Complexity of Locally Injective $k$-Colourings of Planar Graphs

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Abstract
A colouring of the vertices of a graph is called injective if every two distinct vertices connected by a path of length 2 receive different colours, and it is called locally injective if it is an injective proper colouring. We show that for $k \geq 4$, deciding the existence of a locally injective $k$-colouring, and of an injective $k$-colouring, are $NP$-complete problems even when restricted to planar graphs. It is known that every planar graph of maximum degree $\leq \frac{3}{5}k - 52$ allows a locally injective $k$-colouring. To compare the behaviour of planar and general graphs we show that for general graphs, deciding the existence of a locally injective $k$-colouring remains $NP$-complete for graphs of maximum degree $2\sqrt{k}$ (when $k \geq 7$).

1. Definitions and Preliminaries
All graphs in this note are simple undirected graphs without loops. We write $N_G(v) = \{x \mid vx \in E(G)\}$ ($N_G[v] = \{v\} \cup N_G(v)$) to denote the open (closed, respectively) neighbourhood of a vertex $v \in V(G)$ in a graph $G$, and more generally $N_G[A] = \bigcup_{v \in A} N_G[v]$ and $N_G(A) = N_G[A] \setminus A$ for a set $A \subseteq V(G)$ of

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vertices of $G$. For an integer $b$, a graph is $b$-bounded if no vertex has more than $b$ neighbours.

A proper $k$-colouring of a graph $G$ is a map from the vertex set $V(G)$ of $G$ to the set of $k$ colours $[k] = \{1, \ldots, k\}$ such that adjacent vertices are mapped to distinct colours. We refer to the decision problem of determining if an instance graph $G$ has a $k$-colouring as $k$-COL.

A locally injective $k$-colouring of a graph $G$ is a proper $k$-colouring $\chi$ that is injective when restricted to the neighbourhood $N_G[v]$, for every vertex $v$ of $G$. We refer to the decision problem for locally injective $k$-colourings as $k$-LIHOM.

A locally bijective $k$-colouring is defined by replacing 'injective' with 'bijective'. The decision problem is $k$-LBHOM.

An injective $k$-colouring of a graph $G$ is a vertex mapping $\chi : V(G) \to [k]$ which is not necessarily a proper $k$-colouring, but such that $\chi$ is injective on the neighbourhood $N_G(v)$, for every vertex $v$ in $G$. The decision problem for injective $k$-colourings is referred to as $k$-INJ.

The notation is not chosen randomly. An edge preserving vertex mapping between two graphs is called a homomorphism, and homomorphisms are often called generalised colourings since a proper $k$-colouring of a graph is a homomorphism from it into the complete graph on $k$ vertices. A homomorphism $f : G \to H$ is called locally bijective (locally injective) if for every vertex $v \in V(G)$, the restricted mapping $f : N_G[v] \to N_H[f(v)]$ is bijective (injective) [8]. It is easy to see that locally injective $k$-colourings are exactly locally injective homomorphisms into the complete graph $K_k$, and similarly for the locally bijective ones.

Locally bijective homomorphisms were intensively studied under the equivalent notion of graph covers [3, 4, 14, 19, 24]. Locally injective homomorphisms are closely related to distance constrained graph labelings [8], and in particular, many papers on $L(2,1)$-labeling of graphs provide relevant results [5, 10, 15]. The computational complexity of deciding if an input graph allows a locally bijective (injective) homomorphism to a fixed target graph has been considered in [2, 4, 17, 18], but in neither of the two cases has a complete characteriza-
tion been obtained. Even less is known when the input graph is assumed to be planar. The restriction of graph colouring problems to planar inputs has been investigated in [22].

Another perspective is provided by distance powers of graphs. The $d$-th distance power of a graph $G$ (usually denoted by $G^{(d)}$) is the graph with the same vertex set, and edges connecting every two vertices whose distance in $G$ is at most $d$. It is straightforward to see that a locally injective $k$-colouring of a graph is a proper $k$-colouring of its second power, and vice versa. This approach can be found in [1, 7, 13].

Finally, injective colourings have been introduced rather recently in [11] and further results have been obtained in [6, 12, 21].

2. Background and Complexity Results

It is not hard to see that 3-LIHOM is polynomial time solvable. Indeed a graph $G$ has a locally injective 3-colouring if and only if every component is a path or a cycle whose length is a multiple of 3.

In [8], it is shown that any instance of $k$-LBHOM can be reduced, in polynomial time, to an instance of $k$-LIHOM. Since it is shown in [16] that $k$-LBHOM is $NP$-complete for any $k \geq 4$, this shows that $k$-LIHOM is $NP$-complete for any $k \geq 4$.

We consider the complexity of $k$-LIHOM and $k$-INJ for planar graphs. While for $k \geq 4$, the $NP$-complete problem $k$-COL becomes polynomial time solvable when restricted to planar instances, the same does not happen for $k$-LIHOM. It is a consequence of [7] that 4-LIHOM is $NP$-complete for planar instances. Our first result, which we prove in Section 3, is to extend this to all $k \geq 5$.

**Theorem 2.1.** For $k \geq 4$, $k$-LIHOM is $NP$-complete for planar instances.

Since the only graphs that have locally injective $k$-colourings are $(k-1)$-bounded, Theorem 2.1 actually says that $k$-LIHOM is $NP$-complete for $(k-1)$-bounded planar graphs. It is shown in [23] that every $(\frac{k}{5}k - 52)$-bounded planar graph has a locally injective $k$-colouring. Further, it is conjectured by
Wegner that the same is true of every \((\frac{3}{4} k)\)-bounded planar graph. It would be interesting to find the minimum \(b\), \(\frac{3}{4} k - 52 < b \leq k - 1\) for which \(k\)-LIHOM is \(NP\)-complete for \(b\)-bounded graphs.

Theorem 2.1 suggests that in terms of the complexity of \(k\)-LIHOM, restricting to planar instances has no effect. Our next result shows that this is not the case. It shows that compared to the non-planar problem, the planar problem is polynomial time solvable for graphs having much larger degrees. We prove the following in Section 5.

**Theorem 2.2.** For \(k \geq 7\), \(k\)-LIHOM is \(NP\)-complete when restricted to instances of maximum degree at most \(2\sqrt{k} - 1\).

Finally, by providing a polynomial time reduction of any planar instance of \(k\)-LIHOM to a planar instance of \(k\)-INJ, we transport Theorem 2.1 to \(k\)-LIHOM. That is, we prove the following in Section 4.

**Theorem 2.3.** For \(k \geq 4\), \(k\)-INJ is \(NP\)-complete for planar instances.

### 3. Proof of Theorem 2.1

The proof we give here of Theorem 2.1 for \(k \geq 5\), can be slightly altered to give a constructive proof for the case \(k = 4\). However, since this case follows from [7], we forgo the proof.

It was shown in [20] that the problem 3-COL is \(NP\)-complete even when restricted to planar instances. Using this result, we prove Theorem 2.1 as follows. For fixed \(k\), we reduce any planar instance \(G\) of 3-COL, to a planar instance \(G'\) of \(k\)-LIHOM, in polynomial time. That is, we construct a planar graph \(G'\) such that \(G'\) has a locally injective \(k\)-colouring if and only if \(G\) has a 3-colouring. By the result of [20], this will prove the theorem.

The construction of \(G'\) is fairly complicated. In Subsections 3.1 to 3.6, we break it down into several smaller constructions, or gadgets. After constructing each gadget we observe certain properties, in the form of Lemmas 3.1 to 3.6. The proofs of these properties are all similar, and fairly straightforward, so we omit all but the first. After doing this, we formally begin the proof of Theorem 2.1 in Subsection 3.7.
We now define these gadgets for all $k \geq 5$.

3.1. The Doubling Gadget $D = D(k)$

Let $D = D(k)$ be as in Figure 1, where the dashed sets each contain $k - 3$ vertices.

![Diagram of the Gadget $D$](image)

Figure 1: The Gadget $D = D(k)$

**Lemma 3.1.** A mapping $\chi : \{a_1, a_2, a'_1, a'_2\} \rightarrow [k]$ can be extended to a locally injective $k$-colouring of $D$ if and only if the following conditions are satisfied.

i. $\chi(a_1) = \chi(a_2)$

ii. $\chi(a_i) \neq \chi(a'_i)$ for $i = 1, 2$

**Proof.** Suppose $\chi$ is a locally injective $k$-colouring of $D$. Since $a'_1$ and $a''_1$ have degree $k - 1$, their closed neighbourhoods contain all colours, each exactly once. Since $N_D[a'_1] \setminus \{a_1\} = N_D[a''_1] \setminus \{a_3\}$, it must be that $\chi(a_1) = \chi(a_3)$. Similarly, $\chi(a_3) = \chi(a_2)$. This proves (i), while (ii) follows immediately from the fact that $\chi$ is a proper colouring. On the other hand, a simple case analysis shows that any colouring of $a_1, a_2, a'_1, a'_2$ satisfying (i-ii) can be extended to a locally injective $k$-colouring of the whole gadget.

Given an edge $uv$ in a graph, we say that we replace $uv$ with $D$ to mean that we remove the edge $uv$, and then identify $u$ with $a_1$ and $v$ with $a_2$ of $D$.

3.2. The Switching Gadget $S = S(k)$

Let $S = S(k)$ be as in Figure 2 where the sets $A$ and $B$ each contain $k - 3$ vertices. Here we use several copies of the gadget $D(k)$ from the previous subsection. These are represented by the schematic of $D = D(k)$ given in Figure 1.
A

A

B

B

Schematically

Figure 2: The Gadget $S = S(k)$

$a_1$  $a_3$

$a_2$  $a_4$

$k - 4$ vertices

Schematically

Figure 3: The Gadget $X = X(k)$

Lemma 3.2. A mapping $\chi$ from $N_S[A \cup B]$ to $[k]$ can be extended to a locally injective $k$-colouring of $S$ if and only if the following conditions are satisfied.

i. The sets \{\chi(x) \mid x \in A\} and \{\chi(x) \mid x \in B\} contain the same $k - 3$ distinct colours.

ii. For $x \in A \cup B$, $\chi(x) \neq \chi(x')$ where $x'$ is the unique neighbour of $x$ in $S$.

\[\square\]

3.3. The Crossing Gadget $X$

Let $X = X(k)$ be as in Figure 3 where the dashed set contains $k - 4$ vertices.

Lemma 3.3. A mapping $\chi$ from $N_X[a_1, \ldots, a_4]$ to $[k]$ can be extended to a locally injective $k$-colouring of $X$ if and only if the following conditions are satisfied.

i. $\chi(a_1) = \chi(a_4) \neq \chi(a_3) = \chi(a_2)$
ii. For \( x = a_1, \ldots, a_4, \chi(x) \neq \chi(x'), \) where \( x' \) is the unique neighbour of \( x \) in \( X \).

\[ \square \]

3.4. The Splitting Gadget \( Y \)

Let \( Y = Y(k) \) be the graph shown for \( k = 7 \) in Figure 4. For general \( k \), the sets \( A_i \) for \( i = 1, \ldots, 3 \) will each have \( k - 3 \) vertices. The graph \( Y \) is obtained from the three copies of the gadget \( S = S(k) \) by drawing paths from the each vertex of \( B \) from one copy of \( S \) to a vertex of \( B \) in each of the other copies of \( S \). Then each crossing is replaced by a gadget \( X(k) \), and the two paths that do not cross any other paths are replaced by a gadget \( D(k) \). (Hence the graph contains \((k - 4) + (k - 5) + \cdots + 1\) copies of \( X(k) \).)

Lemma 3.4. A mapping \( \chi \) from \( N_Y[A_1 \cup A_2 \cup A_3] \) to \([k]\) can be extended to a locally injective \( k \)-colouring of \( Y \) if and only if the following conditions are satisfied.

i. The sets \( \{ \chi(x) \mid x \in A_i \} \), contain the same \( k - 3 \) distinct colours for \( i = 1, 2, 3 \).

ii. For \( x \in A_1 \cup A_2 \cup A_3, \chi(x) \neq \chi(x') \), where \( x' \) is the unique neighbour of \( x \) in \( Y \).

\[ \square \]
Using the above gadgets, we will now define gadgets that we will use to replace the vertices and edges of a planar instance $G$ of 3-COL, in order to reduce it to a planar instance of $k$-LIHOM.

### 3.5. Vertex Expansion $v \rightarrow V_v = V_v(k)$

Given integer $d \geq 2$, and $k \geq 5$, let $V_d(k)$ be a binary tree with $d$, in which we replace each edge with a copy of $D = D(k)$. Given a vertex $v$ of degree $d$ in some graph $G$, we say that we expand $v$, or replace $v$ with $V_v$, to mean the following process. (See Figure 5).

- Remove $v$ from $G$.
- Add a copy $V_v$ of $V_d(k)$ to $G$ and label the leaves $v_1, \ldots, v_d$.
- Add a matching between the neighbourhood, $N_G(v)$, of $v$ in $G$, and the set $\{v_1, \ldots, v_d\}$ of vertices of $V_v$.

**Lemma 3.5.** A mapping $\chi$ from $N_{V_v}[v_1, \ldots, v_d]$ to $[k]$ can be extended to a locally injective $k$-colouring of $V_v$ if and only if the following conditions are satisfied.

1. $\chi(v_1) = \chi(v_2) = \cdots = \chi(v_d)$
2. For $x \in \{v_1, \ldots, v_d\}$, $\chi(x) \neq \chi(x')$, where $x'$ is the unique neighbour of $x$ in $V_v$. 

$\square$
3.6. Edge Replacement $e \rightarrow E_e = E_e(k)$

Given an edge $e = uv$ in some graph $G$, we say that we replace $e$ (or $uv$) with $E_e$ (or $E_{uv}$) to mean the following process.

- Remove the edge $e = uv$.
- Add the graph $E_{uv}$ shown in Figure 6, between $u$ and $v$.

### Lemma 3.6

A mapping $\chi$ from $N_{E_e}[A_e \cup B_e \cup \{u, v\}]$ to $[k]$ can be extended to a locally injective $k$-colouring of $E_e$ if and only if the following conditions are satisfied.

i. The sets $\{\chi(x) \mid x \in A_e\}$ and $\{\chi(x) \mid x \in B_e\}$ contain the same $k - 3$ colours.
ii. $\chi(u) \neq \chi(v)$
iii. $\chi(u), \chi(v) \notin \{\chi(x) \mid x \in A_e\}$
iv. For $x \in A_e \cup B_e \cup \{u, v\}$, we have $\chi(x) \neq \chi(x')$, where $x'$ is the unique neighbour of $x$ in $E_e$.

3.7. Proof of Theorem 2.1

Let $G$ be a planar instance of 3-COL with a given embedding. We will construct a planar graph $G'$ such that $G$ has a 3-colouring if and only if $G'$ has a locally injective $k$-colouring. We construct $G'$ from $G$ as follows. (See Figure 7 for an example of the construction.)
For each edge $uv$ of $G$ replace $uv$ with $E_{uv}$.

Expand each original vertex $v$ of $G$. (Thus for every edge $uv$ of $G$, the graph $E_{uv}$ now contains the vertices $u_i$ and $v_j$ for some $i$ and $j$.)

For each edge $e$ of each original face $f$ of $G$, either $A_e$ or $B_e$ lies on $f$. For each such $e$ and $f$ let $Y_{f,e}$ be a copy of $Y(k)$ and identify the copy of $A_2$ in $Y_{f,e}$ with $A_e$ or $B_e$, whichever lies on $f$. For consecutive edges $e$ and $e'$ of $f$, proceeding clockwise around the face, identify the copy of $A_1$ in $Y_{f,e}$ with the copy of $A_3$ in $Y_{f,e'}$. Do this preserving planarity.

Call the resulting planar graph $G'$.

We now complete the proof by showing that $G'$ has a locally injective $k$-colouring if and only if $G$ has a 3-colouring. Let $\chi'$ be a locally injective $k$-colouring of $G'$, and define the mapping $\chi : V(G) \to [k]$ by $\chi(v) = \chi'(v_1)$ for each $v \in V(G)$.

By the properties of $Y(k)$ and $E(k)$ (Lemmas 3.4 and 3.6), the sets $\{\chi'(x) \mid x \in A_e\} = \{\chi'(x) \mid x \in B_e\}$ are the same for all edges $e$ of $G$. Assume,
without loss of generality, that this set is \(\{4, \ldots, k\}\). Then by Lemma 3.6, 
\(\chi'(v_1) \in \{1, 2, 3\}\) for all \(v\) in \(V(G)\), so \(\chi\) is actually a mapping of \(V(G)\) to \(\{1, 2, 3\}\). Furthermore, for each edge \(uv\) of \(G\), there is an edge \(u_i v_j\) in \(G'\) for some vertex \(u_i\) of \(V_u\) and \(v_j\) of \(V_v\). Thus 
\[\chi(u) = \chi'(u_1) = \chi'(u_i) \neq \chi'(v_j) = \chi'(v_1) = \chi(v),\]
and so \(\chi\) is a 3-colouring of \(G\).

Assume, for the inverse implication, that \(\chi\) is a 3-colouring of \(G\), and define a mapping \(\chi'\) from \(V(G')\) to \([k]\) as follows.

i. For each vertex \(v\) in \(G\) and for \(i = 1, \ldots, \deg(v)\), set \(\chi'(v_i) = \chi(v)\).

ii. For each edge \(e\) of \(G\), define \(\chi'\) on \(A_e\) and \(B_e\) so that 
\[\{\chi'(x) \mid x \in A_e\} = \{\chi'(x) \mid x \in B_e\} = \{4, \ldots, k\}.\]
Do the same for all sets \(A_1, A_2,\) and \(A_3\) (in copies of \(Y(k)\)) on which \(\chi'\) has not yet been defined.

iii. All vertices for which \(\chi'\) has been defined so far have degree at most 3 and are distance at least 6 apart, so we can easily extend \(\chi'\), as a locally injective \(k\)-colouring, to their neighbourhoods. Do this arbitrarily.

For any vertex \(x\) in \(G'\) for which \(N_{G'}[x]\) is not yet fully coloured, \(N_{G'}[x]\) is entirely within \(V_v\) or \(E_e\) for some vertex \(v\) or edge \(e\) of \(G\), or within \(Y_{f,e}\) for some face \(f\) of \(G\) and some edge \(e\) of \(f\). Thus we need only extend \(\chi'\) to a locally injective \(k\)-colouring of these gadgets.

For any edge \(uv\) of \(G\), let \(u_i\) and \(v_j\) be the vertices of \(E_{uv}\) that are shared with \(V_u\) and \(V_v\) respectively. We have by step (ii) of our colouring that 
\[\{\chi'(x) \mid x \in A_e\} = \{\chi'(x) \mid x \in B_e\} = \{4, \ldots, k\}.\]
This gives us condition (i) of Lemma 3.6. Step (i) gives us that \(\chi'(u_i) = \chi(u) \neq \chi'(v_j) = \chi'(v_1) = \chi(v)\), which is conditions (ii) and (iii) of Lemma 3.6. Step (iii) gives us condition (iv) of Lemma 3.6, and so \(\chi'\) can be extended to a locally injective \(k\)-colouring of \(E_{uv}\).

The verifications that \(\chi'\) can be extended to the gadgets \(V_v\) and \(Y_{f,e}\) are even more straightforward, so we omit them. This completes the proof of Theorem 2.1.
4. Proof of Theorem 2.3.

In this section, we reduce planar $k$-LIHOM to planar $k$-INJ, for $k \geq 4$, showing that planar $k$-INJ is NP-complete.

4.1. The Edge Gadget $E(k)$

Given an edge $uv$ of some graph $G$, let $E_{uv}(k)$ be the graph in Figure 8, where the dashed set contains $k - 2$ vertices.

Lemma 4.1. A mapping $\chi$ of $\{u, v, u'_v, v'_u\}$ to $[k]$ can be extended to an injective $k$-colouring of $E_{uv}(k)$ if and only if $\chi(u) = \chi(v) = \chi(v'_u) \neq \chi(u'_v)$.

Proof. Suppose $\chi$ is an injective $k$-colouring of $E_{uv}(k)$. The $k - 2$ vertices in the dashed set all have paths of length 2 between them, so they get distinct colours. All of the other vertices have paths of length 2 to each vertex of the dashed set, so each must get one of the remaining 2 colours. From the path $u, v'_u, u', v, v'_u, v$, it follows that $\chi(u) = \chi(v') \neq \chi(v'_u)$ and $\chi(u) \neq \chi(u') \neq \chi(u'_v)$ and thus $\chi(u) = \chi(u') = \chi(v'_u) \neq \chi(v) = \chi(u') = \chi(v'_u)$.

4.2. Proof of Theorem 2.3.

Let $G$ be a planar instance of $k$-LIHOM. Construct a planar graph $G'$ from $G$ by replacing each edge $uv$ of with the graph $E_{uv} = E_{uv}(k)$ of Subsection 4.1. We now show that $G$ has a locally injective $k$-colouring if and only if $G'$ has an injective $k$-colouring. By Theorem 2.1, this will prove the theorem.

Let $\chi'$ be an injective $k$-colouring of $G'$, and let $\chi$ be its restriction to $V(G)$. For any vertex $v$ of $G$, Lemma 4.1 gives us that $\chi(v) = \chi'(v) \neq \chi'(u) = \chi(u)$ for any $u \in N_G(v)$. Moreover, $\chi'$ (and so $\chi$) is injective on $N_G(v)$. Indeed if $\chi'(u_1) = \chi'(u_2)$ for $u_1 \neq u_2 \in N_G(v)$, then $\chi'(v'_u) = \chi'(u_1) = \chi'(u_2) = \chi'(v'_u)$,
contradicting the fact that \( \chi' \) is injective on \( N_{G'}(v) \). Thus \( \chi \) is a locally injective \( k \)-colouring of \( G \).

On the other hand, let \( \chi \) be a locally injective \( k \)-colouring of \( G \), and define the mapping \( \chi' : V(G') \to [k] \) as follows.

- For each vertex \( v \in V(G) \subset V(G') \) let \( \chi'(v) = \chi(v) \).
- For each edge \( uv \) of \( G \), let \( \chi'(u') = \chi'(u) \) and \( \chi'(v'_u) = \chi(v) \) and extend \( \chi' \) to an injective \( k \)-colouring of \( E_{uv} \).

Since \( \chi' \) is defined to be locally injective on copies of \( E_{uv} \) we just have to check that it is locally injective with respect to vertices whose neighbourhoods are not entirely within a single copy of \( E_{uv} \). These are the vertices of \( G \).

Let \( v \) be a vertex of \( G \) and assume that \( \chi'(v'_u) = \chi'(v'_u) \) for \( v'_u \neq v'_u \in N_{G'}(v) \). Then \( \chi(u_1) = \chi'(v'_u) = \chi'(v'_u) = \chi(u_2) \), contradicting the fact that \( \chi \) is injective on \( N_{G'}(v) \). Thus \( \chi' \) is a locally injective \( k \)-colouring of \( G' \). This completes the proof of the theorem.

5. Proof of Theorem 2.2.

We show that for \( k \geq 7 \), \( k \)-LIHOM is \( \mathsf{NP} \)-complete for \( (2\sqrt{k-1}) \)-bounded graphs. We do this by taking any instance \( G \) of \( k \)-COL and reducing it to a \( (2\sqrt{k-1}) \)-bounded instance \( G' \) of \( k \)-LIHOM. Since \( k \)-COL is \( \mathsf{NP} \)-complete, this is enough.

For the proof we will construct a vertex replacement gadget \( V = V(k) \), which depends on an auxiliary gadget \( D = D(k) \).

5.1. Gadget \( D = D(k) \)

The gadget \( D \) varies slightly depending on \( k \). There are two cases.

**Case 1:** \( \ell^2 \leq k - 1 \leq \ell(\ell + 1) \) for some integer \( \ell \).

Let \( r = (k - 1) - \ell^2 \), and define \( D \) as follows.

- Let \( A = \{a_{i,j} \mid 1 \leq i, j \leq \ell\} \cup \{a_{\ell+1,j} \mid 1 \leq j \leq r\} \) be a set of \( k - 1 \) independent vertices.
• Put an edge between \(a_{i,j}\) and \(a_{i',j'}\) if \(i = i'\) or \(j = j'\).

• Let \(C = \{c_1, c_2, c_3\}\) be new vertices, and put an edge between \(a_{i,j}\) and \(c_{i'}\) if \(i = i'\). (Since \(k \geq 7, \ell \geq 3\), all vertices of \(C\) get neighbours in \(A\).)

**Lemma 5.1.** For \(\ell^2 \leq k - 1 \leq \ell(\ell + 1)\), the following properties hold for \(D\).

i. Under any locally injective \(k\)-colouring, the vertices of \(C\) all get the same colour.

ii. The vertices of \(A\) have degree at most \(2\ell \leq 2\sqrt{k - 1}\).

iii. The vertices of \(C\) have degree at most \(\ell \leq \sqrt{k - 1}\).

**Proof.** For item (i) observe that the \(k - 1\) vertices of \(A\) are distance at most 2 apart, so all get distinct colours, and the vertices of \(C\) are distance 3 apart and distance at most 2 from everything in \(A\). The rest of the proof is straightforward.

**Case 2:** \(\ell(\ell + 1) < k - 1 < (\ell + 1)^2\) for some integer \(\ell\).

Let \(r = (k - 1) - \ell(\ell + 1)\), and define \(D\) as follows.

• Let \(A = \{a_{i,j} \mid 1 \leq i \leq \ell + 1, 1 \leq j \leq \ell\} \cup \{a_{i,\ell + 1} \mid (\ell + 1) - (r - 1) \leq i \leq \ell + 1\}\) be a set of \(k - 1\) independent vertices.

• Put an edge between \(a_{i,j}\) and \(a_{i',j'}\) if \(i = i'\) or \(j = j'\).

• For \(i' = (\ell + 1) - (r - 1), \ldots, \ell + 1,\) and \(i = 1, \ldots 3,\) if there is no vertex \(a_{i,\ell + 1}\), then put an edge between \(a_{i',\ell + 1}\) and \(a_{i,\ell + 1}\).

• Let \(C = \{c_1, c_2, c_3\}\) be a set of new vertices, and put an edge between \(a_{i,j}\) and \(c_{i'}\) if \(i = i'\).

**Lemma 5.2.** For \(\ell(\ell + 1) < k - 1 < (\ell + 1)^2\), the following properties hold for \(D\).

i. Under any locally injective \(k\)-colouring, the vertices of \(C\) all get the same colour.

ii. The vertices of \(A\) have degree at most \(2\ell + 1 \leq 2\sqrt{k - 1}\).

iii. The vertex \(c_1\) has degree \(\ell\), and the other vertices of \(C\) have degree at most \(\ell + 1\).

**Proof.** We only observe how \(2\ell + 1 \leq 2\sqrt{k - 1}\) follows from the condition \(\ell(\ell + 1) < k - 1\). Indeed, by completing the square one gets \(\ell + 1/2 < \sqrt{k - 3}/4\), so \(2\ell + 1 < \sqrt{4k - 3}\). Now if \(2\ell + 1 > 2\sqrt{k - 1} = \sqrt{4k - 4}\), then \(\sqrt{4k - 4} < 2\ell + 1 < \sqrt{4k - 3}\), and so there is a proper square strictly between \(4k - 4\) and \(4k - 3\), which is impossible.
5.2. Vertex replacement \( v \rightarrow V_v = V_v(k) \).

Let \( k \geq 7 \) be fixed. Given an integer \( d \geq 2 \), let \( V_d \) be the graph constructed from \( d \) mutually vertex disjoint copies \( D_1, \ldots, D_d \) of the graph \( D(k) \) by identifying the copy of \( c_3 \) in \( D_i \) and the copy of \( c_1 \) in \( D_{i+1} \) for \( i = 1, \ldots, d-1 \).

Given a vertex \( v \) of degree \( d \) in a graph \( G \), we say that we expand \( v \) or replace \( v \) with \( V_v \) to mean the following process.

- Remove \( v \) from \( G \), and add \( d \) edgeless vertices \( v_1, \ldots, v_d \).
- Add a copy \( V_v \) of \( V_d \) to \( G \), for \( i = 1, \ldots, d \), identify the copy of \( c_2 \) in \( D_i \) in \( V_v \) with \( v_i \).
- Add a matching between \( \{v_1, \ldots, v_d\} \) and \( N_G(v) \).

**Lemma 5.3.** The following properties hold for \( V_v \).

i. For any locally injective \( k \)-colouring \( \chi \), \( \chi(v_1) = \cdots = \chi(v_d) \).

ii. The maximum degree in \( V_v \) is \( 2\sqrt{k-1} \).

iii. The maximum degree in \( V_v \) of any vertex of \( V_v \) with neighbours outside of \( V_v \) (i.e., any copy of \( c_2 \)) is at most \( \sqrt{k-1} + 1 < 2\sqrt{k-1} \).

iv. For any choice of colours \( x_0, x_1, \ldots, x_d \) from \([k]\) with \( x_0 = x_i \) for \( i = 1, \ldots, d \), there is a locally injective \( k \)-colouring \( \chi \) of \( V_v \) such that \( \chi(v_1) = \cdots = \chi(v_d) = x_0 \), and for \( i = 1, \ldots, d \), no neighbour of \( v_i \) in \( V_v \) gets colour \( x_i \), i.e., \( (*) \)

\[ x_j \notin \{\chi(u) \mid u \in N_{V_v}(v_i)\} \]

**Proof.** The first three properties follow easily from Lemmas 5.1 and 5.2. To see property (iv) for Case 1, define \( \chi \) as follows. Set \( \chi \) to \( x_0 \) on all vertices of \( C \) in all copies of \( D(k) \) in \( V_v \). For \( i = 1, \ldots, \ell \), set \( \chi \) to \( x_i \) on the copy of \( a_{1,1} \) in \( D_i \). This gives us \( (*) \). It is easy now to extend \( \chi \) to a locally injective \( k \)-colouring of \( V_v \). We define \( \chi \) greedily, beginning on \( D_1 \), and proceeding to \( D_2 \) and continuing through to \( D_{\ell} \). We must only be careful that when we define \( \chi \) on \( D_i \) we let \( \chi(a_{1,j}) = x_{i+1} \) for some \( j \), so that it is distance 3 from the vertex \( a_{1,1} \) of \( D_{i+1} \). The proof in Case 2 is similar.

5.3. Proof of Theorem 2.2.

Let \( G \) be an instance of \( k \)-COL, and let \( G' \) be the graph constructed by expanding each vertex of \( G \). By Lemma 5.3 the maximum degree of \( G' \) is at most \( 2\sqrt{k-1} \). Thus, using the fact that \( k \)-COL is \( NP \)-complete, we prove the theorem by showing that \( G \) has a \( k \)-colouring if and only if \( G' \) has a locally injective \( k \)-colouring.
Let $\chi'$ be a locally injective $k$-colouring of $G'$, and define $\chi : V(G) \to [k]$ by setting $\chi(v) = \chi'(v_1)$ for all $v \in V(G)$. Then for every edge $uv \in G$, there exists an edge $u_i v_j$ in $G'$ for some $u_i \in V_u$ and $v_j \in V_v$, so

$$\chi(u) = \chi'(u_1) = \chi'(u_i) \neq \chi'(v_j) = \chi'(v_1) = \chi(v).$$

Thus $\chi$ is a $k$-colouring of $G$.

On the other hand, let $\chi$ be a $k$-colouring of $G$, and define $\chi' : V(G') \to [k]$ as follows. For $v \in V(G)$, property (iv) of Lemma 5.3 implies that there is a locally injective $k$-colouring $\chi_v'$ of $V_v$ such that $\chi_v'(v_1) = \cdots = \chi_v'(v_{d_v})$ where $d_v = |N_G(v)|$, and for each $w \in N_G(v)$, $\chi(w) \notin \{ \chi_v'(u) \mid u \in N_{V_v}(v_i) \}$, where $v_i$ is the neighbour of $w$ in $V_v$. Let $\chi'$ restricted to $V(V_v)$ be this $\chi_v'$.

By its definition via the $\chi_v'$, $\chi'$ is locally injective on any vertex whose neighbourhood is entirely within $V_v$ for some $v \in V(G)$. We just have to check that it is locally injective with respect to $v_i$ for $v \in V(G)$ and $i = 1, \ldots, |N_G(v)|$.

But again by definition of $\chi'$ the neighbours of $v_i$ in $V_v$ get distinct colours that are different from $\chi(w)$, which is the colour of the only other neighbour of $v_i$ in $G'$.

Thus $\chi'$ is a locally injective $k$-colouring of $G'$. This completes the proof of the theorem.

### 6. Conclusion and Final Remarks

We have shown that for every $k \geq 4$ deciding the existence of a locally injective $k$-colouring is NP-complete for planar inputs. This can be viewed as a start to discussing the computational complexity of the question of locally injective homomorphisms into fixed graphs for planar inputs. It is plausible to conjecture that for every fixed target graph $H$, the problem restricted to planar inputs is as difficult as the general one. However, if this is true, it may not be so easy to prove, since it would imply that $L(2,1)$-labeling of span 4 is NP-complete for planar graphs, which is a well known open problem in the area of Frequency Assignment Problems (known to be hard only for span 8 [5]).
Another natural question is the complexity of locally bijective homomorphisms restricted to planar inputs. Here the first open question is already posed by $K_4$ as the target graph. Or equivalently, how difficult is to decide existence of locally injective 4-colouring for planar cubic graphs? (The result of [7] concerns only subcubic graphs, i.e., graphs of maximum degree 3).

We have further shown that for general graphs, locally injective $k$-colouring remains NP-complete for graphs of substantially smaller maximum degree (for $(2\sqrt{k-1})$-bounded graphs). This is (upto a multiplicative constant) best possible, since every $(\sqrt{k-1})$-bounded graph is locally injective $k$-colourable. However, any improvement on the bound of maximum degree would be interesting even for small values of $k$. For instance, the construction in the proof of our Theorem 2.2 works for $k \geq 7$. So what is the smallest $d$ such that 6-LIHOM is NP-complete for $d$-bounded graphs? Our current best bound is 5.

In case of planar inputs, our constructions yield graphs that have maximum degree $k - 1$. It would be interesting to know if this bound can be improved.

References


[14] P. Hliněný, $K_{4,4} - e$ has no finite planar cover. J. Graph Th. 32 (1998) 51-60


