Note on Robust Critical Graphs with Large Odd Girth

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Abstract

A graph $G$ is $(k+1)$-critical if it is not $k$-colourable but $G-e$ is $k$-colourable for any edge $e \in E(G)$. In this paper we show that for any integers $k \geq 3$ and $\ell \geq 5$ there exists a constant $c = c(k, \ell) > 0$, such that for all $\tilde{n}$, there exists a $(k+1)$-critical graph $G$ on $n$ vertices with $n > \tilde{n}$ and odd girth at least $\ell$, which can be made $(k-1)$-colourable only by the omission of at least $cn^2$ edges.

Key words: Graph, colour critical, odd girth.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A $k$-colouring $\chi$ of $G$ is a function $\chi : V(G) \rightarrow [k]$ such that $\chi(u) \neq \chi(v)$ whenever $\{u, v\}$ is an edge of $G$. The graph $G$ is called $k$-colourable if there exists a $k$-colouring of $G$, and is called $k$-chromatic if $k$ is the smallest integer such that $G$ has a $k$-colouring. $G$ is called $(k+1)$-critical if it is not $k$-colourable, but $G-e$ is $k$-colourable for any edge $e$ of $G$. The girth of $G$, denoted by $g(G)$, is the length of the shortest cycle in $G$.

In 1970, Toft [9] extended a result of Dirac [2] to show that for $k \geq 3$, there exists some constant $c_k$ such that for any $n$, there exists a $(k+1)$-critical graph on $n$ vertices with at least $c_kn^2$ edges. These graphs constructed by Toft and

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Dirac had large bipartite subgraphs and contained small odd cycles. Erdős [4] suggested that this may always be the case.

For \( k = 3 \), the above mentioned 4-critical graphs produced by Toft, could be made bipartite by removing \( cn \) edges, for some constant \( c > 0 \). This led Toft to ask whether there exists some constant \( d > 0 \) such that for \( n \) large, there exists a 4-critical graph on \( n \) vertices from which one must remove \( dn^2 \) edges in order to make it 2-colourable. This question was answered by Stiebitz and Rödl.

In 1987, Stiebitz [8], showed the existence of \((k+1)\)-critical graphs, for \( k \geq 3 \), from which one must remove at least \( d_kn^2 \) edges to make them \((k-1)\)-colourable. The graphs that Stiebitz constructed contained many triangles.

At about the same time, Rödl [7] (unpublished) showed that for any \( \ell \geq 3 \) there are 4-critical graphs with odd girth at least \( \ell \), from which one must remove at least \( d_kn^2 \) edges in order to make them 2-colourable.

In this paper, we extend Rödl’s unpublished construction and prove the following theorem:

**Theorem 1** For any integers \( k \geq 3 \) and \( \ell \geq 5 \) there exists a constant \( c = c(k, \ell) > 0 \), such that for all \( \tilde{n} \), there exists a \((k+1)\)-critical graph \( G \) on \( n \) vertices with \( n > \tilde{n} \) and odd girth at least \( \ell \), which can be made \((k-1)\)-colourable only by the omission of at least \( cn^2 \) edges.

Recall that by a result of Kővári, Sós, and Turán [5], any graph with more than \( \frac{1}{2}(n^2 + n - \frac{n}{2}) \) edges has a 4-cycle, so we cannot hope to replace ‘odd girth’ in the above theorem with ‘girth’.

Unfortunately, the proof of Theorem 1 gives only a weak lower bound on the constant \( c(k, \ell) \), and we make no effort to evaluate it. In section 5 we give an easy upper bound on \( c(3, \ell) \) as consequence of a result of Andrásfai, Erdős, and Sós [1].

## 2 Preliminaries

Before we proceed with the proof of Theorem 1 we give two preliminary lemmas that will be used later in the proof. Note that when \( \chi \) is a map with domain \( V \), and \( U \subset V \), we will write \( \chi(U) \) for the set \( \{\chi(u) \mid u \in U\} \).

A set \( \mathcal{A} \) of mappings from \( W \) to \([k]\) is said to be closed under permutations of \([k]\) if for every \( \alpha \in \mathcal{A} \) and any permutation \( \sigma \) of \([k]\), \( \sigma \circ \alpha \in \mathcal{A} \).
We start with Lemma 2 which is a slight variation of a result of Müller [6]. Müller proved that under the assumptions of Lemma 2, conditions (i) and (ii) hold.

**Lemma 2** Let \( k, \ell \in \mathbb{N}, k \geq 3, \) and let \( \mathcal{A} \subseteq \{ \alpha \mid \alpha : W \to [k] \} \) be closed under permutation of \([k]\). Then, there exists a \( k \)-chromatic graph \( M = M(W, \mathcal{A}, k, \ell) \), with \( W \subset V(M) \), which satisfies the following properties.

(i) \( g(M) \geq \ell \)

(ii) \( \{ \chi_W \mid \chi \text{ is a } k\text{-colouring of } M \} = \mathcal{A} \)

(iii) The distance between any two vertices of \( W \) is at least \( \ell \).

**PROOF.**

Given \( k, \ell, \mathcal{A}, \) and \( W \), as in the statement of the lemma, the result in [6] guarantees the existence of a \( k \)-chromatic graph \( M_0 \), with \( W \subset V(M_0) \), which satisfies properties (i) and (ii). To prove the lemma, we will construct a new graph \( M \) from \( M_0 \), which satisfies properties (i) and (ii), and also satisfies property (iii). This construction is a standard technique.

Before we construct \( M \), we will define an auxiliary graph \( R \). First, let \( R' \) be a \((k + 1)\)-critical graph with \( g(R') \geq \ell \); such graphs exist by [3]. Let \( e' = \{x, x'\} \) be an edge of \( R' \). Then \( R' - e' \) is a \( k \)-chromatic graph such that for any \( k \)-colouring \( \chi \) of \( R' - e' \), \( \chi(x) = \chi(x') \). Now let \( R = (R' - e') \cup \{x', y\} \), where \( y \) is a new vertex adjacent to \( x' \). Then the graph \( R \) has \( g(R) \geq \ell \), is \( k \)-chromatic, and has distance \( d_R(x, y) \geq \ell \). Moreover, for any \( k \)-colouring \( \chi \) of \( R \), \( \chi(x) = \chi(x') \neq \chi(y) \).

We now construct the graph \( M \) from \( M_0 \) and several copies of \( R \).

**Construction 3** Let \( E_W \) be the set of edges of \( M_0 \) with at least one vertex in \( W \). For each \( e = \{u, w\} \in E_W \), let \( R_e \) be a copy of \( R \), and let \( x_e \) and \( y_e \) denote the copies of \( x \) and \( y \) in \( R_e \). Replace the edge \( e \) in \( M_0 \) with the graph \( R_e \), by identifying \( u \) with \( x_e \) and \( w \) with \( y_e \). Let the resulting graph be called \( M \).

Note that by Construction 3 all of the vertices of \( M_0 \) are vertices of \( M \), and so in particular \( W \subset V(M_0) \subset V(M) \). We now verify that \( M \) satisfies properties (i) - (iii) of the lemma.

(i): Recall that the graph \( M_0 \) and all the copies of \( R \) have girth at least \( \ell \). Therefore, any cycle in \( M \) which is not entirely in \( M_0 \) or a copy of \( R \), must contain the vertices \( x_e \) and \( y_e \) of \( R_e \) for some \( e \in E_W \). Since \( d(x_e, y_e) \geq \ell \) in \( R_e \) and hence in \( M \), it follows that \( g(M) \geq \ell \).
We first show that any $k$-colouring of $M_0$, can be extended to a $k$-colouring of $M$. Then conversely, we show that any $k$-colouring of $M$ induces a $k$-colouring of $M_0$. This will then give property (ii) since by [6],

$$\{\chi_0|_{W} : \chi_0 \text{ is a } k\text{-colouring of } M_0\} = A.$$ 

Let $\chi$ be a $k$-colouring of $M_0$. For any edge $\{a, b\} \in E(M_0)$, we clearly have that $\chi(a) \neq \chi(b)$. Since the only edges of $M$ that are not in $M_0$ are those which are in copies of $R$, it remains to show that for any $e = \{u, w\} \in E_w$, $\chi$ can be extended to a proper $k$-colouring of $R_e$. Since $\chi$ is a $k$-colouring of $M_0$, and $e$ is in $M_0$, $\chi(x_e) = \chi(u) \neq \chi(w) = \chi(y_e)$. Since $\chi(x_e) \neq \chi(y_e)$, $\chi$ can be extended to a $k$-colouring of $R_e$.

On the other hand, let $\chi$ be a $k$-colouring of $M$. The only edges of $M_0$ that are not in $M$ are those of $E_w$, and for any $e = \{u, w\} \in E_w$ we have that $\chi(u) = \chi(x_e) \neq \chi(y_e) = \chi(w)$. Thus $\chi$ induces a $k$-colouring of $M_0$.

Every edge $e$, incident to a vertex of $W$, was replaced in Construction 3 with a copy $R_e$ of $R$, by identifying the endpoints of $e$ with the vertices $x_e$ and $y_e$ of $R_e$. Since $d(x_e, y_e) \geq \ell$ in $R_e$, any two vertices of $W$ in the graph $M$ are also at a distance of at least $\ell$.

This completes the proof of the lemma. □

Lemma 2 gives no relation between the size of $W$ and the number of vertices of $M$. In the proof of Theorem 1 we will need Lemma 4, which is a modification of Lemma 2. In Lemma 4 the number of vertices of the graph will be bounded by a constant (depending only on $k$ and $\ell$) times the size of the input set $U \cup \{u^*\}$ (which replaces the set $W$). It will be sufficient to prove such a strengthening only for sets $A$ of a special form, as specified by condition (ii) below.

**Lemma 4** Let $d, k, \ell \in \mathbb{N}$ be such that $d \geq 1$, $k \geq 3$, $\ell \geq 5$, and $\ell$ is odd. Then there exists a graph $T = T(d, k, \ell)$, with a distinguished subset of vertices $U \cup \{u^*\} \subset V(T)$, where $u^* \not\in U$ and $|U| = k^d$, and a constant $m(k, \ell)$, such that the following properties hold.

(i) $g(T) \geq \ell$

(ii) Any mapping $\chi : U \cup \{u^*\} \rightarrow [k]$ can be extended to a $k$-colouring of $T$ if, and only if, $\chi(u^*) \in \chi(U)$.

(iii) The distance between any two vertices of $U \cup \{u^*\}$ is at least $\ell$.

(iv) $|V(T)| < k^d \cdot m(k, \ell)$
Let $k$ and $\ell$ be given. Let $V = \{v_1, \ldots, v_k\}$ be a set of independent vertices and let $v \notin V$. Set $W = V \cup \{v\}$ and define the set of mappings
\[
\mathcal{A} = \{\alpha : W \rightarrow [k] \mid \alpha(v) \in \alpha(V)\}.
\]
We now apply Lemma 2 and obtain the graph $M = M(W, \mathcal{A}, k, \ell)$. The graph $M$ thus contains all the vertices of $V \cup \{v\}$ and has the property that $\chi : V \cup \{v\} \rightarrow [k]$ can be extended to a proper $k$-colouring of $M$ if, and only if, $\chi(v) \in \chi(V)$.

We will construct the graph $T = T(d, k, \ell)$ by taking several copies of the graph $M$ and gluing them together as described in Construction 5 below. The copies of $M$ will be pairwise vertex disjoint except for those vertices identified in Construction 5.

**Setup for Construction 5** For all $i = 1, \ldots, d$, let $U^i = \{\xi_1 \ldots \xi_i \mid 1 \leq \xi_j \leq k, \text{ for } j = 1, \ldots, i\}$. Furthermore, let $U = U^d$. We will view $U^i$ as a set of independent vertices, with which we will identify some of the vertices of $T$.

**Construction 5** For all $i = 1, \ldots, d - 1$ and $1 \leq \xi_j \leq k$, where $j = 1, \ldots, i$, let $M_{(\xi_1, \ldots, \xi_i)}$ be a copy of $M$. For each $M_{(\xi_1, \ldots, \xi_i)}$, identify the copy of $V$ with $\{(\xi_1, \ldots, \xi_i, 1), \ldots, (\xi_1, \ldots, \xi_i, k)\}$ and the copy of $v$ with $(\xi_1, \ldots, \xi_i)$. Moreover, let $M_{u^*}$ be a copy of $M$ where we identify $v$ with $u^*$ and $V$ with $U^1$.

Observe that with these identifications any two copies of $M$ share at most one vertex (see Figure 1). We verify now that $T$ satisfies properties (i) - (iv) required by the lemma.

(i): Let $C$ be some cycle in $T$. Since every edge of $T$ is in a copy of $M$, either
$W = g$ because (because the distance between any two vertices of $W$ is at least $\ell$, because $g(M) \geq \ell$ by Lemma 2 (i). In the second case, $C$ has girth at least $C$ because the distance between any two vertices of $W$ is at least $\ell$ by Lemma 2 (iii). This was for any cycle $C$ in $T$, so $g(T) \geq \ell$.}

**(ii):** Let $(1 \ldots 1)$ denote a vector of $i$ ones. By the definition of $A$ from the beginning of the proof, any mapping $\chi : U \cup \{u^*\} \rightarrow [k]$ can be extended to a $k$-colouring of $M_{(\xi_1 \ldots \xi_i)}$ for all $i = 1, \ldots, d-1$, if and only if

$$\chi((\xi_1 \ldots \xi_i)) \in \{\chi(\xi_1 \ldots \xi_i 1), \ldots, \chi(\xi_1 \ldots \xi_i k)\}. \quad (a)$$

Similarly, $\chi$ can be extended to a $k$-colouring of $M_{u^*}$ if, and only if,

$$\chi(u^*) \in \{\chi(1), \ldots, \chi(k)\}. \quad (b)$$

To prove the ‘only if’ implication of (ii), we consider $\chi : U \cup \{u^*\} \rightarrow [k]$, such that $\chi(u^*) \in \chi(U)$. Assume w.l.o.g. that $\chi(u^*) = 1 = \chi((1 \ldots 1)_d)$, where $(1 \ldots 1)_d \in U$. We must show that $\chi$ can be extended to a $k$-colouring of $T$.

Proceeding backwards for $i = d-1, \ldots, 1$ we do as follows. We define $\chi$ on the set $U^i$ by setting $\chi((\xi_1, \ldots, \xi_i)) = \chi((\xi_1, \ldots, \xi_i, 1))$ for for all $(\xi_1 \ldots \xi_i) \in U^i$. Since this makes (a) true, we can extend $\chi$ to a $k$-colouring of $M_{(\xi_1 \ldots \xi_i)}$ for all $(\xi_1 \ldots \xi_i) \in U^i$. Furthermore, we get that $\chi((1, \ldots, 1)_d) = \chi((1, \ldots, 1)_{d+1}) = 1$. Having done this for $i = 1$, we have $\chi(1) = 1$. Since by assumption $\chi(u^*) = 1$, this makes (b) true, and so we can extend $\chi$ to a $k$-colouring of $M_{u^*}$. We have thus extended $\chi$ to a $k$-colouring of all of $T$.

To prove the ‘if’ implication of (ii), let $\chi : U \cup \{u^*\} \rightarrow [k]$ be such that $\chi(u^*) = 1 \notin \chi(U)$. Towards a contradiction, assume that $\chi$ can be extended to a $k$-colouring of $T$. Since there is a vertex of colour 1 in $\{u^*\}$ and there are none in $U^d = U$, there is a maximal $i$ with $i < d$ such that $1 \in \chi(U^{i-1})$ and $1 \notin \chi(U^i)$. Without loss of generality let $\chi((1 \ldots 1)_{i-1}) = 1$. Thus, $M_{(1 \ldots 1)_{i-1}}$ is $k$-coloured with $\chi((1 \ldots 1)_{i-1}) = 1$ while,

$$1 \notin \{\chi((1 \ldots 1 1)_i), \ldots, \chi((1 \ldots 1 k)_i)\}.$$ 

This, however, contradicts property (a).

**(iii):** Since every edge of $T$ is in a copy of $M$, the shortest path between two vertices of $U \cup \{u^*\}$ is either completely within some copy of $M$, or passes through two vertices of $W = V \cup \{v\}$ in some copy of $M$. Again by Lemma 2
(iii), the length of such a path is at least $\ell$.

(iv): Set $m(k, \ell) = |V(M)|/(k - 1)$. Since every vertex of $T$ is in at least one of the $(k^d - 1)/(k - 1)$ copies of $M$, we have that $|V(T)| \leq \left(\frac{k^d - 1}{k - 1}\right) \cdot |V(M)| < k^d \cdot m(k, \ell)$.

Therefore, properties (i) - (iv) hold, and thus, the proof of the lemma is complete. $\square$

3 The Graph $\hat{G} = \hat{G}(d, k, \ell)$

In this section, we construct a graph $\hat{G} = \hat{G}(d, k, \ell)$, which depends on the integers $d, k$, and $\ell$. In Lemma 7, we observe some important properties of $\hat{G}$.

Given $k$ and $\ell$, any $(k + 1)$-critical subgraph $G$ of $\hat{G}(d, k, \ell)$, for an appropriate choice of $d$, will then satisfy the properties of Theorem 1. This will be proved in Section 4.

Graph $\hat{G}$ will be constructed from several components. We now define these components before giving the actual construction.

Setup for Construction 6 Let the integers $k \geq 3$, $\ell \geq 3$ and $d$ be fixed.

(i) Erdős [3] showed that there exist $k$-critical graphs with girth at least $\ell$. Among such graphs let $F = F(k, \ell)$ be one with the fewest vertices (thus $F$ is connected). Set $f(k, \ell) = |V(F)|$ and $V(F) = \{v_1, \ldots, v_{f(k, \ell)}\}$.

(ii) Let $F_B$ be the blowup of $F$, that is, the graph defined by replacing each vertex $v_i \in V(F)$ with a set $B_i$, of $k^d$ independent vertices, and each edge $\{v_i, v_j\} \in E(F)$ with the complete bipartite graph between the sets $B_i$ and $B_j$. In other words, the graph $F_B$ has vertex set $V(F_B) = \bigcup_{i=1}^{f(k, \ell)} B_i$ and edge set

$$E(F_B) = \left\{ \{b_i, b_j\} \mid b_i \in B_i, b_j \in B_j, \{v_i, v_j\} \in E(F) \right\}.$$ 

(iii) For all $i = 1, \ldots, f(k, \ell)$, let $T_i$ be a copy of the graph $T(d, k, \ell)$ provided by Lemma 4. Let $U_i$ and $u^*_i$ be the copies of $U$ and $u^*$ respectively, in $T_i$.

(iv) Set $W = \{w_1, \ldots, w_{f(k, \ell)}\}$ and let $\mathcal{A}$ be the set of all mappings $\alpha : W \to [k]$, that are ‘inconsistent with respect to $F$’; that is, $\alpha \in \mathcal{A}$ if, and only if, the mapping $\chi : V(F) \to [k]$ defined by $\chi(v_i) = \alpha(w_i)$ for all $i = 1, \ldots, f(k, \ell)$, is not a proper $k$-colouring of $F$. Let $H = M(W, \mathcal{A}, k, \ell)$
be the $k$-chromatic graph returned by Lemma 2 for this choice of $\mathcal{A}, k, \ell,$ and $W,$ and set $h(k, \ell) = |V(H)|.$

We now give the construction.

**Construction 6** Given integers $d, k,$ and $\ell,$ let $F = F(k, \ell)$ and $f(k, \ell)$ be defined as in item (i) of Setup for Construction 6.

Construct $\hat{G} = \hat{G}(d, k, \ell)$ from pairwise disjoint components $F_B, T_1, \ldots, T_{f(k,\ell)},$ and $H,$ defined in items (ii), (iii) and (iv) of Setup for Construction 6, by making the following identifications.

(i) For $i = 1, \ldots, f(k, \ell),$ identify $U_i$ of $T_i$ with $B_i$ of $F_B.$

(ii) For $i = 1, \ldots, f(k, \ell),$ and $j = 1, \ldots, k^d,$ identify $u^{*}_i$ of $T_i$ with $w_i$ of $H.$

Thus, the graph $\hat{G}$ has vertex set

$$V(\hat{G}) = \bigcup_{i=1}^{f(k,\ell)} V(T_i) \cup (V(H) - W)$$

and edge set

$$E(\hat{G}) = (\bigcup_{i=1}^{f(k,\ell)} E(T_i)) \cup E(F_B) \cup E(H).$$

Set $C(k, \ell) = f(k, \ell) \cdot m(k, \ell) + h(k, \ell)$ where $m(k, \ell)$ is the constant from Lemma 4, which is independent of $d.$ In the following lemma, we observe some important properties of Construction 6.

**Lemma 7** Let the integers $k$ and $\ell$ be given. Let $F$ and $H,$ be as in Setup for Construction 6. Then for any positive integer $d,$ the graph $\hat{G} = \hat{G}(d, k, \ell),$ of Construction 6, has the following properties:

(i) $\hat{G}$ is not $k$-colourable.

(ii) For any edge $e \in F_B,$ $\hat{G} - e$ is $k$-colourable.

(iii) $\hat{G}$ has odd girth at least $\ell.$

(iv) $|V(\hat{G})| < C(k, \ell) \cdot k^d.$

(v) $\hat{G}$ cannot be made $(k - 1)$-colourable without removing at least $k^{2d}$ edges of $F_B.$

**Proof.** We will now show that for any positive integer $d,$ the graph $\hat{G} = \hat{G}(d, k, \ell)$ satisfies properties (i) - (v).

(i) $\hat{G}$ is not $k$-colourable.
Towards a contradiction, assume that $\hat{G}$ has a $k$-colouring $\chi$. For $i = 1, \ldots, f(k, \ell)$, $\chi$ induces a $k$-colouring of $T_i$, so by property (ii) of Lemma 4, $\chi(u_i^*) \in \chi(U_i)$. By the identification of $u_i^*$ with $w_i$, and $U_i$ with $B_i$, there is thus some $b_i \in B_i$ such that $\chi(w_i) = \chi(b_i)$.

The set of vertices $\{b_i \mid i = 1, \ldots, f(k, \ell)\}$ induces a copy $F_1$, of $F$, in $\hat{G}$, and $\chi$ induces a $k$-colouring of $F_1$. Thus, by the definition of $A$ in Setup for Construction 6 (iv), $\chi$ restricted to $W = \{w_1, \ldots, w_{f(k, \ell)}\}$ is not in $A$. This contradicts the fact that $\chi$ induces a $k$-colouring of $H$. Therefore, $\hat{G}$ is not $k$-colourable.

(ii) For any edge $e \in F_B$, $\hat{G} - e$ is $k$-colourable.

Without loss of generality assume that $e = \{b_1, b_2\}$, where $b_1$ is in $B_1$ and $b_2$ is in $B_2$. Recall that $F_B$ is the blowup of $F$, so $B_1$ and $B_2$ correspond to vertices $v_1$ and $v_2$ in $F$.

Since $g(F) = \ell \geq 5$ there is some vertex $v_i$ of $F$ that is distance at least two from both vertices $v_1$ and $v_2$. Assume, w.l.o.g., that $i = f(k, \ell)$. Thus, there are no edges from $B_{f(k, \ell)}$ to $B_1$ or $B_2$. Since $F$ is $k$-critical, there exists a $k$-colouring $\chi'$ of the graph $F$ in which the only vertex that gets the colour $k$ is $v_{f(k, \ell)}$.

Consequently, setting

(a) $\chi(b) = \chi'(v_i)$ where $b \in B_i$, for any $b \in V(F_B) - \{b_1, b_2\}$, and
(b) $\chi(b_1) = \chi(b_2) = k$,

defines a $k$-colouring $\chi$ of $F_B - e$. Since $U_i$ is identified with the vertex set $B_i$ of $F_B$, this defines $\chi$ on $U_i$ for all $i = 1, \ldots, f(k, \ell)$.

We then extend $\chi$ to the rest of $\hat{G} - e$ as follows.

(e) For $i = 1, 2$, since $k \in \chi(U_i)$, we can extend $\chi$ to a $k$-colouring of $T_i$ in which $\chi(u_i^*) = k$.
(f) For $i \neq 1, 2$, we extend $\chi$ to any $k$-colouring of $T_i$.
(g) The vertices $w_1$ and $w_2$, by their identification with $u_1^*$ and $u_2^*$, both get the colour $k$ under $\chi$. Since $\{v_1, v_2\}$ is an edge in $F$, $\chi$ then induces on $\{w_1, \ldots, w_{f(k, \ell)}\}$ a mapping that is inconsistent with $F$. Thus, by the choice of $A$ in Construction 6 (iii), $\chi$ can be extended to a $k$-colouring of $H$.

This exhibits a $k$-colouring of $\hat{G} - e$, so $\hat{G} - e$ is $k$-colourable.
(iii) $\hat{G}$ has odd girth at least $\ell$.

First note that the components of $\hat{G}$: the blowup $F_B$, the graphs $T_i$, for all $i = 1, \ldots, f(k, \ell)$, and the graph $H$, have odd girth at least $\ell$. Now let $C$ be some odd cycle of $\hat{G}$ that is not completely contained within one of the above components. The cycle $C$ must contain a vertex $v$ in $V(T_i) - (V(B) \cup V(H))$ for some $i$. Since the cycle $C$ is not entirely within the graph $T_i$, $v$ must be contained on a path of $C$ containing at least two vertices of $U_i \cup \{u^*_i\}$. By property (iii) of Lemma 4 these vertices are at least a distance $\ell$ apart. Thus $C$ must have length at least $\ell$.

(iv) $|V(\hat{G})| < C(k, \ell) \cdot k^d$.

Since every vertex of $F_B$ was identified with a vertex of $T_i$, for some $i \in \{1, \ldots, f(k, \ell)\}$, and $\hat{G}$ was constructed from $F_B$, $H$, and $f(k, \ell)$ copies of $T$ each of which had $|V(T)| < k^d \cdot m(k, \ell)$ vertices by Lemma 4 (iv), it follows that $|V(\hat{G})| < f(k, \ell) \cdot k^d \cdot m(k, \ell) + h(k, \ell) < C(k, \ell) \cdot k^d$.

(v) $\hat{G}$ cannot be made $(k-1)$-colourable without removing at least $k^{2d}$ edges of $F_B$.

To allow the graph $\hat{G}$ to be $(k-1)$-colourable, we need to remove at least one edge from each copy of $F$. Restricting our attention to copies of $F$ in $F_B \subset \hat{G}$ with at most one vertex in each set $B_i$, we find that there are at least $(k^d)^{f(k,\ell)}$ copies of $F$. As any edge in $F_B$ is in at most $(k^d)^{f(k,\ell)-2}$ of these copies of $F$, we need to remove at least $k^{2d}$ edges to be sure we have removed at least one from each copy of $F$. \hfill $\Box$

4 Proof of Theorem 1

We now proceed with the proof of the main theorem.

PROOF. Given $k \geq 3$ and $\ell \geq 5$, let $f(k, \ell)$ and $F_B$ be as in Setup for Construction 6 (i) and (ii), and let $C(k, \ell)$ be the constant from Lemma 7. Set $c(k, \ell) = \left(\frac{1}{C(k,\ell)}\right)^2$. For $\tilde{n}$ given, let $d$ be any integer such that

$$\tilde{n} \leq f(k, \ell) \cdot k^d.$$

Let $G$ be a $(k + 1)$-critical subgraph of the graph $\hat{G} = \hat{G}(d, k, \ell)$ provided by Construction 6, and let $n = |V(G)|$. Note that by property (ii) of Lemma 7,
$G$ has all the vertices and the edges of $F_B$. Therefore,
\[ f(k, \ell) \cdot k^d \leq n, \]
and by property (v) of Lemma 7, we must remove at least $k^{2d}$ edges to make $G$ $(k-1)$-colourable. Since property (iii) of Lemma 7 gives that the odd girth of $G$ is at least $\ell$, it follows that the same holds for the odd girth of $G$. Furthermore,
\[ n \leq |V(\hat{G})| < C(k, \ell) \cdot k^d \]
by property (iv) of Lemma 7, and thus the choice of $c(k, \ell)$ yields that
\[ c(k, \ell) \cdot n^2 = \left( \frac{1}{C(k, \ell)} \right)^2 \cdot n^2 < k^{2d}. \]
Therefore, we must remove at least $c(k, \ell) \cdot n^2$ edges from $G$ to make it $(k-1)$-colourable. We have now verified all the necessary properties and thus the proof is complete. \( \square \)

5 Concluding Remarks

The construction of the graph $\hat{G}$ together with Theorem 1 gives a lower bound for the constant $c$. It may be of interest to obtain a better estimate. A straightforward upper bound on $c(3, \ell)$ follows from a result of Andrásfai, Erdős, and Sós [1] who proved that every nonbipartite graph $G$ on $n$ vertices with odd girth at least $\ell$, must have minimum degree $\delta \leq \frac{2}{\ell} \cdot n$. Indeed, sequentially deleting vertices of minimum degree as long as each remaining graph is nonbipartite, we delete at most
\[ 2\left( \frac{n}{\ell} + \frac{n-1}{\ell} + \cdots + \frac{1}{\ell} \right) \approx \frac{n^2}{\ell} \]
edges. Thus, $c(3, \ell) \leq \left( 1 + o(1) \right)/\ell$, where $o(1)$ is a function of $n$ that goes to 0 as $n$ goes to infinity.

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