phase 2). Hence once we prove the third phase will always successfully terminate, Theorem 6.1 follows.

In Phase 3 the dominant steps in terms of running time are Steps 4 and 7. Each of  $n$  vertices can move at most  $O(n)$ . After moving a vertex  $u$ , updating  $\mathbb{A}^+$  of  $\mathbb{A}^-$  requires  $\deg(u) < n$  time. Thus there is a total running time of  $O(n^3)$ .

**Lemma 6.2.** *Let  $\delta$  be a topologically valid difference function for  $\gamma$  and  $\psi$  in  $\text{Hom}(G, B)$ . If  $\delta(u) > 0$  and  $u \rightarrow v$  in  $\mathbb{A}_\gamma^+$  then  $\delta(v) > 0$ .*

*Proof.* Assume that  $\delta(u) > 0$  and  $u \rightarrow v$  in  $\mathbb{A}_\gamma^+$ .

If  $\gamma(v) = \gamma(u) - 1$ , then where  $W = v \sim u$ , we have  $\text{nl}_\gamma(W) = 1$  and of course  $\text{nl}_\psi(W) \leq 1$ , so by the topological validity of  $\delta$  we have  $\delta(v) \geq 1 + \delta(u) - 1$ . As  $\delta(u) \geq 1$  this yields  $\delta(v) \geq 1$ .

So we may assume that  $\gamma(v) = \gamma(u)$  and that  $v \mapsto \gamma(u)$ ,  $u \mapsto \gamma(u) + 1$  is not a homomorphism. Again where  $W = v \sim u$  we have that  $\text{nl}_\gamma(W) = 0$  and  $\text{nl}_\psi(W) \leq 1$ .

---

**Input:** A digraph  $G$ , two homomorphisms  $\phi, \psi : G \rightarrow B$ , and a topologically valid  $\delta$  for  $\phi$  and  $\psi$  where  $\delta(v) \in \text{range}_\phi(v)$  for all  $v$ .  
**Output:** A walk  $\mathcal{P}$  from  $\phi$  to  $\psi$  in  $\mathbf{Col}(G, B)$ .

1.  $\delta^* \equiv \delta$ ,  $\mathcal{P}^* \leftarrow \phi$ ,  $\gamma \leftarrow \phi$ .
  2. **Construct**  $\mathbb{A}_\gamma^+$ .
  3. **Let**  $S = \{u \mid u \text{ is a sink in } \mathbb{A}_\gamma^+\}$
  4. **While**  $(\exists u \in S, \text{ s.t. } \delta^*(u) > 0)$  **do**
    - 4.1.  $\gamma(u) \leftarrow \gamma(u) + 1$ . *Move u up.*
    - 4.2.  $\mathcal{P} \leftarrow \mathcal{P} + \gamma$  *Append new  $\gamma$  to  $\mathcal{P}$*
    - 4.3.  $\delta^*(u) \leftarrow \delta^*(u) - 1$  *Decrease demand for u*
    - 4.4. Update  $\mathbb{A}_\gamma^+$  and  $S$
  5. **Construct**  $\mathbb{A}_\gamma^-$ .
  6. **Let**  $S = \{u \mid u \text{ is a sink in } \mathbb{A}_\gamma^-\}$
  7. **While**  $(\exists u \in S, \text{ s.t. } \delta^*(u) < 0)$  **do**
    - 7.1.  $\gamma(u) \leftarrow \gamma(u) - 1$ . *Move u down.*
    - 7.2.  $\mathcal{P} \leftarrow \mathcal{P} + \gamma$  *Append new  $\gamma$  to  $\mathcal{P}$*
    - 7.3.  $\delta^*(u) \leftarrow \delta^*(u) + 1$
    - 7.4. Update  $\mathbb{A}_\gamma^-$  and  $S$
  8. **Return**  $\mathcal{P}$
- 

FIGURE 7. Recolouring Algorithm Phase 3

By the topological validity of  $\delta$  we have  $\delta(v) \geq 0 + \delta(u) - 1$ , giving that  $\delta(v) > 0$  unless  $\text{nl}_\psi(W) = 1$  and  $\delta(u) = 1$ . But in this case,  $\delta(u) = 1$  implies  $\psi(u) = \gamma(u) + 1$  and so  $\text{nl}_\psi(W) = 1$  implies  $\psi(u) = \psi(v) + 1$ , which yields  $\psi(v) = \gamma(u)$ . But then  $\psi$  maps  $v$  to  $\gamma(u)$  and  $u$  to  $\gamma(u) + 1$ , which contradicts the fact that this is not a homomorphism.  $\square$

The above lemma gives us the following.

**Lemma 6.3.** *Phase 3 (Figure 7) terminates with  $\delta^* \equiv 0$ .*

*Proof.* At the start of Phase 3, we know  $\text{range}_\phi(v) = (-\infty, \infty)$  for all  $v$ , or  $\delta$  has been shifted so that  $\delta(v) \leq \text{maxrange}_\phi(v)$  for all  $v$ . The first part of Phase 3 moves vertices  $v$  with  $\delta^*(v) > 0$  up until each vertex  $v$  has  $\delta^*(v) \leq 0$  or  $\mathbb{A}_\gamma^+$  has no sinks.

In fact, it must terminate with  $\delta^*(v) \leq 0$  for each  $v$ . To see this we observe that if there is a vertex with  $\delta^*(u) > 0$  then there is some sink  $s \in S$  with  $\delta^*(s) > 0$ , which we can move up. Indeed if no vertex in  $U = \{u \mid \delta^*(u) > 0\}$  is a sink, we have by Lemma 6.2 that  $U$  contains a cycle  $C$ . As  $\mathbb{A}_{\gamma|_C}^+ = (\mathbb{A}_\gamma^+)|_C$  contains no sinks, we have by Lemma 5.10 that  $\text{nl}(\mathcal{P}(v)) = \text{maxrange}_{\phi|_C}(v) \geq \text{maxrange}_\phi(v)$  for all  $v \in C$ . But this contradicts  $\delta^*(v) > 0$ .  $\square$

Now we are ready for the formal proof of our main theorem.

*Proof.* If there is a path between  $\phi$  and  $\psi$  in  $\mathbf{Col}(G, B)$  then by Corollary 2.3 we have that  $w_\phi(C) = w_\psi(C)$  for every cycle  $C$  of  $G$ , so by Lemma 4.3 there is a

difference function  $\delta$  that is topologically valid for  $\phi$  and  $\psi$ . By Fact 5.4 we have that it respects ranges.

On the other hand, assume that there is a topologically valid difference function  $\delta'$  for  $\phi$  and  $\psi$  which respects ranges. Phase 1 of the algorithm terminates with a topologically valid function  $\delta$  or a cycle  $C$  with  $w_\phi(C) \neq w_\psi(C)$ . The latter cannot occur by Lemma 4.3. Phase 2 terminates with a shift of  $\delta$  that respects ranges, or a certificate that no such  $\delta$  exists (by Corollary 5.13). By assumption the latter cannot occur, thus Phase 3 of the algorithm runs with the shifted  $\delta$  from Phase 2.

By Lemma 6.3, Phase 3 terminates with a walk  $\mathcal{P}$  from  $\phi$  to  $\gamma$  and  $\delta^*(v) = 0$  for all  $v$ . This implies  $\gamma = \psi$ , as required.  $\square$

Note that Phase 3 of the algorithm actually finds a shortest path  $\mathcal{P}$  of *non-jumping arcs* between  $\phi$  and  $\psi$  of all of those with  $\delta_{\mathcal{P}} = \delta$ . Certainly one can shorten the path by using jumping arcs, however, it seems difficult to find the shortest such path. We wonder if this can also be done in polynomial time.

## 7. FINDING DIRECTED AND SYMMETRIC PATHS

In the introduction we mentioned the problems of deciding if there are directed or symmetric paths in  $\mathbf{Col}(G, B)$  between given maps  $\phi$  and  $\psi$ . We now get this as an easy corollary of our main results. The main point is the following easily verifiable fact which tells us when an arc of  $\mathbf{Col}(G, B)$  is not symmetric.

**Fact 7.1.** *An edge  $\phi \sim \phi'$ , where  $\phi' = \phi + \uparrow v$ , of  $\mathbf{Col}(G, B)$ , is symmetric if  $v$  is irreflexive. If  $v$  has a loop, then  $\phi \rightarrow \phi'$  if and only if  $\phi(v) \rightarrow \phi(v) + 1$ .*

Using this and Corollary 5.8 it is not hard to show the following.

**Lemma 7.2.** *If there is a path  $\phi$  and  $\psi$  in  $\mathbf{Col}(G, B)$  with a difference function  $\delta$ , then there is one that contains a forward directed path if and only if:*

- (i) *For each reflexive vertex of  $G$  with  $\delta(v) > 0$ , there is a directed path  $\phi(v) \rightarrow (\phi(v) + 1) \rightarrow \dots \rightarrow (\phi(v) + \delta(v))$  in  $B$ .*
- (ii) *For each reflexive vertex of  $G$  with  $\delta(v) < 0$ , there is a directed path  $(\phi(v) - \delta(v)) \rightarrow (\phi(v) - \delta(v) + 1) \rightarrow \dots \rightarrow \phi(v)$  in  $B$ .*

Observe that for a difference function  $\delta$  with all values greater than  $b$ , this lemma tells us that there is a forward directed path with this difference function if and only if  $G$  is irreflexive or  $B$  is forward directed. With such considerations, we see that the conditions of the lemma are easily checked in polynomial time for *all* topologically valid difference functions that respect ranges. We then quickly get the following corollary of Theorem 6.1.

**Corollary 7.3.** *Let  $(G, B) \in \mathcal{FGB}$ . There is a polynomial time algorithm to decide if given maps  $\phi$  and  $\psi$  in  $\mathbf{Hom}(G, B)$  are connected in  $\mathbf{Col}(G, B)$  by a directed or a symmetric path.*

## 8. SPECIAL CASES

When  $B$  is symmetric or directed, the tight cycles are very simple to characterize, and so Theorem 6.1 reduces to a characterisation that is very easy to check. In both cases, we let  $\text{fix}_\phi(G)$  be the set of *fixed vertices*  $v$  of  $G$  with  $\text{range}_\phi(v) = \{0\}$ .

**8.1. Symmetric Cycles.** When  $B$  is a symmetric cycle, the second point of condition (3) doesn't happen, so a cycle  $C$  of  $G$  induces a directed cycle in  $\mathbb{A}_\phi^+$  if and only if all arcs are decreasing. Thus a vertex  $v$  is in  $\text{fix}_\phi(G)$  if it is in a cycle  $C$  of girth  $|w_\phi(C) \cdot b|$ ; and otherwise has  $\text{range}_\phi(v) = (-\infty, \infty)$ .

If there are fixed vertices  $v$  then the condition on a difference function  $\delta$  that  $\delta(v) \in \text{range}_\phi(v)$  becomes  $\delta(v) = 0$ , and condition (2) of Definition 4.1 becomes that  $\text{nl}_\phi(W) = \text{nl}_\psi(W)$  for walks  $W$  between fixed vertices. This gives the following.

**Corollary 8.1.** *There is a path between  $\phi$  and  $\psi$  in  $\mathbf{Col}(G, C_n)$  for symmetric cycle  $C_n$  with  $n \geq 5$ , or  $n = 4$  if  $G$  is reflexive, if and only if*

- (i)  $w_\phi(C) = w_\psi(C)$  for all cycles  $C$  in  $G$ ,
- (ii)  $\phi(v) = \psi(v)$  for any  $v \in \text{fix}_\phi(G)$ , and
- (iii)  $\text{nl}_\phi(P) = \text{nl}_\psi(P)$  on all paths  $P$  between vertices of  $\text{fix}_\phi(G)$ .

**8.2. Directed cycles.** When  $B$  is a directed cycle, the second condition of (3) reduces to

$$\gamma(v) = \gamma(u) \text{ and } v \leftarrow u,$$

so a cycle  $C$  in  $G$  is tight if and only if it has wind  $c$  and exactly  $c \cdot b$  forward arcs. Where  $\text{fix}_\phi(G)$  is the set of vertices in such cycles, we get the following.

**Corollary 8.2.** *There is a path between  $\phi$  and  $\psi$  in  $\mathbf{Col}(G, C_n)$  for directed cycle  $C_n$  with  $n \geq 3$  if and only if conditions (i) - (iii) of Corollary 8.1 hold.*

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